Çark Groupoids and Thompson's Groups

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Abstract

In this work, we introduce a groupoid, called the class groupoid and denoted by $\mathscr{CG}$, which generalizes the Ptolemy groupoid constructions of Penner both in the finite case, i.e. for surfaces of genus g with n punctures, and the universal case, i.e. the disk. Objects of $\mathscr{CG}$ are certain configurations of cosets of subgroups Γ of the modular group $\mathrm{PSL}_2(\mathbf{Z})$. $\mathscr{CG}$ has a sub-category $\mathcal{M}$, called the modular groupoid, whose objects are boundaries of $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$. Both categories are disconnected. For the trivial subgroup, the fundamental group of the corresponding connected component of $\mathcal{M}$ turns out to be the universal Teichmüller group (i.e. Thompson's group) T. We study $\mathscr{CG}$ and $\mathcal{M}$ further and prove several results for subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ generated by one element.

Contents

1 Introduction

We aim to both unify and generalize the Ptolemy groupoid construction due to Penner, [9, 10, 11] and study the simplest non-trivial instance of this generalization. To this end, the class groupoid, denoted $\mathscr{CG}$, is defined. Objects of $\mathscr{CG}$ are certain configurations, called admissible, of so-called coset pairs $(\Gamma \cdot M, \gamma \cdot M)$; where Γ is a subgroup of the modular group $\mathrm{PSL}_2(\mathbf{Z})$, $\gamma \in \Gamma$ and $\mathcal{M} \in \mathrm{PSL}_2(\mathbf{Z})$. Flips (or HI moves or Whitehead moves) act on the set of such configurations allowing us to obtain the groupoid structure. Flips leave the genus, the number of punctures and the number of boundary components homeomorphic to circle fixed. Therefore, the groupoid is disconnected. We let $\mathscr{CG}_\Gamma$ denote the subgroupoid of all admissible configurations in $\mathscr{CG}$ that are obtained by applying flips to an admissible configuration for Γ. This subgroupoid turn out to be connected.

Each admissible configuration gives rise to a bipartite ribbon graph on the surface $\Gamma\backslash\mathcal{H}$; where $\mathcal{H}$ is the upper half plane. If Γ is of finite index then the surface is a genus g surface with finitely many punctures and without any boundary homeomorphic to the circle. In this case $\mathscr{CG}_\Gamma$ admits an embedding of the mapping class group Mod_g^n of the surface $\Gamma\backslash\mathcal{H}$.

If the surface $\Gamma\backslash\mathcal{H}$ has at least one boundary component homeomorphic to the circle then Γ is of infinite index. It turns out that if $\Gamma = \{I\}$ then the groupoid $\mathscr{CG}_\Gamma$ has only one object, and hence is a group isomorphic to the Thompson's group T of piecewise Möbius transformations of the boundary of the unit disk with finitely many rational breakpoints, or in other words the universal mapping class group.

The group $\mathscr{CG}$ admits a subgroupoid $\mathcal{M}$, called the modular groupoid, whose objects are boundary of subgroups of the modular group. Flips induce maps between the boundaries of these groups. This groupoid is also disconnected. As above by $\mathcal{M}_\Gamma$ we denote corresponding connected component. If Γ is the trivial subgroup, then $\mathcal{M}_\Gamma$ contains only one object, and is isomorphic to the universal mapping class group. In fact it turns out that boundaries of modular groups are closely related to the classical constructions such as ends of trees and continued fractions.

We carry previous computation to the next step. That is, we consider subgroups of the modular group generated by one element. They form a subcategory called the çark groupoid[a], **Ç**. It has seven connected components. In each case the fundamental group of the connected component turns out to be closely related to the Thompson's group F of piecewise linear homeomorphism of the unit interval having finitely many rational breakpoints.

Plan of the paper is as follows. In Section 2 we develop the appropriate graph terminology, beginning with the bipartite Farey tree and modular graphs. In Section 3 we define and investigate ends of the bipartite Farey tree and show that its ends is equal to the boundary of the unit disk, up to an equivalence relation. We show that this phenomenon can be treated in terms of continued fractions. Section 4 is devoted to the class groupoid. We give the definition and reprove Penner's theorem. We also define the boundaries of subgroups of the modular group and discuss its relation to continued fraction map. In Section

[a]The terminology is stemming from the one to one correspondence between subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ generated by one hyperbolic element and certain bipartite ribbon graphs called çarks, we refer to [14] for further details.

5 we introduce the çark groupoid and discuss the fundamental groups of its connected components. We end the paper by some concluding remarks.

2 Bipartite Farey tree $\mathcal{F}$ and modular graphs

In this section, we will define the bipartite Farey tree, denoted by $\mathcal{F}$, and modular graphs. Then discuss categories of subgroups of the modular group. For further details on these we refer to [13].

2.1 Modular group action on $\mathcal{F}$.

The *modular group* is the projective group $\mathrm{PSL}_2(\mathbf{Z})$ of two by two integral matrices having determinant 1. It is well known that the two elliptic transformations $S(z) = -1/z$ and $L(z) = (z-1)/z$, respectively of orders 2 and 3, freely generate a group of Möbius transformations which is isomorphic to $\mathrm{PSL}_2(\mathbf{Z})$. From $\mathrm{PSL}_2(\mathbf{Z})$ we construct the *bipartite Farey tree* $\mathcal{F}$, whose edges are elements of the modular group. The set of vertices of $\mathcal{F}$ are defined as follows:

$$V(\mathcal{F}) = V_\otimes(\mathcal{F}) \sqcup V_\bullet(\mathcal{F});$$

where $V_\otimes(\mathcal{F}) = \{\{W, WS\} \colon W \in \mathrm{PSL}_2(\mathbf{Z})\}$ is the set of degree-2 vertices and $V_\bullet(\mathcal{F}) = \{\{W, WL, WL^2\} \colon W \in \mathrm{PSL}_2(\mathbf{Z})\}$ is the set of degree-3 vertices. Two vertices v and v' are joined by an edge if and only if the intersection $v \cap v'$ is non-empty and in this case the edge between the two vertices is the only element in the intersection. The edges incident to the vertex $\{W, WL, WL^2\} \in V_\bullet$ are W, WL and WL^2, and these edges inherit a natural cyclic ordering which we fix for all vertices as (W, WL, WL^2). Thus $\mathcal{F}$ is an infinite bipartite ribbon graph. It is a tree since $\mathrm{PSL}_2(\mathbf{Z})$ is freely generated by S and L. We will always assume that our words are reduced, i.e. no cancellation occurs within the words. Empty word will stand for the identity element. For further use, we also introduce the following notation: given two words W and W' in the modular group by $W \cap W'$ we denote the longest word (i.e. having the largest number of letters) that is common both in W and W'. That is, $W \cap W'$ is the longest word so that if we write $W = (W \cap W')W_o$ and $W' = (W \cap W')W'_o$, then $W_o \cap W'_o$ is the empty word.

$M \in \mathrm{PSL}_2(\mathbf{Z})$ acts on $\mathcal{F}$ from the left by ribbon graph automorphisms as follows:

$$\begin{aligned} W \in E(\mathcal{F}) &\mapsto MW \in E(\mathcal{F}) \\ \{W, WS\} \in V_\otimes(\mathcal{F}) &\mapsto \{MW, MWS\} \in V_\otimes(\mathcal{F}) \\ \{W, WL, WL^2\} \in V_\bullet(\mathcal{F}) &\mapsto \{MW, MWL, MWL^2\} \in V_\bullet(\mathcal{F}) \end{aligned}$$

The action is free on $E(\mathcal{F})$ since this is no other than the left-regular action of $\mathrm{PSL}_2(\mathbf{Z})$ on itself.

2.2 Modular graphs.

Let Γ be any subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. Then Γ acts on $\mathcal{F}$ from the left and to Γ we associate a quotient graph $\Gamma\backslash\mathcal{F}$ whose edges and vertices are defined as:

$$\begin{aligned} E(\Gamma\backslash\mathcal{F}) &= \{\Gamma \cdot W \colon W \in \mathrm{PSL}_2(\mathbf{Z})\} \\ V(\Gamma\backslash\mathcal{F}) &= V_\otimes(\mathcal{F}\backslash\Gamma) \cup V_\bullet(\mathcal{F}\backslash\Gamma); \end{aligned}$$

where

$$V_{\otimes}(\Gamma\backslash\mathcal{F}) = \{\Gamma\cdot\{W, WS\} : W \in \mathrm{PSL}_2(\mathbf{Z})\}$$
$$V_{\bullet}(\Gamma\backslash\mathcal{F}) = \{\Gamma\cdot\{W, WL, WL^2\} : W \in \mathrm{PSL}_2(\mathbf{Z})\}$$

The incidence relation induced from $\mathcal{F}$ gives a well-defined incidence relation and we obtain a bipartite graph.

Definition 2.1. *Let Γ be* any *subgroup of the modular group. The graph $\Gamma\backslash\mathcal{F}$ is called a* ***modular graph****.*

The edges incident to the vertex $\Gamma\cdot\{W, WL, WL^2\}$ are $\Gamma\cdot\{W\}, \Gamma\cdot\{WL\}, \Gamma\cdot\{WL^2\}$, and these edges inherit a natural cyclic ordering $(\Gamma\cdot\{W\}, \Gamma\cdot\{WL\}, \Gamma\cdot\{WL^2\})$ from the vertex. Hence $\Gamma\backslash\mathcal{F}$ is a ribbon graph possibly with pending (or terminal) vertices that corresponds to the conjugacy classes of elliptic elements of Γ.

The set of edges of $\Gamma\backslash\mathcal{F}$ is identified with the set of right-cosets of Γ, so that the graph $\Gamma\backslash\mathcal{F}$ has $[\mathrm{PSL}_2(\mathbf{Z}) : \Gamma]$ many edges. For instance, for $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, the quotient graph $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathcal{F}$ is a graph with one edge, see Figure 1. We call this graph the *modular arc* and denote as $\otimes\!\!-\!\!\bullet$.

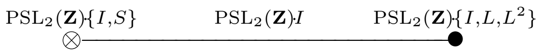

Figure 1: The modular arc.

Note that if Γ_1, Γ_2 are two distinct isomorphic subgroups of the modular group, then $\Gamma_1\backslash\mathcal{F}$ and $\Gamma_2\backslash\mathcal{F}$ are isomorphic as abstract graphs. They are isomorphic as ribbon graphs only if Γ_1 and Γ_2 are conjugates in $\mathrm{PSL}_2(\mathbf{Z})$, see [14, Proposition 2.1]. Therefore modular graphs parametrize conjugacy classes of subgroups of the modular group, whereas the edges of a modular graph parametrize subgroups in the conjugacy class represented by the modular graph. In conclusion we get:

Theorem 2.2. *There is a 1-1 correspondence between modular graphs with a base edge (G, e) (modulo ribbon graph isomorphisms of pairs (G, e)) and subgroups of the modular group.*

Theorem 2.3. *There is a 1-1 correspondence between modular graphs with two base edges (G, e, e') (modulo ribbon graph isomorphisms of triples (G, e, e')) and cosets of subgroups of the modular group.*

In this paper we are mostly interested in the case where Γ is a cyclic subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. Corresponding modular graphs were named *çarks* (pronounced: "chark") in [14].

Every modular graph may be regarded at the same time as a graph of groups, [12], in which vertices of type $\otimes$ are associated with the group $\mathbf{Z}/2\mathbf{Z}$, vertices of type $\bullet$ are associated with the group $\mathbf{Z}/3\mathbf{Z}$ and edges are associated with the trivial group. In this setup, every modular graph has a well-defined fundamental group, and a universal cover. For instance, the fundamental group of the modular arc is the modular group itself. Thus any modular graph can be viewed as a covering of the modular arc and $\mathcal{F}$ as the universal cover of any modular graph. Therefore, any connected bipartite ribbon graph G, with $V(G) = V_{\otimes}(G) \sqcup V_{\bullet}(G)$, such that every $\otimes$-vertex is of degree 1 or 2 and every $\bullet$-vertex is of degree 1 or 3, is modular. There is a canonical isomorphism $\pi_1(\Gamma\backslash\mathcal{F}, \Gamma\cdot I) \simeq \Gamma < \mathrm{PSL}_2(\mathbf{Z})$, with the canonical choice of $\Gamma\cdot I$ as a base edge. In general, subgroups Γ of the modular group (or equivalently the fundamental groups $\pi_1(\Gamma\backslash\mathcal{F})$) are free products of copies of $\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/3\mathbf{Z}$, see [6].

Every modular graph has a "thickening", i.e. a punctured topological surface, possibly with boundary, in which it embeds. Given a subgroup Γ of $\mathrm{PSL}_2(\mathbf{Z})$, we define the genus of the modular graph (and the genus of Γ) as the genus of the topological surface into which the graph $\Gamma\backslash\mathcal{F}$ embeds. Punctures and the boundaries which are homeomorphic to the circle are defined in a similar fashion. Remark also that these properties can be defined in a purely combinatorial manner. For instance, punctures on the topological surface are nothing but finite loops on the modular graph. In fact, modular graphs are precisely the dual graphs of triangulations of punctured topological surfaces, where the vertices of the triangles are situated at punctures and along boundary components.

2.3 Modular graphs and the modular curve

There is the well-known action of the modular group $\mathrm{PSL}_2(\mathbf{Z})$ on the upper half plane $\mathcal{H}$, simultaneously by conformal transformations and by hyperbolic isometries. The bipartite Farey tree admits a realization on the upper half plane, such that the action of $\mathrm{PSL}_2(\mathbf{Z})$ on $\mathcal{F}$ is compatible with the action on $\mathcal{H}$. This realization is obtained by identifying an edge of $\mathcal{F}$ with the geodesic arc, which is the half of the lower boundary component of the standard fundamental region of the $\mathrm{PSL}_2(\mathbf{Z})$-action on $\mathcal{H}$.

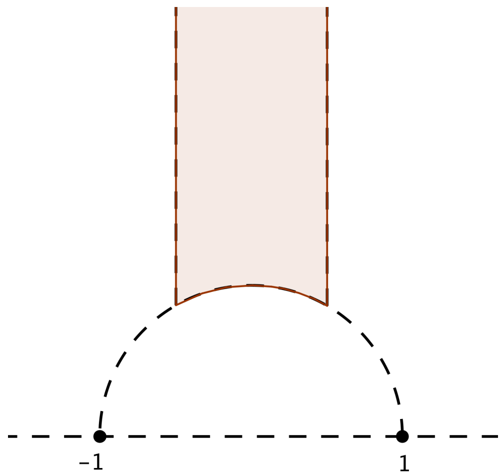

Figure 2: The fundamental region of the $\mathrm{PSL}_2(\mathbf{Z})$-action on the upper half plane.

The Farey tree $\mathcal{F}$ is then identified with the $\mathrm{PSL}_2(\mathbf{Z})$-orbit of this geodesic arc. By the compatibility of actions, every modular graph has a canonical realization as a piecewise analytic and geodesic subset of a Riemann surface with a canonical hyperbolic metric. (This last sentence is to be understood in the extended sense of orbifolds and cone-metrics, due to the existence of fixed points of the action.) Note that the analytic and the hyperbolic structures of the ambient surface are determined by the graph.

2.4 Covering categories of the modular arc

Our aim in this section is to introduce the category of finitely generated subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ of infinite index and the equivalent category of coverings of the modular arc with a base-edge. The simplest examples are $\mathbf{Z}$-subgroups, but there are many others and this system will be our playground in this paper.

The category (directed poset) of coverings of the modular arc with a base edge, is equivalent to the category of subgroups of the modular group with inclusions as morphisms. We denote the former system with $\mathbf{Cov}^*(\otimes\!\!-\!\!\bullet)$ and the latter with $\mathbf{Sub}(\mathrm{PSL}_2(\mathbf{Z}))$. Since intersection of subgroups is a subgroup, both of these systems are directed. The category $\mathbf{Cov}^*(\otimes\!\!-\!\!\bullet)$ is the category of modular graphs with a base edge, where morphisms are coverings of modular graphs with a base edge. Forgetting base edges yields the covering category of the modular arc, denoted as $\mathbf{Cov}(\otimes\!\!-\!\!\bullet)$. Both three of these categories has uncountably many objects as there exists uncountably many modular graphs.

The category $\mathbf{Cov}^*(\otimes\!\!-\!\!\bullet)$ has the full sub-category $\mathbf{FCov}^*(\otimes\!\!-\!\!\bullet)$, which consists of base-edged coverings of finite degree, equivalent to the category of finite-index subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ under inclusion. We denote this latter category by $\mathbf{FSub}(\mathrm{PSL}_2(\mathbf{Z}))$. The category $\mathbf{FCov}^*(\otimes\!\!-\!\!\bullet)$ is countable, since its objects are finite modular graphs with a base edge. Forgetting base edges yields the category finite coverings of the modular arc, denoted as $\mathbf{FCov}(\otimes\!\!-\!\!\bullet)$. The finite modular graphs are realized on Riemann surfaces with a positve number of punctures and orbifold points. These graphs have several peculiar properties, for example, they parametrize arithmetic Riemann surfaces (Belyi theorem). If we metrize them (by associating a length to each edge), they parametrize decorated Teichmüller spaces of punctured surfaces (Penner's work). Moreover, mapping class groups of punctured surfaces can be described in terms of these graphs, see Theorem 4.4 below.

The category $\mathbf{Sub}(\mathrm{PSL}_2(\mathbf{Z}))$ has another, (much neglected!) full sub-category, denoted as $\mathbf{FGISub}(\mathrm{PSL}_2(\mathbf{Z}))$, which consists of **F**initely **G**enerated **I**nfinite-index **Sub**groups of the modular group. This category is equivalent to the category of infinite base-edged modular graphs of finite topology (i.e. with a finitely generated fundamental group). We denote this latter category by $\mathbf{FGICov}^*(\otimes\!\!-\!\!\bullet)$. It has countably many objects. Forgetting base edge yields the category of infinite modular graphs with finite topology, denoted henceforth as $\mathbf{FGICov}(\otimes\!\!-\!\!\bullet)$. Note that unlike the categories $\mathbf{Cov}(\otimes\!\!-\!\!\bullet)$ and $\mathbf{FCov}(\otimes\!\!-\!\!\bullet)$, the category $\mathbf{FGICov}(\otimes\!\!-\!\!\bullet)$ has no final object. Its initial object is $\mathcal{F}$; the corresponding subgroup being $\{I\} \subset \mathrm{PSL}_2(\mathbf{Z})$.

Finite coverings of an object of $\mathbf{FGICov}(\otimes\!\!-\!\!\bullet)$ are always in $\mathbf{FGICov}(\otimes\!\!-\!\!\bullet)$; however, this is not true for infinite coverings in general. Note that there do exist some infinite coverings among its objects. For example, $\langle L\rangle$ is an infinite index subgroup in $\langle L, XSX^{-1}\rangle$ and the latter is a finitely generated subgroup which is of infinite index for $X \neq I, S, L, L^2, LS, L^2S$. Hence

$$\langle L\rangle\backslash\mathcal{F} \longrightarrow \langle L, XSX^{-1}\rangle\backslash\mathcal{F}$$

is an infinite covering in $\mathbf{FGICov}(\otimes\!\!-\!\!\bullet)$ for generic X.

A modular graph of finite topology may have at most a finite number of punctures or pending (terminal) vertices. There may be no punctures at all. Besides these, an infinite modular graph of finite topology must have some *Farey branches*; i.e. subgraphs which are trees with only one terminal vertex. Note that Farey branches are simply connected so these don't effect the finiteness of the topology of the modular graph. These modular graphs are realized on Riemann surfaces with finitely many punctures, orbifold points and boundary components. These boundary components are in a sense created by the Farey branches attached to the modular graph.

As an example, any $\mathbf{Z}$-subgroup generated by an element of infinite order in $\mathrm{PSL}_2(\mathbf{Z})$ falls into the category $\mathbf{FGISub}(\mathrm{PSL}_2(\mathbf{Z}))$.

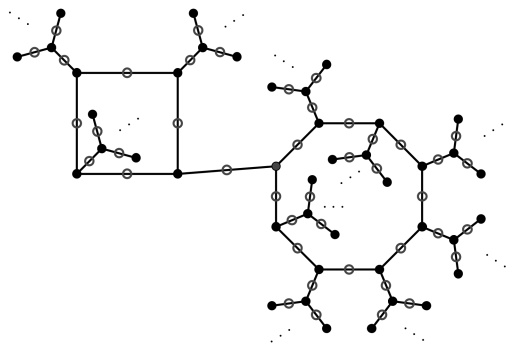

Figure 3: A pair of pants.

3 Boundary of $\mathcal{F}$ and continued fraction map

Here our goal is to prepare the ground for the study of the flip action on the boundary of $\mathcal{F}$, which gives rise to an avatar of Thompson's group as the group $P\mathrm{PSL}_2(\mathbf{Z})$ of piecewise-$\mathrm{PSL}_2(\mathbf{Z})$ homeomorphisms of S^1 with rational break points. In the last part, it will be shown that flips on the çark groupoid act on çark boundaries. We will capture a nice representation of the çark groupoid.

3.1 Paths on $\mathcal{F}$ and the boundary of $\mathcal{F}$.

A *path* on a graph $\mathcal{G}$ is a sequence of edges $e_1, e_2, \dots, e_k$ of $\mathcal{G}$ such that e_i and e_{i+1} are coincident at a vertex, for each $1 \leq i < k$. The objects of the *fundamental groupoid* of $\mathcal{G}$ are the edges of $\mathcal{G}$ and the morphisms are defined to be non back-tracking oriented paths. Composition in the groupoid is defined as the concatenation of paths wherever possible.

Since the edges of $\mathcal{F}$ are labeled by reduced words in the letters L and S, a path in the bipartite Farey tree $\mathcal{F}$ is a sequence of reduced words (W_i) in L and S, such that $W_i^{-1}W_{i+1} \in \{L, L^2, S\}$ for every i. Since $\mathcal{F}$ is connected and simply connected, there is a unique non-backtracking path through any two edges, and the fundamental groupoid is identified with the pair groupoid of

$\mathrm{PSL}_2(\mathbf{Z})$. To be more precise, one has

$$\begin{aligned} Obj(\Pi_1(\mathcal{F})) &= E(\mathcal{F}) = \{W \,:\, W \in \mathrm{PSL}_2(\mathbf{Z})\}, \\ Mor(\Pi_1(\mathcal{F})) &= E(\mathcal{F}) \times E(\mathcal{F}) \end{aligned}$$

An *end* of $\mathcal{F}$ is an equivalence class of infinite (but not bi-infinite) non-backtracking paths in $\mathcal{F}$, where eventually coinciding paths are considered to be equivalent, [4]. In other words, an end of $\mathcal{F}$ is the equivalence class of an infinite sequence of finite reduced words (W_i) in L and S with $W_i^{-1}W_{i+1} \in \{L, S\}$ for every i, where sequences with coinciding tails are equivalent. One may also view the end (W_i) of $\mathcal{F}$ as the pair (W_1, W), where W_1 is the starting edge and W is the infinite word

$$W = \prod_{i=1}^{\infty} W_i^{-1} W_{i+1}$$

in L and S. Here infinite words with different starting edges are taken to be equivalent.

Let us denote by $\partial\mathcal{F}$ the set of ends of $\mathcal{F}$. The action of $\mathrm{PSL}_2(\mathbf{Z})$ on $\mathcal{F}$ extends to an action on the set $\partial\mathcal{F}$, the element $M \in \mathrm{PSL}_2(\mathbf{Z})$ sending the path (W_i) to the path (MW_i). (Note that this is not the $\mathrm{PSL}_2(\mathbf{Z})$-action on the set of paths of $\mathcal{F}$, introduced in the above remark.) This action is neither free nor transitive, see the next section for details.

Given an edge e of $\mathcal{F}$ and an end b of $\mathcal{F}$, there is a unique path in the class b which starts at e. Hence for any edge e, we may canonically identify the set $\partial\mathcal{F}$ with the set of infinite non-backtracking paths that start at e. We denote this set latter set by $\partial\mathcal{F}_e$ and endow it with the product topology. Intuitively, closeness of two paths is determined by the number of common edges. Note that for an arbitrary graph $\mathcal{G}$; given an edge e, the sets $\partial\mathcal{G}$ and $\partial_e\mathcal{G}$ are not in bijection.

Given any edge e' of $\mathcal{F}$, the spaces $\partial\mathcal{F}_e$ and $\partial\mathcal{F}_{e'}$ are canonically homeomorphic. This homeomorphism is given by pre-composing with the unique path joining e to e'. The above-mentioned action of the modular group on the set $\partial\mathcal{F}$ induces an action of $\mathrm{PSL}_2(\mathbf{Z})$ by homeomorphisms of the topological space $\partial\mathcal{F}_e$, for any choice of a base edge e.

$\partial\mathcal{F}_e$ is an uncountable, compact, totally disconnected, Hausdorff topological space. Hence it is homeomorphic to the Cantor set. We want to "contract the holes" of this Cantor set to obtain the continuum, as follows. Define a *rational end* of $\mathcal{F}$ to be an eventually left-turn or eventually right-turn path. Now introduce the equivalence relation $\sim$ on $\partial\mathcal{F}$ as: left- and right- rational paths which bifurcate from the same vertex are equivalent[b] see Figure 4. We will see that (Theorem 3.1) the set of rational ends modulo this equivalence relation is in natural correspondence with the set of rational numbers.

3.2 Continued fraction map.

Recall that, as sets, $\partial\mathcal{F}_e$ and $\partial\mathcal{F}$ are in bijection. On the quotient space $\partial\mathcal{F}_e/\!\sim$ there is the quotient topology induced by the topology on $\partial\mathcal{F}_e$ such that the projection map

$$\partial\mathcal{F}_e \longrightarrow \partial\mathcal{F}_e/\!\sim$$

[b]One may consider these two paths as forming a horocycle at a rational point at infinity

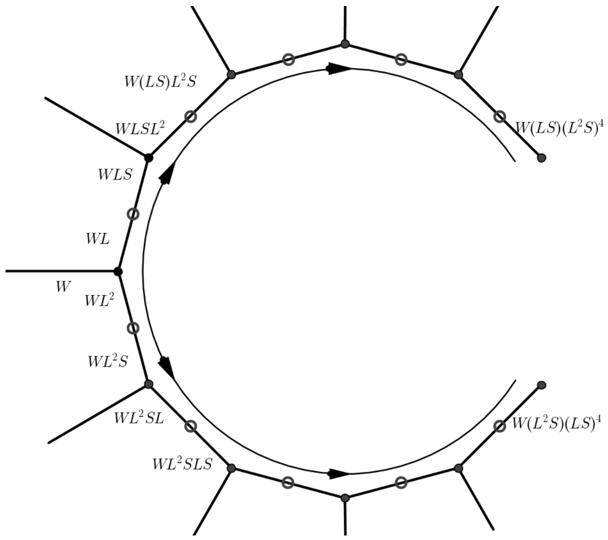

Figure 4: The left and right turn paths which bifurcate from the vertex $\{W, WL, WL^2\}$.

is continuous. We shall denote this quotient space by S^1_e. This equivalence relation is preserved under the canonical homeomorphisms $\partial\mathcal{F}_e \longrightarrow \partial\mathcal{F}_{e'}$ and is also respected by the $\mathrm{PSL}_2(\mathbf{Z})$-action. Therefore we have the commutative diagram

$$\begin{array}{ccc} \partial\mathcal{F}_e & \longrightarrow & \partial\mathcal{F}_{e'} \\ \downarrow & & \downarrow \\ S^1_e & \longrightarrow & S^1_{e'} \end{array}$$

where the horizontal arrows are canonical homeomorphisms and the vertical arrows are projections. Moreover, $\mathrm{PSL}_2(\mathbf{Z})$ acts by homeomorphisms on S^1_e, for any e.

Now, $\mathcal{F}$ comes equipped with a distinguished edge, the edge marked I, the identity element of the modular group. Hence all spaces S^1_e are canonically homeomorphic to S^1_I.

Any element of S^1_I can be represented by an infinite word in L and S. Regrouping occurrences of LS and L^2S, any such word x of S^1_I thus can be written in one of the following forms:

$$x = (LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\cdots \text{ or}$$
$$x = S(LS)^{n_0}(L^2S)^{n_1}(LS)^{n_2}(L^2S)^{n_3}(LS)^{n_4}\cdots,$$

where $n_0, n_1 \cdots \geq 0$. Since our paths do not have any backtracking we have $n_0 \geq 0$ and $n_i > 0$ for $i = 1, 2, \cdots$. The pairs of words

$$(LS)^{n_0}\cdots(LS)^{n_k+1}(L^2S)^\infty \text{ and } (LS)^{n_0}\cdots(LS)^{n_k}(L^2S)(LS)^\infty, \quad (k \text{ even})$$
$$(LS)^{n_0}\cdots(L^2S)^{n_k+1}(LS)^\infty \text{ and } (LS)^{n_0}\cdots(L^2S)^{n_k}(LS)(L^2S)^\infty, \quad (k \text{ odd})$$

correspond to pairs of rational ends and represent the same element of S^1_I. For irrational ends this representation is unique.

Set $U(z) = 1/z$. Noting that

$$LS(z) = T = 1 + z \implies (LS)^n(z) = n + z \text{ and}$$
$$L^2S(z) = 1/(1 + 1/z) = UTU(z) \implies (L^2S)^n = UT^nU,$$

we can rewrite the element $x \in S^1_I$ of the form

$$x = T^{n_0}UT^{n_1}UT^{n_2}T^{n_3}UT^{n_4}\cdots$$
$$x = ST^{n_0}UT^{n_1}UT^{n_2}T^{n_3}UT^{n_4}\cdots$$

We shall employ the usual notation for the continued fractions

$$[n_0; n_1, n_2, \dots] := n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots}}}$$

Define the continued fraction map $\kappa : S^1_I \to \hat{\mathbf{R}}$ by

$$\kappa(x) = \begin{cases} [n_0, n_1, n_2, \dots] & \text{if } x = T^{n_0}UT^{n_1}UT^{n_2}T^{n_3}U\dots \\ -1/[n_0, n_1, n_2, \dots] & \text{if } x = ST^{n_0}UT^{n_1}UT^{n_2}T^{n_3}U\dots \end{cases}$$

Theorem 3.1. *The continued fraction map is a homeomorphism.*

Proof. First note that this map is well defined as it respects the equivalence of pairs of rational ends: $[n_0, \dots n_k + 1, \infty]$ and $[n_0, \dots n_k, 1, \infty]$ represent the same number. Moreover, it is bijective from the set of rational ends modulo equivalence of pairs onto the set of rational numbers. Now an infinite path determines a unique Dedekind cut. Precisely, given an infinite path $x = T^{n_0}UT^{n_1}UT^{n_2}\cdots) \in S^1_I$ the set A is defined to be the set of all rational paths which are to the right[c] of x and to the left of the path $x_{-\infty} = (SL^2)^\infty$, and the set B is defined to be the set of all paths which are to the left of x and to the right of the path $x_\infty = (LS)^\infty$. □

As a consequence of this result, we see that the continued fraction map conjugates the $\mathrm{PSL}_2(\mathbf{Z})$-action on S^1_I to its action on $\hat{\mathbf{R}}$ by Möbius transformations.

3.3 Flip action on S^1_I

Constructing $\mathcal{F}$ from the modular group endows $\mathcal{F}$ with a distinguished base edge, I. On the other hand, every degree two vertex, say $\{W, WS\}$, and the two edges that are incident to this vertex, namely W and WS, may be represented uniquely by an ordered pair (W, WS), where without loss of generality we assume that W is a word not ending with S, which is equivalent to saying that the length of W is strictly less than that of WS. Now, given any such pair (W, WS), except (I, S), the action of the flip, $f_{(W,WS)} = f_W$, on an infinite path x on $\mathcal{F}$ is defined as follows:

[c] By a path, y, to the right of a given path x we mean that x and y agree in a *finite* word, W, and the first letter of x after W is L whereas that of y is L^2.

$$f_W(x) = \begin{cases} WL^2SLSW' & \text{, if } x \text{ is of the form } WL^2SW' \text{ for some } W' \\ W(L^2S)^2W' & \text{, if } x \text{ is of the form } WSLSW' \text{ for some } W' \\ WSW' & \text{, if } x \text{ is of the form } WSL^2SW' \text{ for some } W' \\ x & \text{, otherwise} \end{cases}$$

in the case where the edge I is *closest* to the lower left edge, see Figure 5. If the edge labeled I is closest to the upper left edge, then

$$f_W(x) = \begin{cases} WLSL^2SW' & \text{, if } x \text{ is of the form } WLSW' \text{ for some } W' \\ W(LS)^2W' & \text{, if } x \text{ is of the form } WSL^2SW' \text{ for some } W' \\ WSW' & \text{, if } x \text{ is of the form } WSLSW' \text{ for some } W' \\ x & \text{, otherwise} \end{cases}$$

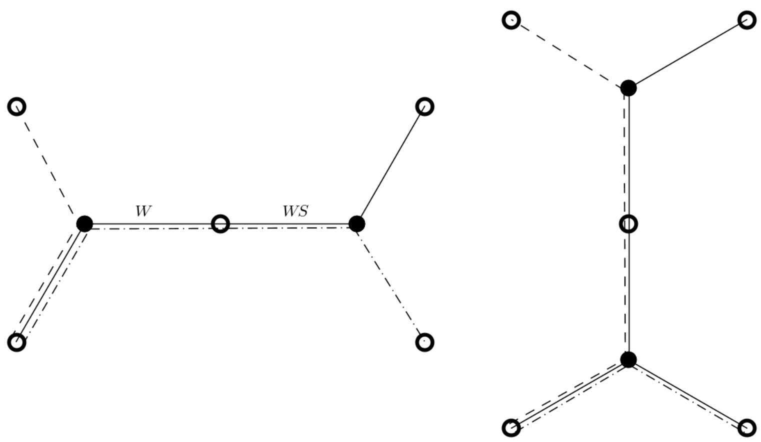

Figure 5: The action of flip on the pair (W, WS) on infinite paths.

Finally, flip on the pair (I, S) is defined as:

$$f_{(I,S)}(x) = \begin{cases} L^2SW' & \text{, if } x \text{ is of the form } SL^2SW' \text{ for some } W' \\ SL^2SW' & \text{, if } x \text{ is of the form } SLSW' \text{ for some } W' \\ L^2SW' & \text{, if } x \text{ is of the form } LSW' \text{ for some } W' \\ SLSW' & \text{, if } x \text{ is of the form } L^2SW' \text{ for some } W' \end{cases}$$

In summary, the action of a flip is trivial on an end x if x does not agree with all but the last two letters of the word W. If this is indeed the case, that is if x agrees with W except the final letter or all W then the action is merely an HI-move on the path.

The inverse of a flip on the pair (W, WS) is a flip on the pair (WL^2, WL^2S) except when $(W, WS) = (I, S)$. The flip on the pair (I, S) is of order four. Moreover, flips applied to (WL^2, WL^2S), $(WLSL^2, WLSL^2S)$, (WL, WLS),

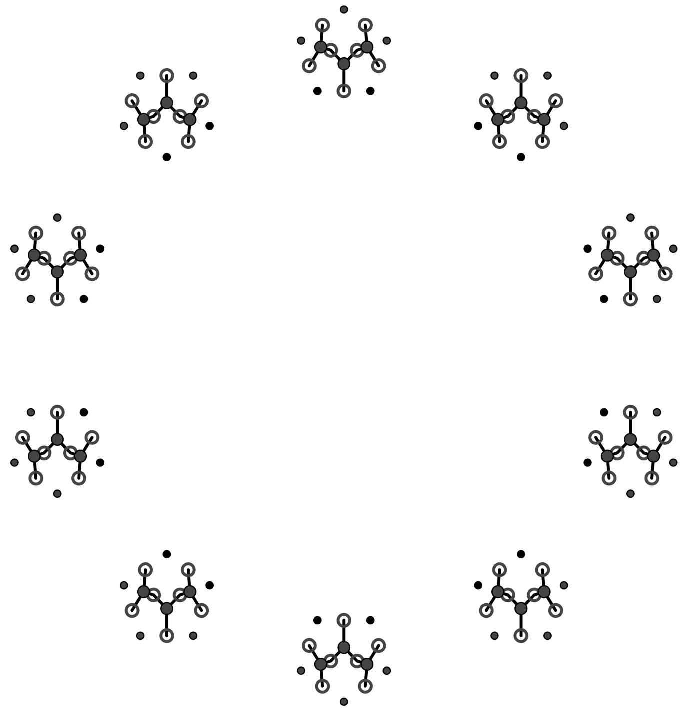

Figure 6: The pentagon relation.

(WL, WLS) and finally to (WL^2SL^2, WL^2SL^2S) in this order is identity. Remark that this is nothing but the pentagon relation, see Figure 6.

Given any two pairs (W, WS) and $(W', W'S)$ different from (I, S), by $W \cap W'$ let us denote the word in S, L and L^2 which is the maximal common part of both W and W'. Let us define the difference $d[(W, WS), (W', W'S)]$ of the pair (W, WS) and $(W', W'S)$ as another pair of integers (d_1, d_2), where d_1 (resp. d_2) is the number of edges of the finite non-backtracking path whose initial edge is the word $W \cap W'$ and final word is W (resp. W') minus 2. The distance, $\delta[(W, WS), (W', W'S)]$, or $\delta[W, W']$ for short, between two pairs (W, WS) and $(W', W'S)$ is then defined to be:

$$\delta[W, W'] = \begin{cases} d_1 + d_2 & \text{, if both } d_1 \text{ and } d_2 \text{ are positive} \\ 0 & \text{, otherwise} \end{cases}$$

Now, two flips (W, WS) and $(W', W'S)$ commute if the distance $\delta[W, W'] \geq 4$. In particular, if $W \cap W'$ is I, then the corresponding flips commute.

Lemma 3.2. *The action of flips on boundary of $\mathcal{F}$ induces a well-defined action on S^1_I by homeomorphisms.*

Proof. Let x and y be the two elements of the equivalence class. Then by definition there is an element $x \cap y \in \mathrm{PSL}_2(\mathbf{Z})$ such that $x = (x \cap y)LS(L^2S)^\infty$ and $y = (x \cap y)L^2S(LS)^\infty$. As is observed above, for any flip determined by the word $W = (x \cap y)LS(L^2S)^kL$ or $W = (x \cap y)LS(L^2S)^kL^2$, where k is assumed to be a positive integer, acts trivially on y because the distance δ is larger than

4 and fixes x. Similarly, $W = (x \cap y)L^2S(LS)^kL$ or $W = (x \cap y)L^2S(LS)^kL^2$ acts trivially on x because the distance δ is larger than 4 and fixes y. The only remaining case is to consider the action on $(x \cap y)L$ and on $(x \cap y)L^2$. Because of symmetry, it is enough to look at $W = (x \cap y)L$:

$$\begin{aligned} f_W(x) &= (x \cap y)L(LS)^2(L^2S)^\infty \\ &= (x \cap y)(L^2S)LS(L^2S)^\infty \\ &= X(LS)(L^2S)^\infty \end{aligned}$$

and

$$\begin{aligned} f_W(y) &= (x \cap y)L\,LS\,L^2S\,(LS)^\infty \\ &= (x \cap y)L^2S\,L^2S(LS)^\infty \\ &= X(L^2S)(LS)^\infty; \end{aligned}$$

where $X = (x \cap y)\,L^2S(LS)^\infty$. This action is by homeomorphisms since flips preserve right-left pairs of rational ends. $\square$

3.4 The action of flips on $\widehat{\mathbf{R}}$

In Theorem 3.1, we have seen that the spaces $\widehat{\mathbf{R}}$ and S^1_I are homeomorphic. Using Lemma 3.2 we deduce that flips act on $\widehat{\mathbf{R}}$. To describe the action, let us choose an arbitrary pair (W, WS) different from (I, S), and say $W = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ so that $WS = \begin{pmatrix} q & -p \\ s & -r \end{pmatrix}$. Then the continued fraction map induces the unique piecewise fractional linear transformation sending

$$\begin{aligned} \left(\frac{p}{r}, \frac{p-q}{r-s}\right) &\mapsto \left(\frac{p}{r}, \frac{q}{s}\right) \\ \left(\frac{p-q}{r-s}, \frac{q}{s}\right) &\mapsto \left(\frac{q}{s}, \frac{p+2q}{r+2s}\right) \\ \left(\frac{q}{s}, \frac{p+q}{r+s}\right) &\mapsto \left(\frac{p+2q}{r+2s}, \frac{p+q}{r+s}\right) \end{aligned}$$

and constant everywhere else, if WL is closer to I than WL^2. In the remaining case, the map on $\widehat{\mathbf{R}}$ is given by the piecewise linear fractional transformation sending:

$$\begin{aligned} \left(\frac{p+q}{r+s}, \frac{p}{r}\right) &\mapsto \left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \\ \left(\frac{p}{r}, \frac{p-q}{r-s}\right) &\mapsto \left(\frac{2p+q}{2r+s}, \frac{p}{r}\right) \\ \left(\frac{p-q}{r-s}, \frac{q}{s}\right) &\mapsto \left(\frac{p}{r}, \frac{q}{s}\right) \end{aligned}$$

and constant elsewhere. One computes also that the mapping defined on $\widehat{\mathbf{R}}$ by the pair (I, S) is the unique piecewise fractional linear transformation given as follows, compare [5]:

$$\begin{array}{lllllll}
(-\infty,-1) & \mapsto & (-1,0) & \text{given by} & z & \mapsto & \frac{z+1}{z} \\
(-1,0) & \mapsto & (0,1) & \text{given by} & z & \mapsto & z+1 \\
(0,1) & \mapsto & (1,\infty) & \text{given by} & z & \mapsto & \frac{-1}{z-1} \\
(1,\infty) & \mapsto & (-\infty,-1) & \text{given by} & z & \mapsto & \frac{z}{1-z}
\end{array}$$

In fact, the piecewise linear fractional transformations obtained above are elements of $\mathrm{PSL}_2(\mathbf{Z})$. Furthermore, we have:

Theorem 3.3. *The fundamental group of the groupoid whose single object is the set of infinite paths based at I and the set of morphisms is generated by flips is isomorphic to Thompson's group T.*

For the proof it is enough to consider the map sending the flip on the pair (I, S) to α and the sequence of flips given by (L, LS), (I, IS), (SL, SLS), (I, IS) and (L, LS) (with this order) to β, [7, Theorem 1]. Another set of generators of T is given in [2] and the corresponding flip sequences, which we denote in terms of α and β for brevity, are as follows:

$$\begin{array}{lll}
A & \mapsto & \beta\alpha^2 \\
B & \mapsto & \beta^2\alpha \\
C & \mapsto & \beta^2
\end{array}$$

Moreover, we have the following:

Corollary 3.4. *Given any class of an infinite path, call x, based at I, the fundamental group of the groupoid whose set of objects consists of equivalence classes of infinite paths on $\mathcal{F}$ based at I and the set of morphisms is generated by flips away from x is a subgroup of T isomorphic to Thompson's group F.*

By morphisms away from the path we mean the set of morphisms which fix the class of the path x. Moreover, if we use the piecewise fractional linear transformation presentation of the T, then this subgroup coincides with those fractional linear transformations which fix the real number corresponding to the path x.

4 The class groupoid

In this section, our aim is to generalize the construction of the so called *Ptolemy groupoid*, see for instance [10], or [8]. The construction (i.e. the class groupoid) we propose unifies both into the finite case and the infinite case. We will observe that the previous section concerning boundary of the Farey tree, its ends and continued fractions (which is related very closely to the universal Teichmüller space $\mathrm{Homeo}^+/\mathrm{PSL}_2(\mathbf{Z})$) are merely particular cases of certain the class groupoid. We will also recover the Ptolemy groupoids associated to surfaces of genus g with n punctures and hence mapping class group of a surface of genus g with n punctures as a connected component of the class groupoid.

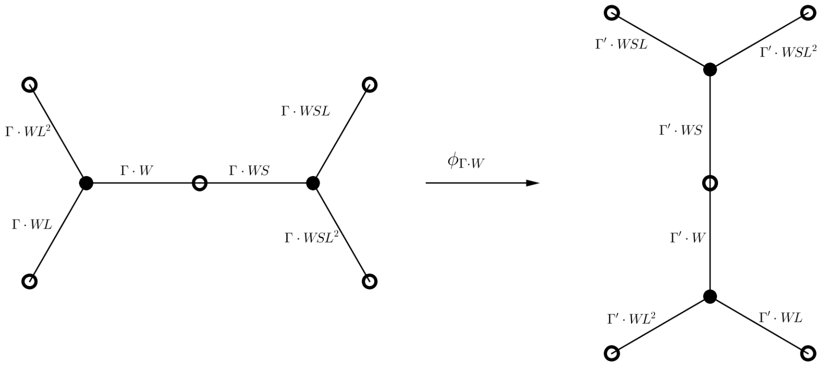

Figure 7: Flip on an edge of a modular graph.

4.1 Flips and admissible configurations

For subgroup Γ of the modular group, $\mathrm{PSL}_2(\mathbf{Z})$, the flip associated to an edge $(\Gamma\backslash\mathcal{F}) \cdot W$ of the corresponding modular graph $\Gamma\backslash\mathcal{F}$ is defined as the HI-move (or Whitehead move) at this edge. This flip will be denoted by $\phi_{\Gamma \cdot W}$[d], see Figure 7.

In particular, flip on the edge $\Gamma \cdot W$ induces the following map on the edges:

$$\begin{array}{rcl}
\Gamma \cdot W & \mapsto & \Gamma' \cdot W \\
\Gamma \cdot WS & \mapsto & \Gamma' \cdot WS \\
\Gamma \cdot WL & \mapsto & \Gamma' \cdot WL^2 \\
\Gamma \cdot WL^2 & \mapsto & \Gamma' \cdot WSL \\
\Gamma \cdot WSL & \mapsto & \Gamma' \cdot WSL^2 \\
\Gamma \cdot WSL^2 & \mapsto & \Gamma' \cdot WL
\end{array}$$

where Γ' is the image group $\phi_{\Gamma \cdot W}(\Gamma)$. The flip defined by the edge WS induces the same map on the set of edges. So there is a one to one correspondence between pair of cosets $(\Gamma \cdot W, \Gamma \cdot WS)$ and flips from Γ.

Flips preserve valencies of vertices but does not respect the incidence relation, hence is not a graph homomorphism. But flips leave the genus, the number of punctures and the number boundary components which are homeomorphic to circle of a modular graph fixed. Hence the topological type of the corresponding surface obtained by thickening the modular graph is the same.

A pair $(\Gamma \cdot M, \gamma \cdot M)$; where $\gamma \in \Gamma$ and $M \in \mathrm{PSL}_2(\mathbf{Z})$ is called a *coset pair*. Two coset pairs $(\Gamma \cdot M, \gamma \cdot M)$ and $(\Gamma' \cdot M', \gamma' \cdot M')$ are called equal if $\Gamma = \Gamma'$, $\Gamma \cdot M = \Gamma' \cdot M'$ and $\gamma \cdot M = \gamma \cdot M'$.

[d]Some care must be taken in defining flips whenever $\Gamma \cdot W = \Gamma \cdot I$, see for instance [5, §2]. We follow the positive orientation convention in defining the flip on the base edge. Whenever $\Gamma \cdot W$ is a dangling edge (or an orbifold point), see for instance [11, Definition 2.14].

Definition 4.1. *An* admissible configuration *of coset pairs for* Γ *is a set* A *of coset pairs so that*

i. for each coset, say $\Gamma \cdot M_o$ *of* Γ *in* $PSL_2(\mathbf{Z})$*,* A *contains exactly one coset pair of the form* $(\Gamma \cdot M_o, \gamma \cdot M_o)$ *for some* $\gamma \in \Gamma$*,*

ii. A *can be decomposed into a disjoint union of sets of size* 3 *each of which can be written as* $\{(\Gamma \cdot M, \gamma \cdot ML), (\Gamma \cdot ML, \gamma \cdot ML), (\Gamma \cdot ML^2, \gamma \cdot ML^2)\}$

iii. for every element $(\Gamma \cdot M, \gamma \cdot M) \in A$ *there is a corresponding element* $(\Gamma \cdot MS, \gamma' \cdot MS)$ *so that* $(\gamma MS)^{-1}\gamma' MS \in \Gamma$*.*

In fact, the above definition is equivalent to the choice of a fundamental domain in the universal cover $\mathcal{H}$, so that the part of the Farey tree (realized as the orbit of the geodesic connecting the fixed point of L to that of S) lying in this fundamental domain is isomorphic to the modular graph $\Gamma\backslash\mathcal{F}$. Given A, one may recover the graph $\Gamma\backslash\mathcal{F}$ and the group Γ. The disjoint sets of size 3 correspond to edges of the modular graph lying in a single triangle in an appropriate ideal triangulation of the corresponding thickening. The pairing explained in part iii. determines gluing of the triangles in order to recover the surface. The number of elements in an admissible configuration is exactly the number of distinct coset of Γ in $\mathrm{PSL}_2(\mathbf{Z})$.

Example 4.2. *Let us consider the group* $\Gamma = \langle L^2SLS, LSL^2S\rangle$*. We refer to Figure 8 for the corresponding modular graph where the bold edge indicates the base edge. The sets*

$$\begin{aligned} A_1 &= \{(\cdot I, I), (\cdot L, L), (\cdot L^2, L^2), (\cdot S, S), (\cdot SL, SL), (\cdot SL^2, SL^2)\} \\ A_2 &= \{(\cdot I, I), (\cdot L, L), (\cdot L^2, L^2), (\cdot S, LSL^2), (\cdot SL, LS), (\cdot SL^2, LSL)\} \end{aligned}$$

are both admissible configurations; where for sake of brevity we dropped Γ *from the first component. However,*

$$A = \{(\cdot I, I), (\cdot L, L), (\cdot L^2, L^2), (\cdot S, S), (\cdot SL, LS), (\cdot SL^2, SL^2)\}$$

is not an admissible configuration. It violates ii.

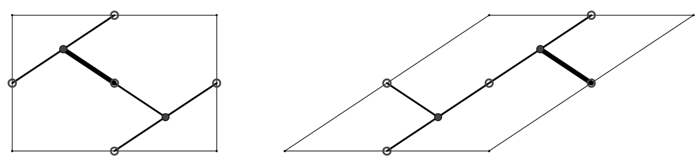

Figure 8: Modular graphs of the admissible configurations A_1 and A_2 on the torus with one puncture.

The definition of flips on the set of cosets can be used to define a flip on an admissible configuration for Γ, where the action is on both the coset $\Gamma \cdot W$ and the element γW of the coset as defined earlier. We also keep the previously introduced notation $\phi_{\Gamma \cdot W}$ for flips. We have $\phi_{\Gamma \cdot W} = \phi_{\Gamma \cdot WS}$. Given and admissible configuration of a subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$ there are $|A|/2$ many possible flips. A flip $\phi_{\Gamma \cdot W}$ for the subgroup Γ sends an admissible configuration A for Γ to an admissible configuration for the group $\Gamma' := \phi_{\Gamma \cdot W}(\Gamma)$.

We define the *class groupoid* as the groupoid $\mathscr{C}\mathscr{G}$ whose set of objects is the set of all admissible configurations of all subgroups of the modular group. The morphisms in this groupoid is generated by flips. That is,

$$\mathrm{Mor}(\mathscr{C}\mathscr{G})(A, A') := \{(A_1, A_2, \ldots A_k),\ A = A_1,\ A' = A_k\}$$

where $(A, A, \ldots A_k)$ is a sequence of admissible configurations for subgroups of $\mathrm{PSL}_2(\mathbf{Z})$, such that the A_{i+1} is obtained from A_i by the flip ϕ_i. We will denote such sequence by the triple (A, A', ϕ), where ϕ is the composition of all the flips $\phi_k \circ \ldots \circ \phi_1$. We will call two flip sequences (A_1, A_1', ϕ_1) and (A_2, A_2', ϕ_2) to be equivalent if $A_1 = A_2$, $A_1' = A_2'$ and $\phi_1 = \phi_2$. This is an equivalence relation, and each equivalence class will be denoted by $[A, A', \phi]$.

Flips do not change the invariants like genus, number of punctures of a modular graph so this groupoid is not connected. Given an element $\Gamma \in \mathbf{Sub}(\mathrm{PSL}_2(\mathbf{Z}))$, we denote by $\mathscr{C}\mathscr{G}_\Gamma$ the connected component of Γ in the class groupoid.

4.2 Ptolemy groupoids and $\mathscr{C}\mathscr{G}_\Gamma$.

Let A be an admissible configuration and T be a triple, that is a set of coset pairs in A of the form

$$\{(\Gamma \cdot M, \gamma \cdot ML), (\Gamma \cdot ML, \gamma \cdot ML), (\Gamma \cdot ML^2, \gamma \cdot ML^2)\},$$

for some $\gamma \in \Gamma$. One can find a word M_T so that $M_T \cap g = I$ for any $g \in \Gamma$. In this case, we can write the triple as

$$\{(\Gamma \cdot M, \gamma' \cdot M_T L), (\Gamma \cdot ML, \gamma' \cdot M_T L), (\Gamma \cdot ML^2, \gamma' \cdot M_T L^2)\}.$$

Now for an arbitrary automorphism ω of Γ we define the $\omega(T)$ to be the triple

$$\{(\Gamma \cdot M, \gamma' \cdot M_T' L), (\Gamma \cdot ML, \gamma' \cdot M_T' L), (\Gamma \cdot ML^2, \gamma' \cdot M_T' L^2)\};$$

where $M_T' = M_T$ if $M_T \cap g = I$ for any generator g of Γ. Otherwise, i.e. if for some generator g of Γ, we have $g \cap M_T$ is a non-empty word of length ℓ (i.e. there are ℓ-many letters in this word), then we define M_T' to be word comprising of the first ℓ letters of $\omega(g)$. It is only combinatorially cumbersome to show that this is indeed an action of the group $\mathrm{Aut}(\Gamma)$ on the set of triples, and that the image of an admissible configuration is again an admissible configuration. Also note that the normal subgroup of inner automorphisms of Γ (i.e. maps of the form $\omega_g(x) = g^{-1}xg$; for $g \in \Gamma$) induce the identity mapping on admissible configurations. Therefore, we obtain an action of $\mathrm{Out}(\Gamma) := \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ on the set of admissible configurations.

In the case when $\Gamma \in \mathbf{FSub}(\Gamma)$. The surface $\Gamma \backslash \mathcal{H}$ is of finite type, it has finite genus, say g, and finitely many punctures, say s-many. This means in particular that the associated modular graph is finite. For future reference we note the following:

Proposition 4.3 ([9, Lemma 1.2])**.** *Let $\Gamma, \Gamma' \leq PSL_2(\mathbf{Z})$ be two subgroups whose corresponding modular graphs are finite modular graph of genus g with n punctures. Then*

- *for any admissible configurations A_1, A_2 of Γ, the set $\mathrm{Mor}_{\mathscr{CG}}(A_1, A_2)$ is not empty, and*
- *for any admissible configuration A of Γ and A' of Γ', there set $\mathrm{Mor}_{\mathscr{CG}}(A, A')$ is not empty, whenever $(g, n) \neq (0, 3)$ or $(1, 1)$.*

In practice this says that the sub-groupoid $\mathscr{CG}_\Gamma$ is connected, except for the mentioned two cases, and that the symmetric group on the set of edges of the associated modular graph can be embedded in $\mathscr{CG}_\Gamma$ whenever Γ is of finite index.

Let Mod_g^n denote the mapping class group of the surface $\Gamma\backslash\mathcal{H}$. The well-known Dehn-Nielsen-Baer theorem, [3, Theorem 8.8] establishes the isomorphism between Mod_g^n and $\mathrm{Out}(\Gamma)$. Therefore, we obtain an action of the mapping class group on admissible configurations in $\mathscr{CG}_\Gamma$. Let us declare two admissible configurations, say A_1 and A_2, to be equivalent if and only if there is an element $\omega \in \mathrm{Out}(\Gamma)$ so that $\omega(A_1) = A_2$. Since the group Γ is of finite index, the fundamental group of the groupoid equipped with this equivalence relation is the symmetric group, as a result of Proposition 4.3. In this case, we say that Mod_g^n is a subgroup of $\mathscr{CG}_\Gamma$ of finite index. Let us summarize:

Theorem 4.4 (Mosher, Penner)**.** *For $\Gamma \in \mathbf{FSub}(PSL_2(\mathbf{Z}))$, the sub-groupoid $\mathscr{CG}_\Gamma$ contains Mod_g^n whose index is finite.*

4.3 Boundaries of subgroups of the modular group

Let $E(\Gamma)$ denote the set of all cosets of the subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$. We call two cosets $\Gamma \cdot W$ and $\Gamma \cdot W'$ to be equivalent if and only if for any element γW in $\Gamma \cdot W$, there is at least one element $\gamma' W'$ in $\Gamma \cdot W'$ so that the product $(\gamma W)^{-1}(\gamma' W') \in \langle LS \rangle$. We write $\Gamma \cdot W \sim_\Gamma \Gamma \cdot W'$ to indicate that the two cosets are equivalent. This relation is easily seen to be reflexive and symmetric. To see transitivity, say $\Gamma \cdot W$ is equivalent to $\Gamma \cdot W'$ and $\Gamma \cdot W'$ is equivalent to $\Gamma \cdot W''$. Then for any element $\gamma W \in \Gamma \cdot W$, there is an element $\gamma' W'$ in $\Gamma \cdot W'$ so that $(\gamma W)^{-1}\gamma' W' \in \langle LS \rangle$. Then for the element $\gamma' W'$, there is a corresponding element, say $\gamma'' W''$ in $\Gamma \cdot W''$ with $(\gamma' W')^{-1}\gamma'' W'' \in \langle LS \rangle$. Then $(\gamma W)^{-1}\gamma'' W = (\gamma W)^{-1}\left((\gamma' W')(\gamma' W')^{-1}(\gamma'' W'')\right) \in \langle LS \rangle$. We proved:

Lemma 4.5. *The relation $\sim_\Gamma$ described above is an equivalence relation.*

An equivalence class of this relation will be called a *boundary point* of the group and denoted by $[\Gamma \cdot W]$. The set of all boundary points of the group will be called the *boundary of* Γ and denoted by $\partial\Gamma$.

Let us explain the terminology. For any coset $\Gamma \cdot W$, we consider

$$(\Gamma \cdot W)(\infty) := \{\gamma W(\infty) \colon \gamma \in \Gamma\},$$

that is, the set of values of all the transformations in the coset $\Gamma \cdot W$ at ∞. Since each transformation is in $\mathrm{PSL}_2(\mathbf{Z})$, this is a subset of the rational numbers.

Lemma 4.6. *Two cosets $\Gamma \cdot W$ and $\Gamma \cdot W'$ are equivalent if and only if $(\Gamma \cdot W)(\infty) = (\Gamma \cdot W')(\infty)$.*

Proof. Say $(\Gamma\cdot W)(\infty) = (\Gamma\cdot W')(\infty)$. Let γW be an arbitrary element of $\Gamma\cdot W$. There is an element $\gamma' \in \Gamma$ with the property that $\gamma W(\infty) = a = \gamma' W'(\infty)$. Then the composition $(\gamma W)^{-1}(\gamma W')$ fixes ∞. Therefore, it is an element of $\langle LS\rangle$. Conversely, suppose that $\Gamma\cdot W \sim_\Gamma \Gamma\cdot W'$. For $a \in (\Gamma\cdot W)(\infty)$ arbitrary, let $\gamma W \in \Gamma\cdot W$ so that $\gamma W(\infty) = a$. As $\Gamma\cdot W \sim_\Gamma \Gamma\cdot W'$, there is an element $\gamma' \in \Gamma$ so that $(\gamma W)^{-1}(\gamma' W') \in \langle LS\rangle$. Then $\gamma' W'(\infty) = (\gamma W)(LS)^n(\infty)$ for some $n \in \mathbf{Z}$. Note that LS^n fixes ∞, hence $\gamma' W'(\infty) = a$, that is $(\Gamma\cdot W)(\infty) \subseteq (\Gamma\cdot W')(\infty)$. We obtain the reverse inclusion by symmetry. $\square$

One consequence of the previous lemma is that we can write $[\Gamma\cdot W](\infty)$ instead of $(\Gamma\cdot W)(\infty)$. The equivalence relation $\sim_\Gamma$ induces an equivalence relation on the set of rationals. Namely, two rational numbers x and y are said to be equivalent if there is an element $W \in \mathrm{PSL}_2(\mathbf{Z})$ so that $x, y \in [\Gamma\cdot W](\infty)$.

The above construction can be repeated for any other subgroup P of the modular group which is conjugate to $\langle LS\rangle$. Instead of *evaluation* at ∞, one should consider evaluation at the unique fixed point of P. We have chosen this particular map, merely because it fits perfectly into the continued fractions and other classical subjects. This construction is also in accordance with the constructions made purely on graphs, see [4]. These matters will be treated in detail in Section 3 for the case when $\Gamma = \{I\}$, where we recover a theorem of Penner concerning the universal Teichmüller space. In the next example we treat the order 3 elliptic subgroup. Order 2 case is similar.

Example 4.7. *Set $\Gamma = \{I, L, L^2\}$. The modular graph corresponding to Γ is depicted in Figure 9. The base edge is the root of the tree. The values of the equivalence class of the coset $\Gamma\cdot I = \{I, L, L^2\}$ is $\{\infty, 1, 0\}$ respectively. This is denoted by $+$ in the figure. Similarly, the coset values of the equivalence class of the coset $\Gamma\cdot SL$ is $\{-1, 2, 1/2\}$, respectively. For instance, the same coset can be represented by $\Gamma\cdot SLSL^2$, whose elements assume exactly the same values, i.e. $\{-1, 2, 1/2\}$, in the this order.*

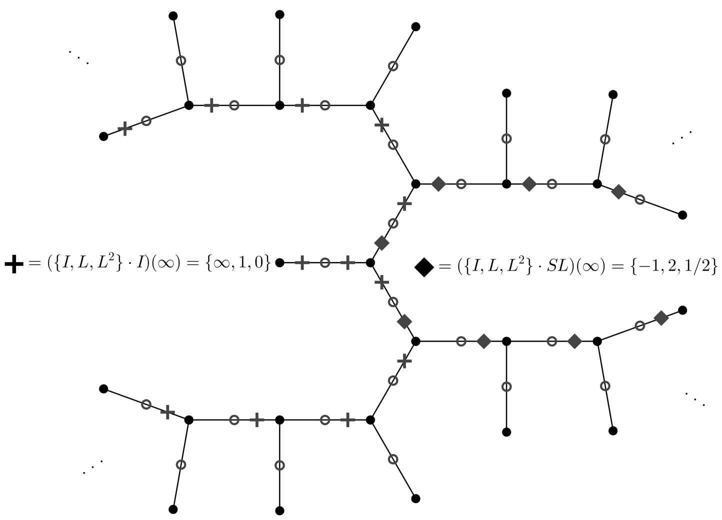

Figure 9: Boundary of the subgroup $\Gamma = \{I, L, L^2\}$.

Given a coset of the group Γ, say $\Gamma \cdot W$, one can use the previously defined flip acting on the corresponding modular graph $\Gamma\backslash\mathcal{F}$, $\phi_{\Gamma \cdot W}$, to define a map on $\partial\Gamma$, see Figure 10. We denote this map by $\phi^{\partial}_{\Gamma \cdot W}$. Explicitly, we have:

$$\begin{array}{rcl}
[\Gamma \cdot WL] & \mapsto & [\Gamma' \cdot WL^2] = [\Gamma' \cdot WS] \\
[\Gamma \cdot WL^2] & \mapsto & [\Gamma' \cdot WSL] \\
[\Gamma \cdot WSL] & \mapsto & [\Gamma' \cdot WSL^2] = [\Gamma' \cdot W] \\
[\Gamma \cdot WSL^2] & \mapsto & [\Gamma' \cdot WL]
\end{array}$$

where $\Gamma' = \phi_{\Gamma \cdot W}(\Gamma)$. Observe that the flip $\phi^{\partial}_{\Gamma \cdot WS}$ induces the same map on $\partial\Gamma$. So there is a one to one correspondence between pairs of cosets $(\Gamma \cdot W, \Gamma \cdot WS)$ and maps flips on the boundary.

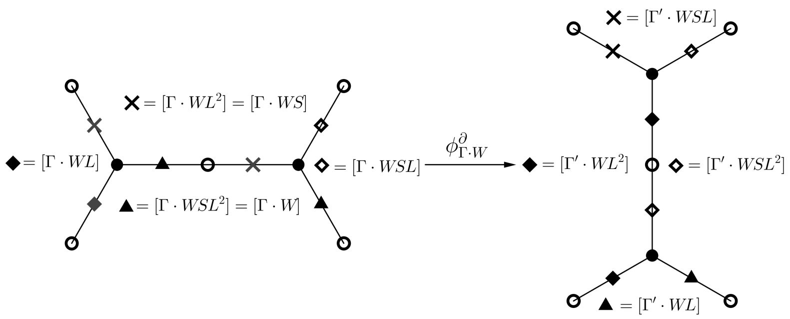

Figure 10: The action of flip on the boundary of $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$.

Suppose now that $\Gamma = \{I\}$. All cosets contain only one element, hence the sets $[\Gamma \cdot W](\infty)$ contain exactly one element. The corresponding equivalence relation on the set of rational numbers is the trivial relation, and therefore flips act on $\mathbf{Q}$. Identifying each equivalence class $[\Gamma \cdot W]$ with the corresponding end gives an identification of rationals in S^1_I. Then the following theorem becomes a direct consequence of Theorem 3.3:

Theorem 4.8 (Penner)**.** *For the trivial subgroup $\Gamma = \{I\} \in \mathbf{Sub}(PSL_2(\mathbf{Z}))$, the fundamental group of the sub-groupoid of $\mathscr{C}\mathscr{G}_\Gamma$ is Thompson's group T.*

First, let us remark that Thompson's group T can be regarded as the universal mapping class group. For the proof, one has to use the fact that the action of flips on the corresponding admissible configuration is equivalent to the action of the flip on $\partial\Gamma$. If this is the case, then the groupoid is closely related to the fundamental group of this groupoid. However, this identification is not possible in all cases. For instance, when $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$ if of genus g with n punctures, $|\partial\Gamma|$ is finite, as it is only comprised of classes corresponding to the punctures. In fact, one can identify an admissible configuration for the group Γ with a choice of a rational number selected from the class $[\Gamma \cdot W](\infty)$ for each puncture in $\partial\Gamma$. This induces an action of Mod^n_g on certain subsets[e] of $\mathbf{Q}$ whenever the

[e]It is possible to give a more precise definition of such sets, but we will not take on this task here.

graph is finite.

We define the *modular groupoid*, denoted $\mathcal{M}$, as the groupoid whose objects are boundaries of subgroups of the modular group and whose morphisms are generated by flips. Given an admissible configuration A, forgetting the second component of each pair $(\Gamma \cdot M, \gamma \cdot M)$ gives a well-defined map from $\mathcal{M}$ and $\mathscr{CG}$ and hence $\mathcal{M}$ becomes a sub-category of $\mathscr{CG}$. By abuse of language we say that $\mathcal{M}$ is a projection of $\mathscr{CG}$. $\mathcal{M}$ has many connected components. For any given subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$ the connected component of $\mathcal{M}$ containing Γ will be denoted by $\mathcal{M}_\Gamma$. The difference between $\mathscr{CG}$ and $\mathcal{M}$ is visible when Γ is a finite index subgroup of genus g with n punctures. We have seen that in this case, Mod_g^n can be embedded into $\mathscr{CG}_\Gamma$ and the image is of finite index, see Theorem 4.4. However, $\mathcal{M}_\Gamma$ does not admit such an embedding merely because it does have only finitely many objects having finitely many cosets $\Gamma \cdot M$, and hence admit only finitely many morphisms.

If, however Γ is the trivial subgroup, then $\mathscr{CG}_\Gamma = \mathcal{M}_\Gamma$. If for a subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$ the surface $\Gamma\backslash\mathcal{H}$ has trivial mapping class group, e.g. when Γ is a finite subgroup, then there is no advantage in considering $\mathscr{CG}_\Gamma = \mathcal{M}_\Gamma$. The case of the trivial subgroup (see Theorem 4.8) shows that the modular groupoid contains the relevant information in this case.

5 The çark groupoid

This section is devoted to the study the connected components of the sub-groupoid $\mathscr{CG}_\Gamma$; where Γ is a subgroup of the modular group generated by one element. A *çark* is defined as the quotient of $\mathcal{F}$ by Γ. Notice that çarks' characteristics are determined by the type of the generator. We investigate each case separately.

Notice that the set of objects in $\mathscr{CG}$ which are generated by one element forms a sub-category of the the class groupoid. There are seven connected components of the çark sub-groupoid: one connected component corresponds to the trivial group which has been treated in the previous section, four components stemming from elliptic elements of order two (denoted $\mathscr{CG}_2$) and three (denoted $\mathscr{CG}_3$), one component corresponding to parabolic subgroups (denoted $\mathscr{CG}_p$) and one corresponding to hyperbolic subgroups, denoted $\mathscr{CG}_h$. Let us investigate each non-trivial component in turn.

5.1 Elliptic Cases.

Any elliptic element in $\mathrm{PSL}_2(\mathbf{Z})$ is conjugate to either S, L or L^2. The associated modular graph is a rooted tree, see Figure 11 for the base edge free versions of the corresponding modular graphs. Since there are only 2 cosets there is only one admissible configuration associated to each modular graph. Hence we will do the computations on $\partial\Gamma$; where Γ is a subgroup generated by an elliptic element.

Thickening of each such graphs gives rise to disk with an orbifold point whose mapping class group is trivial. Therefore, in what follows, as objects, instead of considering the admissible configurations we will consider the action of flips on $\partial\Gamma\backslash\mathcal{F}$, i.e. we will consider the projection of $\mathscr{CG}_\Gamma$ onto $\mathcal{M}_\Gamma$; where Γ is a finite non-trivial subgroup of $\mathrm{PSL}_2(\mathbf{Z})$.

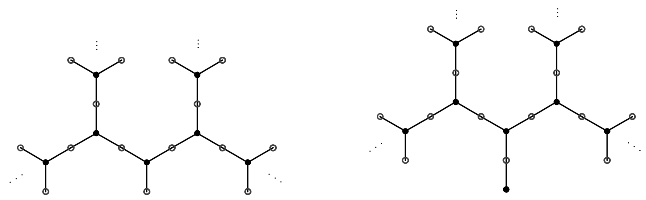

Figure 11: Elliptic çarks of order 2 and 3, respectively.

5.1.1 Order 2 case.

In this case, we assume that the generator is of order 2 and hence is conjugate to S. This means that up to the choice of the base edge, all the associated modular graphs are *same*. If the root of the tree is chosen as the base edge, that is if $\Gamma = \langle S \rangle = \{I, S\}$, then a case by case analysis shows that there is no flip sequence (in fact there are only finitely many flips that can be applied to the root) that moves the base edge. Therefore the group $\{I, S\}$ constitute one connected component of $\mathscr{C}\mathscr{G}_2$ to which we will refer as $\mathscr{C}\mathscr{G}_{2,S}$. If the generator of Γ is not equal to S, then the base edge is not equal to the root. In this case, base edge, say b, is different from the root. Given any other edge b' on the graph which is different from the root, by applying Proposition 4.3 to the finite part of the graph containing b and b', we see that there is a sequence of flips sending the base edge b to b'. Hence, all the remaining order two subgroups form the other connected component.

As for the corresponding groupoids, as $\mathscr{C}\mathscr{G}_{2,S}$ has only one object, this component of the groupoid is indeed a group.

Theorem 5.1. *For the group $\Gamma = \{I, S\}$ the groupoid $\mathscr{C}\mathscr{G}_\Gamma = \mathscr{C}\mathscr{G}_{2,S}$ is isomorphic to Thompson's group F.*

Proof. Recall that the group F is isomorphic to the group of piecewise linear Möbius transformations of $[0, +\infty]$ with finitely many rational break points, see [1, Section 2]. This group is generated by the two transformations

$$x_0(t) = \begin{cases} [0, \frac{1}{2}] & \mapsto [0, 1] \\ [1/2, 1] & \mapsto [1, 2] \\ [1, \infty] & \mapsto [2, \infty] \end{cases}$$

and by

$$x_1(t) = \begin{cases} [0, 1] & \mapsto [0, 1] \\ [1, \frac{3}{2}] & \mapsto [1, 2] \\ [\frac{3}{2}, 2] & \mapsto [2, 3] \\ [2, \infty] & \mapsto [3, \infty] \end{cases}$$

Observe that x_0 is the map on $\partial\Gamma$ induced by applying flip on the edge $\Gamma \cdot L$ and x_1 is induced by the flip on $\Gamma \cdot LSL^2$. The quasi-flip on the root changes only the ordering of the subsets $[\Gamma \cdot M](\infty)$, and hence act trivially on the boundary. $\square$

For the remaining case, let Γ be a subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ of order 2 and different from $\{I, S\}$. As indicated earlier, this is a connected groupoid. The flip on the base edge will result in the change of the ordering of $[\Gamma \cdot M](\infty)$ and hence acts trivially on the set of admissible configurations. Recalling the fact that the conjugation acts as change of base edge on the graph we obtain the following result as a consequence of Theorem 5.1

Corollary 5.2. *The fundamental group of the groupoid $\mathscr{CG}_\Gamma$; where Γ is a subgroup of order two, different from $\{I, S\}$, is isomorphic to Thompson's group F.*

Let us also note that the previous arguments can be carried out in terms of continued fractions and ends of the corresponding modular graphs. Indeed, all the edges of the corresponding çark can be labeled with the word in the cosets beginning with L or L^2 and hence the continued fraction map identifies the ends with the closed interval $[0, +\infty]$. If we use also the same labels to address the flips, then we see that this identifies the fundamental group of this elliptic groupoid with the subgroup of T, see Section 3.4, which fixes $[-\infty, 0]$. This group is easily seen to be isomorphic to Thompson's group F, see [2].

5.1.2 Order 3 case.

The groupoid has two connected components. One connected component has two objects $\{I, L, L^2\}$ and $\{I, SLS, SL^2S\}$. These two are connected by the flip on the root. This flip acts by conjugation at the level of subgroups. In this case, we'll identify Thompson's group F with the piecewise linear Möbius transformations of the unit interval. The generators become the following:

$$x_0(t) = \begin{cases} [0, \frac{1}{3}] & \mapsto [0, \frac{1}{2}] \\ [\frac{1}{3}, \frac{1}{2}] & \mapsto [\frac{1}{2}, \frac{2}{3}] \\ [\frac{1}{2}, 1] & \mapsto [\frac{2}{3}, 1] \end{cases}$$

and by

$$x_1(t) = \begin{cases} [0, \frac{1}{2}] & \mapsto [0, \frac{1}{2}] \\ [\frac{1}{2}, \frac{3}{5}] & \mapsto [\frac{1}{2}, \frac{2}{3}] \\ [\frac{3}{5}, \frac{2}{3}] & \mapsto [\frac{2}{3}, \frac{3}{4}] \\ [\frac{2}{3}, 1] & \mapsto [\frac{3}{4}, 1] \end{cases}$$

If we fix $\Gamma = \{I, L, L^2\}$ then these two maps of the unit interval are induced by flips on the edges $\Gamma \cdot SL^2$ and $\Gamma \cdot SLSL^2$. By the conjugation of $\mathrm{PSL}_2(\mathbf{Z})$ on the modular graph in question we obtain:

Theorem 5.3. *Each connected component of the groupoid $\mathscr{CG}_3$ is isomorphic to Thompson's group F.*

As in the previous sections, similar results can be obtained using the language of continued fractions and ends of graphs.

5.2 Parabolic Case.

Let Γ be a subgroup generated by a parabolic element. Then it is isomorphic to $\mathbf{Z}$ and one of its generators must conjugate to a power of LS. The surface obtained by thickening the corresponding modular graph is a punctured disk, see Figure 12. Once again the mapping class group is trivial, hence we will consider

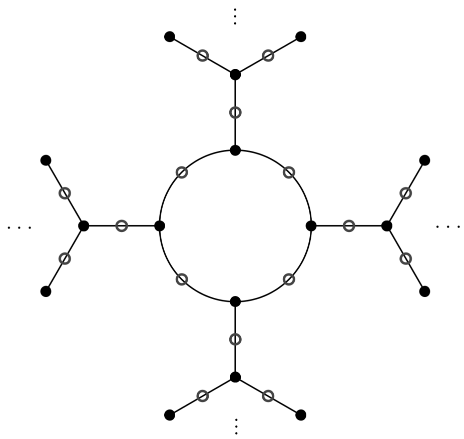

Figure 12: Modular graph of $\Gamma \leq \mathrm{PSL}_2(\mathbf{Z})$ conjugate to $\langle (LS)^4 \rangle$.

the projection of this subgroupoid $\mathscr{C}\mathscr{G}_p$ onto the corresponding component of the modular groupoid, namely $\mathcal{M}_{\langle LS \rangle}$.

Proposition 4.3 tells us that the groupoid in question is connected. Therefore, we will set $\Gamma = \langle LS \rangle$. Let us refer to the unique cycle of the modular graph as the *spine* of the graph. Any flip on the spine changes the group Γ and any other flip off the spine fixes the group. Moreover, flips on the spine commute with any other flip off the spine. Therefore, the class $[\Gamma \cdot I](\infty)$ is fixed. Flips on $\Gamma \cdot L^2SL^2$ and $\Gamma \cdot L^2SLSL^2$ induces the maps x_0 and x_1 viewed as maps from $[0, 1]$ to $[0, 1]$, see previous section. Let us summarize:

Theorem 5.4. *Let Γ be a subgroup of $PSL_2(\mathbf{Z})$ generated by a parabolic element. Then the fundamental group of the connected component $\mathcal{M}_\Gamma$ is isomorphic to Thompson's group F.*

5.3 Hyperbolic Case.

Suppose now that Γ is a subgroup generated by a hyperbolic element. Then it is isomorphic to $\mathbf{Z}$. The surface obtained by thickening the corresponding modular graph is an annulus, see Figure 13.

We may repeat the previous arguments once again to obtain the following

Theorem 5.5. *Let $\Gamma \leq PSL_2(\mathbf{Z})$ be a subgroup generated by one hyperbolic element. Then*

- *the groupoid $\mathscr{C}\mathscr{G}_\Gamma$ admits an embedding (in fact an anti-homomorphism) of $\mathbf{Z}$ (i.e. the mapping class group of annulus), and*
- *the fundamental group of $\mathcal{M}_\Gamma$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus F \oplus F$.*

5.4 Concluding Remarks.

There are certain identifications between the objects of the çark groupoids and binary quadratic form. Among the most interesting is the one between the groupoid $\mathscr{C}\mathscr{G}_{hyp}$ and indefinite primitive binary quadratic forms, see [14]. Such

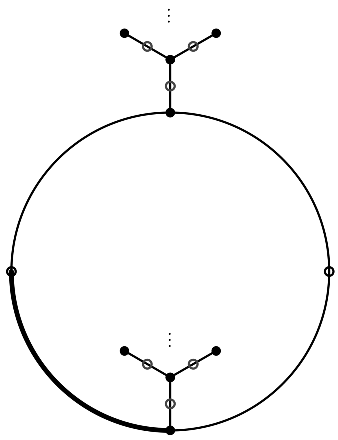

Figure 13: Modular graph of the group $\Gamma = \langle L^2SLS \rangle$; where the bold edge is the base edge.

forms correspond to ideal classes in real quadratic number fields. There are a myriad of arithmetic questions, e.g. class number problems of Gauss, around these objects, however, it must be admitted that we have failed to extract any arithmetic information from the class groupoid.

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