MATH 504
EXERCISES 8

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(1) Let R and S be two rings.
- Define appropriate binary operations on $R \times S$ sot hat $R \times S$ becomes a ring.
- Show that $R \times S$ is commutative if and only if both R and S are commutative.
- Show that $R \times S$ has identity if and only if both R and S have identity.
- Show that $R \times S$ is never an integral domain.
- Show that $\{(x, 0) \in R \times S \mid x \in R\}$ and $\{(0, y) \in R \times S \mid y \in S\}$ are subrings of $R \times S$. Are they ideals?
- Let I be and ideal of R and J be an ideal of S. Is the set $I \times J$ an ideal of $R \times S$

(2) Show that if R is a commutative ring with unity then the commutativity of addition is forced by distributivity.

(3) Decide if the following subsets are subrings of the ring of all functions $f \colon \mathbf{R} \to \mathbf{R}$ under pointwise addition and multiplication, denoted by $R = F(\mathbf{R}, \mathbf{R})$?
- $\{f \in R \mid f(t) = 1 \text{ for all } t \in \mathbf{Q}\} \subseteq R$
- $\{f \in R \mid f(t) = 0 \text{ for all } t \in \mathbf{Q}\} \subseteq R$
- $\{f \in R \mid f(t) = 1 \text{ for all } t \in \mathbf{R} \setminus \mathbf{Q}\} \subseteq R$
- $\{f \in R \mid f(t) = 0 \text{ for all } t \in \mathbf{R} \setminus \mathbf{Q}\} \subseteq R$
- $\{f \in R \mid f \text{ is a polynomial}\} \subseteq R$
- $\{f \in R \mid f \text{ is continuous}\} \subseteq R$
- $\{f \in R \mid f \text{ is differentiable}\} \subseteq R$

(4) Let D be a square-free integer and set $R = \mathbf{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbf{Z}\}$.
- Show that R is a ring with respect to usual addition and multiplication. Deduce that R is a subring of $\mathbf{R}$ if and only if $D > 0$.
- Show that the map $N \colon R \to \mathbf{Z}$ defined as $N(a + b\sqrt{D}) = a^2 - b^2 D$ is multiplicative : that is for $\alpha, \beta \in R$, $N(\alpha \cdot \beta) = N(\alpha)N(\beta)$.
- Deduce that an element $\alpha \in R$ is a unit if and only if $N(\alpha) = \pm 1$.
- Use this information to determine $R^{\times} = (\mathbf{Z}[\sqrt{D}])^{\times}$ when $D \cong 1 \pmod 4$
- Use this information to determine $R^{\times} = (\mathbf{Z}[\sqrt{D}])^{\times}$ when $D \cong 2, 3 \pmod 4$

(5) Let R be a ring with identity. Show that
- if R is non-commutative, the only right ideals of R are $\{0\}$ and R if and only if R is a division ring.
- if R is commutative, the only ideals of R are $\{0\}$ and R if and only if R is a field.

(6) Determine the unit group of the ring $R = (F(\mathbf{R}, \mathbf{R}), +, \circ)$

(7) Let R be a ring and X be any subset of R. Show that
- $\mathrm{Ann}_l(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\}$ is a left ideal of R.
- $\mathrm{Ann}_r(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}$ is a right ideal of R

(8) Let R be a commutative ring with unity and I, J be ideals of R. Show that the following sets are again ideals of R :
- $I \cap J$
- $I + J = \{\alpha + \beta \in R \mid \alpha \in I,\ \beta \in J\}$
- $IJ = \left\{\sum_{i=1}^{n} \alpha_i \beta_i \in R \mid n \in \mathbf{Z}_+ \alpha_i \in I,\ \beta_i \in J \text{for all } i \in \{1, \dots, n\}\right\}$
- $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbf{Z}\}$ (this ideal is called the radical of I.)