MATH 504
EXERCISES 9

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(1) Determine if the following maps are ring homomorphisms or not. If yes, determine $\ker(\varphi)$ and $\mathrm{im}(\varphi)$.
  - ▶ for any non-zero integer n :
$$\begin{aligned}\varphi_n \colon \mathbf{Z} &\to \mathbf{Z}\\ a &\mapsto n \cdot a\end{aligned}$$
  - ▶ for any positive integer n :
$$\begin{aligned}\varphi_n \colon M_n(\mathbf{R}) &\to \mathbf{R}\\ M &\mapsto \det(M)\end{aligned}$$
  - ▶ for any positive integer n :
$$\begin{aligned}\varphi_n \colon M_n(\mathbf{R}) &\to \mathbf{R}\\ M &\mapsto \mathrm{tr}(M)\end{aligned}$$
  - ▶
$$\begin{aligned}\varphi \colon M_2(\mathbf{R}) &\to \mathbf{R}\\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} &\mapsto r + s\end{aligned}$$
  - ▶
$$\begin{aligned}\varphi_n \colon M_2(\mathbf{R}) &\to \mathbf{R}\\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} &\mapsto p\end{aligned}$$

(2) Let $\varphi : R \to S$ be a ring homomorphism. Decide whether the following statements hald true. If yes, give a proof, else give a counter-example :
  - ▶ If $u \in R$ is a unit of R (i.e. an invertible element), then $\varphi(u)$ is a unit of S.
  - ▶ If $\varphi(u)$ is a unit of S then $u \in R$ is a unit of R.
  - ▶ If $v \in R$ is a unit of S, then $\varphi^{-1}(v)$ is a unit of R.
  - ▶ If $r \in R$ is a zero divisor then $\varphi(r)$ is a zero divisor of S. Is the converse true?
  - ▶ If $\varphi(u)$ is a zero divisor of S then $u \in R$ is a zero divisor of R.
  - ▶ If $s \in S$ is a zero divisor then each element of the set $\varphi^{-1}(s)$ is a zero divisor of R.

(3) Prove that $\mathbf{R}[X]/(X^2 + 1)$ and $\mathbf{C}$ are isomorphic.

(4) Prove that the rings $\mathbf{R}[X]$ and $\mathbf{Z}[X]$ are not isomorphic.

(5) Consider the polynomial $p(X) = X^2 + 1 \in (\mathbf{Z}/3\mathbf{Z})[X]$.
  - ▶ Explicitly write each element of the ring $R = (\mathbf{Z}/3\mathbf{Z})[X]/(p(X))$.
  - ▶ Show that R is an integral domain.
  - ▶ Find the multiplicative inverse of $X + (p(X)) \in R$.
  - ▶ By finding multiplicative inverses of remaining non-zero elements of R, show that R is a field. Deduce that $(p(X))$ is a maximal ideal.
  - ▶ Are R and $\mathbf{Z}/9\mathbf{Z}$ isomorphic?

(6) If R and S are two commutative rings with identity, we have defined a ring homomorphism to send 1_R to 1_S. Show that this need not be the case in general.

(7) Show that if $\varphi \colon R \to S$ and $\psi \colon S \to T$ are ring homomorphisms, then so si $\psi \circ \varphi \colon R \to T$.

(8) Let $\varphi\colon R \to S$ be a ring homomorphism.
- ► Show that if J is an ideal of S then $\varphi^{-1}(J)$ is an ideal of R. Show that $\varphi^{-1}(J)$ contains $\ker(\varphi)$.
- ► Show that if I is an ideal of R then $\varphi(I)$ need not be an ideal of S.

(9) Let R be a commutative ring with identity. Show that R is a field if and only if for any commutative ring with identity every non-trivial homomorphism $\varphi\colon R \to S$ is injective.

(10) Let R be a ring and let I be an ideal of R. Show that the canonical projection $\pi\colon R \to R/I$ induces an order preserving one-to-one correspondence between ideals of R/I and ideal of R containing I.

(11) Let R be a commutative ring. Show that (0) is a prime ideal if and only if R is an integral domain.

(12) Let $\varphi\colon R \to S$ be a ring homomorphism between commutative rings with identities. Show that if J is a prime ideal of S then $\varphi^{-1}(J)$ is a prime ideal of R. Does the analogous claim for maximal ideals hold true?

(13) Let R be a commutative ring with identity. An element $r \in R$ is called nilpotent if $r^n = 0$ for some $n \in \mathbf{Z}_+$. Show that if r is a nilpotent element, then $1 + r$ is a unit of R.

(14) Let R be a commutative ring with identity. Consider the ring of formal power series over A, that is

$$A[[X]] = \left\{ \sum_{n=0}^{\infty} a_n X^n \mid a_n \in A \right\}.$$

- ► Show that $r(X) \in A[[X]]$ is a unit if and only if $a_o = r(0)$ is a unit in A.
- ► Show that if $r(X) \in [[X]]$ is nilpotent, then a_n is nilpotent for all $n \in \mathbf{Z}_{\geq 0}$.