BELYI LATTÈS MAPS

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Abstract. In this work, using elementary tools we determine those Lattès maps which are at the same time Belyi maps by explicitly determining their ramification data. It turns out that in the generic case, i.e. when the automorphism group is $\mathbf{Z}/2\mathbf{Z}$, the corresponding family of Lattès maps are Belyi maps if and only if the isogeny is multiplication by two. Elliptic curves with extra automorphisms also determine families of Belyi maps. We provide examples of some Belyi Lattès maps together with a formula for such maps which may be used to write Belyi maps of arbitrarily high degree. We conclude the paper with a discussion of the field of definition of such Belyi pairs.

1. Introduction

An algebraic curve, X, admitting a model whose defining equations have algebraic coefficients is referred to as an *arithmetic curve*. A celebrated theorem of Belyi, [1], states that arithmetic curves admit a **Belyi map**, ϕ; that is, a meromorphic function ramified at most over 3 points, which are often fixed as 0, 1 and ∞. The pair (X, ϕ) is called a **Belyi pair**. Conversely, if an algebraic curve X admits a Belyi map, then X is an arithmetic curve, [18].

The absolute Galois group, $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, acts on the set of Belyi pairs (X, ϕ) by the action of any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the coefficients of the defining equation(s) of X and on ϕ. Even though studying a group via its actions on well-known objects is a sound principle, understanding the mentioned action in this generality is widely accepted as a task out-of-reach. One is therefore led to work on certain *natural substructures* which are rich enough. It is known that $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on the set of arithmetic elliptic curves faithfully, see [9] for a detailed account. More generally, this action is faithful for arbitrary curves of genus g, see [6], a result which was first proven for genus 0 case, [14]. There are other families which admit (conjecturally) a faithful action, see for instance [5, 17].

A map $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$ is said to be a quotient of an affine map if it is semi-conjugate to an affine self map of $\mathbf{C}/\Omega$; where Ω is an additive subgroup of $\mathbf{C}$, see Section 2 for precise definitions. When Ω is of rank one it turns out that all such maps are Belyi maps. This motivates the analogous question for rank 2 case. To this end, we determine Lattès maps which are at the same time Belyi maps using elementary tools. This approach leads also to an explicit description of associated dessins d'enfants. It turns out that whenever $\mathbf{C}/\Omega$ does not have extra automorphisms, the only isogeny that induces a Belyi map is multiplication by 2 (and its translates). This gives a family of Belyi maps parametrized over the moduli space of elliptic curves. For each elliptic curve with extra automorphisms, we obtain families of Belyi maps. We then address the question of determining the field of definition of Belyi Lattès maps and the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on Belyi Lattès maps. Let us remark that results in the same vein are obtained by Guralnick, Müller and Saxl, [7] where the authors study a classification problem referred as exceptionality of rational maps. Indeed, several Belyi type maps stemming from different underlying

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structures are obtained by the authors. Their results and ours seem to agree in the particular case of Lattès maps.

2. Fundamental notions

This section is devoted to define and collect some properties Lattès maps which are particular cases of quotients of affine maps. Our main aim is to state a classification result (Theorem 2.1).

2.1. **Quotients of affine maps.** A map $\psi\colon \mathbf{C} \longrightarrow \mathbf{C}$ of the form $\psi(z) = \lambda z + \mu$, where $\lambda, \mu \in \mathbf{C}$ with $\lambda \neq 0, 1$, is called an *affine map* of $\mathbf{C}$. We define a map $\phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ to be the *quotient of an affine map* if there is a commutative diagram of holomorphic maps :

$$\begin{tikzcd} \mathbf{C}/\Omega \arrow[r, "\psi"] \arrow[d, "\pi"] & \mathbf{C}/\Omega \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\phi"] & \mathbb{P}^1 \end{tikzcd} \tag{1}$$

where Ω is a non-trivial discrete subgroup of $\mathbf{C}$ and π is a non-constant finite map. As ψ is Ω periodic we have $\lambda\Omega \subseteq \Omega$. This inclusion ensures the holomorphicity of the map ϕ, see [3, Theorem 3.20] for a proof. From now on, we consider lattices up to homotheties. So, without loss of generality, we assume that if rank of Ω is 1 then $\Omega \cong \mathbf{Z}$, and if rank of Ω is 2 then Ω is generated as a $\mathbf{Z}$-module by 1 and τ for some τ in the fundamental region of the action of $\mathrm{PSL}_2(\mathbf{Z})$ on the upper half plane, $\mathbb{H}$. This means, in the rank 1 case $\lambda \in \mathbf{Z}$ and in the rank two case λ is an eigenvalue of an integral non-degenerate 2×2 matrix (whose eigenvector is $(1\,\tau)^t$; where $\Omega = \langle 1, \tau\rangle$) and hence is a quadratic algebraic integer. In particular, for $\Omega = \mathbf{Z}[\sqrt{-1}]$ or $\Omega = \mathbf{Z}[\zeta_3]$, $\lambda \in \Omega$; where $\zeta_k = \exp(2\pi\sqrt{-1}/k)$.

For simplicity of the exposition, we will assume $\mu = 0$. If Ω is generated by one element, then by a change of variable, we may assume that the generator is 2π. The commutativity of the above diagram implies that $\phi(\pi(z)) = \pi(\lambda\, z)$. For $\pi(z) = e^{iz}$ we necessarily have $\phi(w) = w^\lambda$. If $\pi(z) = e^{iz} + e^{-iz}$ then $\phi(w + w^{-1}) = w^\lambda + w^{-\lambda}$. Functions satisfying this functional equation are called Chebyshev polynomials. It turns out that these are the only possibilities up to conjugation in the rank 1 case, [13, Lemma 3.8]. Notice that both w^λ and Chebyshev polynomials are Belyi maps, i.e. ramified at most over 3 points, see [10, Section 1.4.2].

2.2. **Lattès maps.** From now on, we assume that Ω is of rank two generated by 1 and τ so that $\mathbf{C}/\Omega$ is an elliptic curve. In this case, the map ϕ is called a *Lattès map*, a subfamily of which were studied first by Lattès in [12]. The degree of the map ϕ is $|\lambda|^2$. By Riemann-Hurwitz formula, the map ψ is unramified. As ramification behaves multiplicatively in compositions, we have $e_\psi(z)\, e_\pi(\psi(z)) = e_\pi(z)\, e_\phi(\pi(z))$; where $e_f(x)$ denotes the ramification index of f at x. So the ramification values of $\phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ are a subset of the ramification values of $\pi\colon \mathbf{C}/\Omega \longrightarrow \mathbb{P}^1$. The following theorem is due to Milnor :

Theorem 2.1 ([13, Theorem 3.1]). *A rational map $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a Lattès map if and only if there is a lattice Ω, an affine map $\psi\colon \mathbf{C}/\Omega \to \mathbf{C}/\Omega$ and some subgroup G of $Aut(C/\Omega)$ so that ϕ is conjugate, by a Möbius transformation, to $\psi/G\colon (\mathbf{C}/\Omega)/G \to (\mathbf{C}/\Omega)/G$ sending the equivalence class of a point z under the action of G to the class of $\psi(z)$.*

Let us stress at this point that there are Lattès maps where the map π is not a quotient map of the form $\mathbf{C}/\Omega \to \mathbb{P}^1 = (\mathbf{C}/\Omega)/G$. In particular, π may be of arbitrary high degree. The above theorem implies that in all such cases one can find a full lattice, say $\Omega' \subseteq \mathbf{C}$ so that ϕ is a Lattès map of the form :

$$\begin{array}{ccc} \mathbf{C}/\Omega' & \xrightarrow{\psi} & \mathbf{C}/\Omega' \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{P}^1 = (\mathbf{C}/\Omega')/G & \xrightarrow{\phi} & \mathbb{P}^1 = (\mathbf{C}/\Omega')/G \end{array}$$

The following table summarizes the automorphism group and possibilities for the map $\pi\colon \mathbf{C}/\Omega \to (\mathbf{C}/\Omega)/G$.

Ω	$j(\mathbf{C}/\Omega)$	$\mathrm{Aut}(\mathbf{C}/\Omega)$	G	π
$\mathbf{Z}[\zeta_3]$	0	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/6\mathbf{Z}$	$\wp^3$
$\mathbf{Z}[\zeta_3]$	0	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	$\wp'$
$\mathbf{Z}[\zeta_3]$	0	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\wp$
$\mathbf{Z}[\zeta_4]$	1728	$\mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$	$\wp^2$
$\mathbf{Z}[\zeta_4]$	1728	$\mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\wp$
otherwise	$\neq 0$ or 1728	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\wp$

Table 1. Quotient maps from an elliptic curve giving rise to Lattès maps.

3. Ramification of Lattès maps

Theorem 2.1 and Table 1 allow us to treat each case individually. For future reference, let us note some standard facts concerning elliptic functions :

- the functions $\wp$ and $\wp'$ are related by the differential equation :
$$(\wp'(z))^2 = 4\,(\wp(z))^3 - g_2\wp(z) - g_3;$$
where $g_2 = 60\sum_{\omega\in\Omega\setminus\{0\}} \omega^{-4}$ and $g_3 = 140\sum_{\omega\in\Omega\setminus\{0\}} \omega^{-6}$,
- the equation $\wp(z) = \wp(w)$ holds if and only if either $z + w = 0 \pmod{\Omega}$ or $z - w = 0 \pmod{\Omega}$,
- the function $\wp$ has two zeros, denoted d_1 and d_2, in $\mathbf{C}/\Omega$ (see [4, 2] for further information on their explicit computation)
- the roots of $\wp'$ are $\frac{\omega_1}{2} = \frac{1}{2}$, $\frac{\omega_2}{2} = \frac{\tau}{2}$ and $\frac{\omega_3}{2} = \frac{1+\tau}{2}$, i.e. half periods.
- the addition formula of $\wp$, [8, pg. 157], reads :
$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 - \wp(z_1) - \wp(z_2).$$

Let us observe the following:

Lemma 3.1. *If degree of ψ is even (resp. odd) then the cardinality of the intersection $\psi^{-1}(z_o) \cap \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$ is even (resp. odd) when z_o is either a period or a half period.*

Proof. Say z_o is either 0 or a half period. Let $z \in \psi^{-1}(z_o)$ and $w \in \mathbf{C}$ with $z+w = 0 \pmod{\Omega}$. Then, $w \in \psi^{-1}(z_o)$ because $\lambda(z+w) = \lambda z + \lambda w = \psi(z) + \psi(w) \in \Omega$, and $-z_o = z_o \pmod{\Omega}$. This means that the parity of the set $\psi^{-1}(z_o)$ is determined by the number $|\psi^{-1}(z_o) \cap \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}|$. We deduce the result using the fact that ψ is unramified. $\square$

3.1. $G \cong \mathbb{Z}/2\mathbb{Z}$. We have $\pi = \wp$ by Theorem 2.1. So the set of ramification values of ϕ is a subset of $R = \{\wp(\frac{1}{2}), \wp(\frac{\tau}{2}), \wp(\frac{\omega_3}{2}), \wp(0) = \infty\}$. Remark that as λ is a quadratic algebraic integer, $|\lambda|^2 \in \mathbf{Z}$. Let us first note the following :

Lemma 3.2. *Let $z_o \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$ be an arbitrary element. Then $\psi^{-1}(z_o)$ contains exactly one element of $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$ whenever $|\lambda|^2$ is odd.*

Proof. We know that $|\psi^{-1}(z_o)|$ is odd, therefore it has to contain an odd number of elements from the set $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$, by Lemma 3.1. Let $z, w \in \psi^{-1}(z_o)\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$ be two elements. Then $z = |\lambda|^2 z = \overline{\lambda}\lambda z = \overline{\lambda}\lambda w = |\lambda|^2 w = w$ in $\mathbf{C}/\Omega$. □

The following result gives a complete picture of the ramification data in this case summarized in Table 2 :

Proposition 3.3. *If $\deg(\psi) = |\lambda|^2 = 4$; then ϕ is ramified exactly over $\{\wp(\omega_i/2)\}$ with the inverse image of each point containing 2 elements. If $\deg(\psi) = |\lambda|^2 > 4$ is an even integer, then ϕ is ramified exactly over R with the numbers $|\phi^{-1}(\wp(\omega_i/2))|$ being $|\lambda|^2/2$ and $|\phi^{-1}(\infty)|$ being $|\lambda|^2/2 - 2$. Otherwise, that is if $|\lambda|^2$ is odd, then ϕ is ramified exactly over R so that for any $w \in R$, $|\phi^{-1}(w)|$ is $\frac{|\lambda|^2-1}{2} + 1$.*

Proof. Say $|\lambda|^2$ is an odd integer. Then, for any $z_o \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$ there is exactly one element of $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$, say w_o, in $\psi^{-1}(z_o)$ (Lemma 3.2). For any element $z \in \psi^{-1}(z_o) \setminus \{w_o\}$ there is some $w \in \psi^{-1}(z_o) \setminus \{w_o\}$ so that $z + w = 0$ in $\mathbf{C}/\Omega$. For these two elements we have $\wp(z) = \wp(w)$. This means that $|\phi^{-1}(\wp(z_o))| = \frac{|\lambda|^2-1}{2}$. Suppose now that $|\lambda|^2 = 4$. $\psi^{-1}(0) = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\}$ and their images under $\wp$ are distinct, hence 0 is not a ramification point. However, for any $e_i \in \mathbf{C}/\Omega$, elements of $\psi^{-1}(\omega_i/2)$ can paired so that they add up to $0 \in \mathbf{C}/\Omega$, and therefore $|\wp(\psi^{-1}(\omega_i/2))| = 2$. Therefore $|\phi^{-1}(\wp(\omega_i/2))|$ is 2. More generally, if $\deg(\psi) > 4$ and even then a similar diagram chasing argument leads to the fact that $|\phi^{-1}(\infty)| = \frac{|\lambda|^2}{2} + 4$, because in this case $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\omega_3}{2}\} \subset \psi^{-1}(0)$. □

$\deg(\psi)$	$\lvert\phi^{-1}(\wp(0))\rvert$	$\lvert\phi^{-1}(\wp(e_i))\rvert$
4	4	2
even and > 4	$\frac{\lvert\lambda\rvert^2}{2} + 2$	$\frac{\lvert\lambda\rvert^2}{2}$
odd	$\frac{\lvert\lambda\rvert^2-1}{2} + 1$	$\frac{\lvert\lambda\rvert^2-1}{2} + 1$

TABLE 2. Ramification of Lattès maps corresponding to the group $G \cong \mathbf{Z}/2\mathbf{Z}$.

3.2. $G \cong \mathbb{Z}/3\mathbb{Z}$. In order to admit an order 3 automorphism, Ω must be homothetic to the Eisenstein lattice, $\mathbf{Z}[\zeta_3]$. In this case, we have the identity $\wp''(z) = 6(\wp(z))^2$. For $i = 1, 2$ the differential equation satisfied by $\wp$ and $\wp'$ tells us that the value of $\wp'$ at d_i is $\pm\sqrt{-g_3}$. Remark that $\wp'(d_1) \neq \wp'(d_2)$. Indeed, otherwise the degree 3 doubly periodic function $F(z) = \wp'(z) - \wp'(d_1)$ would have two triple zeros, d_1 and d_2. We conclude that ϕ is ramified at most over the set $\{\wp'(d_1), \wp'(d_2), \wp'(0)\}$. The following proposition gives the complete ramification behaviour in this case :

Proposition 3.4. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map whose degree is 3. Then ϕ is ramified exactly over the set $\{\wp'(d_1), \wp'(d_2), \wp'(0)\}$. We refer to Table 3 for the precise ramification data.*

	$\lvert\phi^{-1}(\infty)\rvert$	$\lvert\phi^{-1}(\wp'(d_1))\rvert$	$\lvert\phi^{-1}(\wp'(d_2))\rvert$
$\lvert\psi^{-1}(0)\cap\{d_1,d_2\}\rvert=2$	$\frac{\lvert\lambda\rvert^2-3}{3}+3$	$\frac{\lvert\lambda\rvert^2}{3}$	$\frac{\lvert\lambda\rvert^2}{3}$
$\lvert\psi^{-1}(0)\cap\{d_1,d_2\}\rvert=1$	impossible		
$\lvert\psi^{-1}(0)\cap\{d_1,d_2\}\rvert=0$	$\frac{\lvert\lambda\rvert^2-1}{3}+1$	$\frac{\lvert\lambda\rvert^2-1}{3}+1$	$\frac{\lvert\lambda\rvert^2-1}{3}+1$

TABLE 3. Ramification of Lattès maps corresponding to the group $G \cong \mathbf{Z}/3\mathbf{Z}$.

Proof. For brevity, let us only show that : $|\psi^{-1}(0)\cap\{d_1,d_2\}|=1$ is not possible. The proofs of the remaining cases have been exemplified in the proof of Proposition 3.3. Indeed, as a result of the fact that the sum of zeros (counting multiplicities) of and elliptic function should be equal to the sum of poles (counting multiplicities), we obtain $d_1+d_2\equiv 0 \pmod{\Omega}$. Say $\psi(d_1)=0 \pmod{\Omega}$. Then

$$\psi(d_2)\equiv\psi(-d_1)\equiv-\psi(d_1)\equiv 0 \pmod{\Omega}.$$

□

3.3. $G\cong\mathbb{Z}/4\mathbb{Z}$. This is possible only when the lattice Ω is homothetic to the ring of Gaussian integers, $\mathbf{Z}[\sqrt{-1}]$. To determine the ramification points of ϕ we determine the zeros and poles of $\wp$ and $\wp'$, namely, 0, half periods, $\frac{\omega_i}{2}$, $i=1,2,3$, and d_1 and d_2. As a result of the extra symmetry of $\wp$, i.e. $\wp(\sqrt{-1}\,z)=-\wp(z)$, we have $d_1=d_2=\frac{\omega_3}{2}$ and $\wp(\frac{1}{2})=-\wp(\frac{\sqrt{-1}}{2})$. So, the set $R=\{\wp^2(0)=\infty,\wp^2(\frac{1}{2})=\wp^2(\frac{\sqrt{-1}}{2}),\wp^2(\frac{\omega_3}{2})=\wp^2(d_1)=\wp^2(d_2)=0\}$ is the set of all possible ramification values of ϕ. We further have $\psi^{-1}(0)$ is the lattice generated by $\frac{b+a\sqrt{-1}}{a^2+b^2}$ and $\frac{a-b\sqrt{-1}}{a^2+b^2}$; where $\lambda=a+b\sqrt{-1}$. The set $\psi^{-1}(\{0,1/2,\sqrt{-1}/2,(1+\sqrt{-1})/2\})$ is also a lattice, denoted by F_λ, generated by halves of the generators of $\psi^{-1}(0)$. That is, $\psi^{-1}(0)=2F_\lambda$. To ease notation, elements of $\psi^{-1}(0)$ will be referred to as a vertices of type $\times$, elements of $\psi^{-1}(\frac{1}{2})\cup\psi^{-1}(\frac{\sqrt{-1}}{2})$ (or edge centers) will be referred to as vertices of type $\circ$ and elements of $\psi^{-1}(\frac{1+\sqrt{-1}}{2})$ (or face centers) will be referred to as vertices of type $\bullet$.

The degree 4 morphism $\wp^2\colon\mathbf{C}/\mathbf{Z}[\sqrt{-1}]\to\mathbb{P}^1$ can be described geometrically : $\wp^2$ identifies the line segments $[0,\frac{1}{2}]$ and $[\frac{1}{2},\frac{1+\sqrt{-1}}{2}]$ with $[0,\frac{\sqrt{-1}}{2}]$ and $[\frac{\sqrt{-1}}{2},\frac{1+\sqrt{-1}}{2}]$, respectively. To determine the exact ramification we will determine points of the lattice F_λ that lies in $[0,1/2]\times[0,\sqrt{-1}/2]$, which is denoted by $F_{\lambda,\wp^2}$, paying attention to their types. The points of the set $((0,1/2)\times(0,\sqrt{-1}/2)\cap F_{\lambda,\wp^2}$ will be called interior lattice points, and points that are in $\partial([0,1/2]\times[0,\sqrt{-1}/2])\cap F_{\lambda,\wp^2}$ are called boundary lattice points.

As $\mathbf{Z}[\sqrt{-1}]$ has a multiplication by $\sqrt{-1}$ symmetry, without loss of generality we will assume that $\lambda=a+b\sqrt{-1}$, with $a,b\geq 0$. We would like to remark that elements of F_λ in the fundamental region of $F=\{z\in\mathbf{C}\,|\,0\leq\mathrm{Re}(z),\mathrm{Im}(z)\leq 1\}$ is exactly the set of intersection points of the lines

$$\begin{aligned} 2by&=2ax+j;\quad j=-2a,1-2a,\ldots,2b-1,2b, \text{ and}\\ 2ay&=-2bx+k;\quad k=0,1,\ldots,2(a+b)\end{aligned}$$

We refer to Figure 1 for the case $\lambda=3+4\sqrt{-1}$.

Let us collect some preliminary results :

(PR1) Types of vertices of $F_{\lambda,\wp^2}$ depends on the parities of the integers a and b. Table 4 gives a summary where each claim can be seen by direct computation.

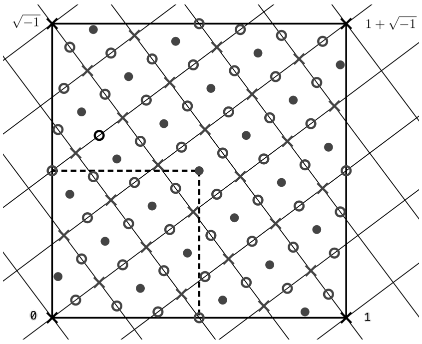

FIGURE 1. Inverse image of the set $\psi^{-1}(0,\infty,\wp^2(1/2))$ in F_λ for $\lambda = 3+4\sqrt{-1}$: $\times$, $\circ$, $\bullet$ represents elements of $\psi^{-1}(0)$, $\psi^{-1}(1/2) = \psi^{-1}(\sqrt{-1}/2), \psi^{-1}(\frac{1+\sqrt{-1}}{2})$, respectively.

a	b	$\frac{1}{2} \in$	$\frac{\sqrt{-1}}{2} \in$	$\frac{1+\sqrt{-1}}{2} \in$
even	even	$\times$	$\times$	$\times$
even	odd	$\circ$	$\circ$	$\bullet$
odd	even	$\circ$	$\circ$	$\bullet$
odd	odd	$\bullet$	$\bullet$	$\times$

TABLE 4. Parities of a and b and values of ψ at half-periods.

(PR2) Each one of four line segments bounding the square $[0,1/2]\times[0,\sqrt{-1}/2]$ can contain at most 2 distinct vertex types. Indeed, parities of integral solution of the equation $a'\alpha + b'\beta = 0$ (where $a = \gcd(a,b)a'$ and $b = \gcd(a,b)b'$) determines the type of the corresponding element $F_{\lambda,\wp^2}$ on the segment $[0,1/2]$. Precisely, the vertex is of type $\times$ (resp. $\bullet$) if α and β are both even (resp. odd). A vertex of type $\circ$ is on the segment $[0,1/2]$ if α and β have opposite parities.

(PR3) if z is an element of $F_{\lambda,\wp^2}$, then $z+1/2$ and $z+\sqrt{-1}/2$ is also a vertex of $F_{\lambda,\wp^2}$. If a and b are both even then the type of the vertices $z+1/2$ and $z+\sqrt{-1}/2$ are the same as that of z. A vertex of type $\times$ becomes a vertex of type $\circ$ if a and b are of opposite parities and a vertex of type $\bullet$ if both a and b are odd. A vertex of type $\bullet$ becomes a vertex of type $\circ$ if a and b are of opposite parities and a vertex of type $\times$ if both a and b are odd. A vertex of type $\circ$ becomes a vertex of type $\times$ if a and b are of opposite parities. We cannot have a vertex of type $\circ$ along the boundary in the other cases. To exemplify the argument, let us suppose that a is even and b is odd with $z = \frac{p}{q}$ (with $p,q \in \mathbf{Z}$ and relatively prime) so that $\psi(z) = 1/2$, then we must have : $(a+b\sqrt{-1})\frac{p}{q} = \frac{1}{2}$ implying $a\frac{p}{q} \equiv \frac{1}{2} \pmod 1$. This means $4|q$, as a is even. Therefore q is even. On the other hand, the vanishing of the imaginary part implies $q|b$, which is a contradiction as b is odd. The other cases are treated similarly.

(PR4) The family of lines $2by = 2ax + j;\quad j = 0,\dots,2b-1,2b$ divide $[0,1]$ into $2b$ equal intervals, and the lines $2ay = -2bx+k; k = 0,1,\dots,2a$ into $2a$ equal intervals. In particular, an element of $F_{\lambda,\wp^2}$ is a point on $[0,1]$ whenever $0 \le \frac{\gamma}{2a} = \frac{\delta}{2b} \le 1$ for some non-negative integers γ and δ.

Lemma 3.5. *For $\lambda = a + b\sqrt{-1}$, we set $d = \gcd(a, b)$ and $a = da'$ and $b = db'$. The number of vertices of types $\times$, $\circ$, $\bullet$ in $F_{\lambda,\wp^2}$ depends on the parities of a and b. More precisely :*

parities of a and b	*vertex in*	*type $\times$ vertices*	*type $\circ$ vertices*	*type $\bullet$ vertices*
both even	*int*	$\frac{a^2+b^2}{4} - d + 1$	$\frac{a^2+b^2}{2}$	$\frac{a^2+b^2}{4} - d$*; a', b' odd*
			$\frac{a^2+b^2}{2} - d$	$\frac{a^2+b^2}{4}$*; a', b' mixed*
	∂	$2d$	*0*	$2d$*; a', b' odd*
			$2d$	*0; a', b' mixed*
mixed	*int*	$\frac{a^2+b^2-2d+1}{4}$	$\frac{a^2+b^2-2d+1}{2}$	$\frac{a^2+b^2-2d+1}{4}$
	∂	d	$2d$	d
both odd	*int*	$\frac{a^2+b^2+2}{4} - d$	$\frac{a^2+b^2}{2}$	$\frac{a^2+b^2+2}{4} - d$
	∂	$2d$	*0*	$2d$

Proof. We'll first assume that both a and b are even. To determine the points of $F_{\lambda,\wp^2}$ we'll use Pick's theorem, which predicts that given such a lattice polygon (i.e. a polygon whose vertices are elements of the lattice) one has $A = i + j/2 - 1$; where A is the number of lattice squares in the polygon, i is the number of interior points and j is the number of boundary points. Among these terms, remark first that there are $\frac{a^2+b^2}{4}$ lattice squares because $\deg(\psi) = a^2 + b^2$. (PR1) implies that the vertices of $F_{\lambda,\wp^2}$ are exactly at 0, $\frac{1}{2}$, $\frac{\sqrt{-1}}{2}$, $\frac{1+\sqrt{-1}}{2}$. (PR3) implies that it is enough to count the vertices along the segment $[0, \frac{1}{2}]$ to determine j. As discussed in (PR2) such lattice points correspond to integral solutions of the equation $a'\alpha + b'\beta = 0$ for which the quantity $\frac{b'\alpha - a'\beta}{2d((a')^2+(b')^2)} \in [0, 1]$. The method described in (PR4), which counts the number of points on $[0, 1]$, implies that the number of such solutions are exactly $2\gcd(a, b) + 1$. From here, we conclude that $j = 4\gcd(a, b)$. We deduce $i = \frac{a^2+b^2}{4} - \frac{4d}{2} + 1$. As both a and b are even, $z \mapsto \frac{1}{2}\lambda z$ also defines a Lattès map for which we have $F_{\frac{1}{2}\lambda} \cap F = F_{\lambda,\wp^2}$, therefore there are exactly $\frac{a^2+b^2}{4}$ many face centers (vertices of type $\bullet$) in $F_{\lambda,\wp^2}$. Each edge of any square is incident to exactly 2 squares, therefore, in total there are $\frac{a^2+b^2}{2}$ many edge centers in $F_{\lambda,\wp^2}$. Among the face centers, only when a' and b' are both odd $2\gcd(a, b)$ of them lies on the boundary of $F_{\lambda,\wp^2}$. Otherwise, that is when a' and b' are of mixed parity there are $2\gcd(a, b)$ many vertices of type $\circ$ on the boundary, see Figure 2. We obtain the number of vertices of type $\times$ on the boundary as $2\gcd(a, b)$. So, we must have $\frac{a^2+b^2}{4} - d + 1$-many interior lattice points of type $\times$.
In the case where both a and b are odd we first note that the boundary lattice points can only be of type $\times$ and $\bullet$ and they always exist, but we cannot have any vertex of type $\circ$ on the boundary. We pass to 2λ, and apply the results obtained above to obtain the number of interior lattice points of $F_{2\lambda,\wp^2}$ of type $\times/\circ/\bullet$ as $a^2 + b^2 - 2d + 1/2(a^2 + b^2)/a^2 + b^2 - 2\gcd(a, b)$. The interior lattice points of $F_{2\lambda,\wp^2}$ contains 4 copies of interior lattice points of $F_{\lambda,\wp^2}$ and 4 copies of boundary lattice points. Therefore, the number of interior lattice points of $F_{\lambda,\wp^2}$ must be $\frac{a^2+b^2-2d}{4} - \frac{d-1}{2} = \frac{a^2+b^2+2}{4} - d$. In a similar fashion, the number of points of type $\bullet$(resp. $\circ$) are counted as : $\frac{a^2+b^2+2}{4} - d$ (resp. $\frac{a^2+b^2}{2}$). For the final case where the parities of a and b are opposite, one computes the points in $F_{\lambda,\wp^2}$ by comparing it with $F_{2\lambda,\wp^2}$. We leave the proof to the reader as no new arguments are involved. $\square$

The following result gives the precise ramification of Lattès maps in question :

Proposition 3.6. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map with the degree of π being 4. Then we have:*

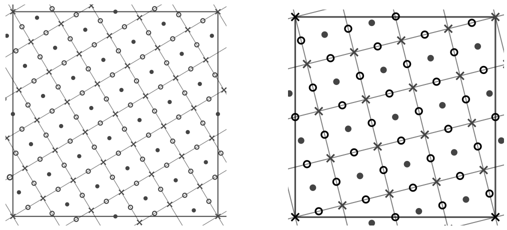

FIGURE 2. The set $F_{\lambda,\wp^2}$ when $\lambda = 6+10\sqrt{-1}$ and $\lambda = 2+8\sqrt{-1}$, respectively.

parities of a and b	$\#\phi^{-1}(\infty)$	$\#\phi^{-1}(\wp^2(1/2))$	$\#\phi^{-1}(0)$
both even	$\frac{a^2+b^2}{4}+2$	$\frac{a^2+b^2}{2}$	$\frac{a^2+b^2}{4}$
mixed	$\frac{a^2+b^2+3}{4}$	$\frac{a^2+b^2+1}{2}$	$\frac{a^2+b^2+3}{4}$
both odd	$\frac{a^2+b^2+2}{4}+1$	$\frac{a^2+b^2}{2}$	$\frac{a^2+b^2+2}{4}$

Proof. Let us treat the case where both a and b are odd and leave the remainder of the proof to the reader. We begin by noting that

$$\#\phi^{-1}(\infty) = \#\{\wp^2(z) \,|\, z \text{ is a vertex of type } \times\}.$$

By Lemma 3.5, there are $\frac{a^2+b^2+2}{4} - d$ many distinct images of vertices of type $\times$ which are in the interior. For the boundary lattice points, as a result of the proof of Lemma 3.5 the $2d$-many give rise to $2\frac{d-1}{2}+2 = d+1$ distinct values under $\wp^2$. Again, using the fact that $\#\phi^{-1}(0) = \#\{\wp^2(z) \,|\, z \text{ is a vertex of type } \bullet\}$, we find number the images inner lattice points as $\frac{a^2+b^2+2}{4} - d$. Concerning the boundary lattice points of type $\bullet$ we obtain $2\frac{d-1}{2}+1 = d$. $\square$

The norm 1 case, that is $\lambda = \pm 1$ or $\lambda = \pm\sqrt{-1}$ does not lead to any interesting Lattès map. For $\lambda = \pm 1 \pm \sqrt{-1}$ is also not interesting for our purposes as in these cases, the map ϕ is of degree 2 and by Proposition 3.6 ϕ is ramified exactly over 0 and $\wp^2(1/2)$. In all the remaining cases, ϕ has exactly 3 ramification values.

3.4. $G \cong \mathbb{Z}/6\mathbb{Z}$**.** This is possible only when the lattice is Eisenstein integers, $\mathbf{Z}[\zeta_3]$. The map π becomes $\wp^3$. Candidates for ramification points are then 0, d_1, d_2, $\omega_1/2$, $\omega_2/2$, $\omega_3/2$. We know, in this case that the zeroes d_1, d_2 of $\wp$ are distinct and satisfy $d_1 + d_2 \equiv 0 \pmod{\Omega}$. Note that these zeroes are distinct from the half periods $\omega_i/2$, for $i = 1, 2, 3$. In fact, an elementary computation shows that in this case $d_1 = \omega_3/3$ and therefore $d_2 = 2\omega_3/3$. We also have $\wp^3(\omega_i/2) = g_3/4$, as $g_2 = 0$. Therefore, there are only 3 candidates for ramification values, namely $\wp^3(d_i) = 0$, $\wp^3(\omega_i/2) = g_3/4$ and $\wp^3(0) = \infty$.

As the map $\wp^3$ is invariant under the action of $\mathbf{Z}/6\mathbf{Z}$, which acts as multiplication by $\exp(2\pi\sqrt{-1})/6$ on $\mathbf{C}/\mathbf{Z}[\zeta_3]$, the set $F_{\lambda,\wp^3}$ becomes the finite closed region whose boundary contains the lines $\arg(z) = 0$, $\arg(z) = \pi/3$, $\mathrm{Re}(z) = 1/2$ and $\arg(z-1) = 2\pi/3$, see Figure 3.

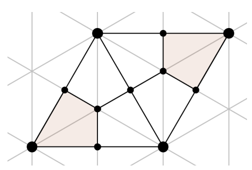

FIGURE 3. Shaded region is the fundamental region of $\wp^3$.

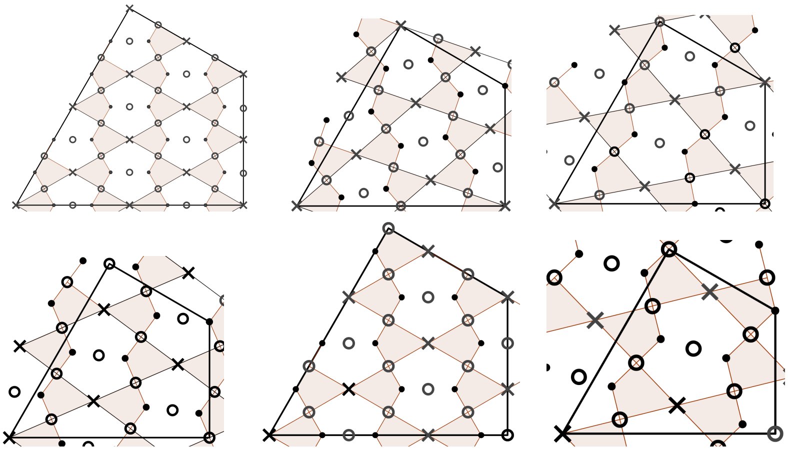

FIGURE 4. The set $F_{\lambda,\wp^3}$ when $\lambda = 4+4\zeta_6$, $4+2\zeta_6$, $1+4\zeta_6$, $2+3\zeta_6$, $3+3\zeta_6$, $1+3\zeta_6$, respectively.

One may follow a line of reasoning similar to the $\mathbf{Z}/4\mathbf{Z}$ case. More precisely, vertices of the fundamental region of $\wp^3$ is determined by values of a and $b \pmod 6$. (PR2) does not have an analogue in this case, see $\lambda = 1+4\zeta_6$ case in Figure 4. (PR3) and (PR4) have analogues with small modifications. All these results lead to a Lemma 3.5 type statement. Consequently, we obtain :

Proposition 3.7. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map with the degree of π being 6. Then we have:*

parities of a and b	$\#\phi^{-1}(\infty)$	$\#\phi^{-1}(\wp^3(\omega_i/2))$	$\#\phi^{-1}(0)$
both even and $a \equiv b \pmod 6$	$\frac{a^2+ab+b^2}{6}+2$	$\frac{a^2+ab+b^2}{2}$	$\frac{a^2+ab+b^2}{3}$
both even and $a \not\equiv b \pmod 6$	$\frac{a^2+ab+b^2+8}{6}$	$\frac{a^2+ab+b^2}{2}$	$\frac{a^2+ab+b^2+2}{3}$
mixed and $b-a \equiv 3 \pmod 6$	$\frac{a^2+ab+b^2+9}{6}$	$\frac{a^2+ab+b^2+1}{2}$	$\frac{a^2+ab+b^2}{3}$
mixed and $b-a \not\equiv 3 \pmod 6$	$\frac{a^2+ab+b^2+5}{6}$	$\frac{a^2+ab+b^2+1}{2}$	$\frac{a^2+ab+b^2+2}{3}$
both odd and $a \equiv b \pmod 6$	$\frac{a^2+ab+b^2+9}{6}$	$\frac{a^2+ab+b^2+1}{2}$	$\frac{a^2+ab+b^2}{3}$
both odd and $a \not\equiv b \pmod 6$	$\frac{a^2+ab+b^2+5}{6}$	$\frac{a^2+ab+b^2+1}{2}$	$\frac{a^2+ab+b^2+2}{3}$

The proof follows a similar reasoning as the proof of Proposition 3.7 and therefore will be omitted.

When we eliminate the norm 1 case, in all the remaining cases, the map ϕ induced by $\wp^3$ has exactly 3 ramification values.

4. EXAMPLES

Recall that a map $\phi\colon X \to \mathbb{P}^1$; where X is an algebraic curve is called a Belyi map if it is ramified at most over 3 points. As a result of Belyi's celebrated theorem such X admit a model over $\overline{\mathbf{Q}}$. Let us summarize the results obtained so far :

Theorem 4.1. *Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map. If $\pi = \wp$, then ϕ is a Belyi map if and only if $\deg(\psi) = 4$. In all the remaining cases, i.e. when $\pi = \wp'$ or $\wp^2$ or $\wp^3$, ϕ is a Belyi map.*

4.1. $G = \mathbf{Z}/2\mathbf{Z}$. As noted above, the only possibility is the case when $|\lambda| = 2$. For simplicity, let us choose ψ to be the multiplication by 2 map on $\mathbf{C}/\Omega$. Then, the corresponding Lattès map becomes :

$$\phi(x) = -2x + \frac{1}{4}\frac{(6x^2 - \frac{g_2}{2})^2}{4x^3 - g_2 x - g_3};$$

where g_2 and g_3 are invariants of the corresponding elliptic curve.

4.2. $G = \mathbf{Z}/3\mathbf{Z}$. We consider ψ to be the multiplication by n map on $\mathbf{C}/\Omega$; where $\Omega = \mathbf{Z}[\zeta_3]$. The corresponding Lattès map, denoted $\phi_{3,n}$, is ramified at most over 3 points, and hence a Belyi map. In fact, for any integer n, these Belyi Lattès maps are visualized as gluing two copies of an equilateral triangle whose edges are divided into n equal segments, see Figure 5.

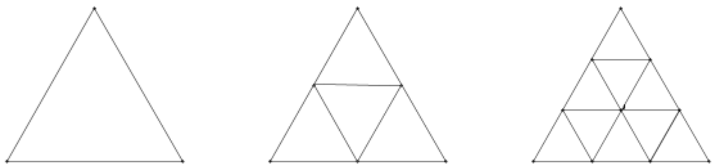

Figure 5. First three subdivisions of an equilateral triangle.

The corresponding Belyi maps can be computed, using addition formula, as:

$$\phi_2(z) = \frac{(z-1)(z+1)^3}{(2z-1)^3}$$
$$\phi_3(z) = \frac{(z^3+3z^2-6z+1)^3}{z(z-1)(z^2-z+1)^3}$$

The following short list of Belyi maps are computed in PARI, [16]. The code is available in author's web page, using which, one may compute Belyi morphisms of arbitrarily large degree.

$$\phi_{3,4}(z) = \frac{(z-2)^3 z(z^4+10z^3-12z^2+4z-2)^3}{(2z-1)^3(2z^4-4z^3+12z^2-10z-1)^3}$$
$$\phi_{3,5}(z) = \frac{(z-1)^1(z^8+17z^7-107z^6+164z^5-155z^4+164z^3-107z^2+17z+1)^3}{(5z^8-20z^7+125z^6-305z^5+275z^4-65z^3-40z^2+25z-1)^3}$$
$$\phi_{3,6}(z) = \frac{(z^3-6z^2+3z+1)^3(z^9+36z^8-99z^7+165z^6-387z^5+666z^4-564z^3+225z^2-45z+1)^3}{(z-2)^3(z-1)^1(z)^1(z+1)^3(2z-1)^3(z^2-z+1)^3(z^6-3z^5+60z^4-115z^3+60z^2-3z+1)^3}$$

We remark that each irreducible factor of the rational function $\phi_{4,n}$ is defined over a proper subfield of the $2n$-division field of the corresponding elliptic curve, which is in general not a Galois extension of $\mathbf{Q}$.

4.3. $G = \mathbf{Z}/4\mathbf{Z}$. Let us again consider the multiplication by n map on $\mathbf{C}/\Omega$; where $\Omega = \mathbf{Z}[\sqrt{-1}]$. The corresponding Belyi maps can be obtained by gluing two copies of isosceles right triangle which is tiled using squares, see Figure 6.

Addition formula of the corresponding $\wp$ help one to compute these Belyi-Lattès maps explicitly, as exemplified below.

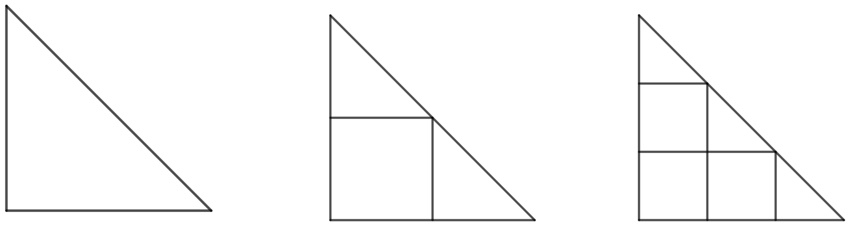

Figure 6. First three subdivisions of an isosceles right triangle.

$$\phi_{4,2}(z) = \frac{(x+1)^4}{(x-1)^2(x)^1}$$
$$\phi_{4,3}(z) = \frac{(x)^1(x^2+6x-3)^4}{(3x^2-6x-1)^4}$$
$$\phi_{4,4}(z) = \frac{(x^4+20x^3-26x^2+20x+1)^4}{(x-1)^2(x)^1(x+1)^4(x^2-6x+1)^4}$$
$$\phi_{4,5}(z) = \frac{(x)^1(x^2-2x+5)^4(x^4+52x^3-26x^2-12x+1)^4}{(5x^2-2x+1)^4(x^4-12x^3-26x^2+52x+1)^4}$$
$$\phi_{4,6}(z) = \frac{(x+1)^4(x^8+104x^7-548x^6+3032x^5-4922x^4+3032x^3-548x^2+104x+1)^4}{(x-1)^2(x)^1(x^2+6x-3)^4(3x^2-6x-1)^4(x^4-28x^3+6x^2-28x+1)^4}$$

4.4. $G = \mathbf{Z}/6\mathbf{Z}$. Let us again consider the multiplication by n map on $\mathbf{C}/\Omega$; where $\Omega = \mathbf{Z}[\zeta_3]$. The corresponding Belyi maps can be obtained by tiling an equilateral triangle by $\pi/6$-$\pi/3$-$\pi/2$ triangles. We exemplify below the corresponding Belyi-Lattès maps

$$\phi_{6,2}(z) = \frac{(x)^1(x+8)^3}{(x-1)^3}$$
$$\phi_{6,3}(z) = \frac{(x^3+96x^2+48x-64)^3}{(x-4)^6(x)^2}$$
$$\phi_{6,4}(z) = \frac{(x)^1(x+8)^3(x^4+536x^3-1344x^2+2048x-512)^3}{(x-1)^3(x^2-20x-8)^6}$$
$$\phi_{6,5}(z) = \frac{(x)^1(x^8+2080x^7+50320x^6-367040x^5+2924800x^4-3491840x^3+2805760x^2-1064960x-327680)^3}{(5x^4-380x^3-240x^2+1600x-256)^6}$$

There are many interesting relations among the irreducible factors that appear in the nominator and denominator of Belyi-Lattès maps. We leave such problems to interested reader. As mentioned before, one prefers to study these maps as they are easy to compute - relevant PARI/gp routines are available on authors web-page. For instance, the computation of Belyi-Lattès map of degree 29^2 took 5min, 35,316 ms on an 8 year old laptop of 2,4 GHz Quad-Core Intel Core i7 processor with 8GB memory. Notice that 29 is chosen in order to make sure that the nominator is an irreducible polynomial of degree 29^2.

In the remaining cases the Belyi maps are parametrized by the lattices $\Omega = \mathbf{Z}[\sqrt{-1}]$ and $\Omega = \mathbf{Z}[\zeta_3]$. The map ϕ admits then the following general form :

$$\phi(z) = \frac{\prod_{z_o \text{is a vertex of type } \bullet}(z-z_o)^{\deg(z_o)}}{\prod_{z_o \text{is a vertex of type } \times}(z-z_o)^{\deg(z_o)}}.$$

Galois action on Belyi Lattès maps. Let us end the paper with a discussion of the Galois action on Belyi Lattès maps. The advantage of Belyi Lattès maps over other suggested families (e.g. trees, regular dessins, etc.) lies in its computability, in the sense that one may explicitly write and study rigorously arbitrarily high degree Belyi Lattès maps, using the general formula given above.

Notice that for $\lambda = a + b\sqrt{-1} \in \mathbf{C}$ satisfying $\lambda\Omega \subseteq \Omega$, points of type $\bullet$, $\circ$ and $\times$ are all torsion points. For instance, when $\Omega = \mathbf{Z}[\sqrt{-1}]$ points of type $\bullet$ and $\circ$ are of order $2|\lambda|^2/\gcd(a,b)$ and points of type $\times$ are of order $|\lambda|^2/\gcd(a,b)$. To this end, we let E be the elliptic curve whose j invariant is either 0 or 1728 and let λ be an element in the endomorphism ring of E. We take $K = \mathbf{Q}(\sqrt{-1})$ when $j(E) = 0$ and $K = \mathbf{Q}(\sqrt{-3})$ when $j(E) = 1728$. We set n_λ to be the smallest positive integer so that elements of F_λ are a subset of the set of points of order n_λ on E, that is the set $E[n_\lambda]$. The following result is a direct consequence of [11, Theorem 2 (pg.126)]:

Theorem 4.2. *The field of definition of the corresponding Belyi Lattès map is a subfield of the Ray class field of K of conductor n_λ.*

The Belyi Lattès maps arising from the sub-family of multiplication by n maps gives rise to functions whose ramification set lies in a division field of $\wp$. This is in a sense the optimal case as one has $n_\lambda = n$ and $E[n] = F_\lambda$. For n sufficiently large these fields are Galois extensions of $\mathbf{Q}$. The Galois groups are $\mathrm{GL}(2, \mathbf{Z}/n\mathbf{Z})$, [15].
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