CLASS NUMBER PROBLEMS AND LANG CONJECTURES

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Abstract. Given a square-free integer d we introduce an affine hypersurface whose integer points are in one-to-one correspondence with ideal classes of the quadratic number field $\mathbf{Q}(\sqrt{d})$. Using this we relate class number problems of Gauss to Lang conjectures.

1. Introduction

A binary quadratic form is a homogeneous polynomial of degree two in two variables with integer coefficients. In Disquisitiones Arithmeticae, [14], without the terminology and tools that are in our disposal today, Gauß defined the action of the modular group, $\mathrm{PSL}_2(\mathbf{Z})$, on binary quadratic forms and showed (among many other things) that discriminant of a binary quadratic form is invariant under this action. He then defined a binary operation on the $\mathrm{PSL}_2(\mathbf{Z})$-classes of binary quadratic forms whose ratio of discriminants is a perfect square. Under this operation, the set of binary quadratic forms for which the ratio of the discriminant is a perfect square forms a group.

Let K be a number field, that is a finite extension of $\mathbf{Q}$. The set of elements of K that are roots of monic polynomials in $\mathbf{Z}[x]$ is called the ring of integers of K and denoted by $\mathcal{O}_K$. $\mathcal{O}_K$ is a ring whose properties have been exploited by the works of many including Kummer, Dirichlet, Dedekind and Weber. It turns out that unlike $\mathbf{Z}$ $\mathcal{O}_K$ fails to be a unique factorization domain. It is classical to measure this by the ideal class group, H_K, of K, defined as the quotient of fractional ideals in K by the principal ideals. The number of elements of H_K is denoted usually by h_K and whenever is equal to 1, $\mathcal{O}_K$ is a unique factorization domain.

Whenever the degree of the extension $K/\mathbf{Q}$ is 2 then K is said to be a quadratic number field. In this case, one can find a square-free integer d so that $K \cong \mathbf{Q}(\sqrt{d})$. Associating each (narrow) ideal class a binary quadratic form and vice versa gives a one to one correspondence between narrow ideal classes in K and binary quadratic forms of discriminant d.

Using this, for negative discriminants[1] (i.e. $d < 0$) Gauß gave a list[2] comprising of 9 imaginary quadratic number fields with class number one which he believed to be complete. This fact was later proved by Heegner[3], [19], Baker, [1], Siegel, [29] and Stark, [31]. There are only 18 imaginary quadratic number fields with class number 2 as shown by Baker, [2], and by Stark, [32, 33]. As far as author is aware,

2010 *Mathematics Subject Classification.* Primary 11E16 11E41; Secondary 11R29.

Key words and phrases. subgroups of the modular group, binary quadratic forms, class number problems, çark hypersurfaces, rational points on hypersurfaces, complex hyperbolic manifolds, Lang conjectures, Kobayashi hyperbolicity.

[1]The discriminant of $\mathbf{Q}(\sqrt{d})$ is $4d$ or d depending on the class of d in $\mathbf{Z}/4\mathbf{Z}$

[2]$\mathbf{Q}(\sqrt{-d})$; where $d = -1, -2, -3, -7, -11, -19 - 43, -67, -163$.

[3]Heegner's proof had some gaps, which are filled by Deuring, [10].

class number problem for imaginary quadratic number fields is known up to 100, [39].

For positive discriminants, i.e. for real quadratic number fields, class number problems (i.e. finding the class number of $\mathbf{Q}(\sqrt{d})$) has a completely different nature, of which Gauß was aware. He conjectured the existence of infinitely many positive integer d so that the class number of $\mathbf{Q}(\sqrt{d})$ is 1. In fact, for any positive integer N, the question of whether the number of quadratic number fields with class number N is finite or not is open. Among others, the existence of $\log \varepsilon$, where ε is a fundamental unit of $\mathbf{Q}(\sqrt{d})$, as a term in the analytic class number formula is a difficulty in attacking problems concerning real quadratic number fields.

Given a complex space X the two intrinsic pseudo-metrics on X, namely Kobayashi and Carathéodory, are in a sense dual to each other[4]. These pseudo-metrics are distance decreasing under holomorphic maps and behave well under topological coverings. This property reflects the geometric nature of the two pseudo-metrics and indeed it is possible to argue that Kobayashi pseudo-metric has its roots in the uniformization theorem and Schwarz Lemma (and its generalizations), [22].

A complex space is called *Kobayashi hyperbolic* if the Kobayashi pseudo-distance is a metric. For instance, $\mathbf{C}$ is not Kobayashi hyperbolic whereas the unit disk in $\mathbf{C}$, denoted $\mathbb{D}$, is. This implies that algebraic curves of genus $g \geq 2$ are all Kobayashi hyperbolic whereas elliptic curves are not. As comes to algebraic varieties, it turns out that hyperbolicity is closely related to the degree of the algebraic variety in question. In fact, Kobayashi conjectures that if the degree of a hypersurface, X, in $\mathbf{P}^{n+1}$ is at least $2n+2$ then X is Kobayashi hyperbolic. In [8] Demailly has claimed a proof of the Kobayashi conjecture when the hypersurface is *very general*, and in [30] Siu proved the statement for hypersurfaces of sufficiently high degree. In [24] Lang conjectures that Kobayashi hyperbolic varieties should have at most finitely many rational points. The case of algebraic curves provides a beautiful set of examples.

In this work, we present a completely different approach to class number problems for real quadratic number fields. The main tool, called çark, is an infinite bipartite ribbon graph embedded in a conformal annulus, see Figure 1 for an example. A çark is an infinite version of a dessin d'enfant, [18]. It has a unique cycle of finite length, called its spine. The fact that relates hyperbolicity to class number problems is that there is a one to one correspondence between the set of $\mathrm{PSL}_2(\mathbf{Z})$-classes of binary quadratic forms positive discriminant and çarks, [35]. In a work in progress, [41], it is shown that for a given square-free positive integer d indefinite binary quadratic forms whose discriminant's square-free part is equal to d can be realized as integral points on an affine hypersurface, $\mathcal{C}_d$, to which we will refer as *çark-hypersurface*. The hypersurface $\mathcal{C}_d$ admits an action of a certain abelian group which depends only on the integer d. By projectivizing $\mathcal{C}_d$ one obtains a one to one correspondence between certain ideal classes and integral points on the projective çark-hypersurface. The degree of a çark hypersurface $\mathcal{C}_d$ is exactly equal to twice the dimension of its ambient space, and hence by the result of Demailly, see Theorem 3.5, it is Kobayashi hyperbolic, and by Lang these surfaces has finitely many rational points and thus finitely many integral points.

[4]Reader interested in intrinsic metrics on complex spaces may consult the article by Kobayashi, [20]

The paper is organized as follows: In the next section we will outline basic facts and correspondences mentioned in the introduction, namely those between binary quadratic forms, narrow ideal classes and çarks. In Section 3, we provide necessary information to state Lang's conjecture relating geometry to arithmetic. More precisely, we introduce the Kobayashi pseudo-metric, investigate the hyperbolicity of algebraic curves and then review results about the hyperbolicity of hypersurfaces of high degree. In the final section we will define çark-hypersurfaces and the abelian group in question. We conclude by giving explicit explicit equations for small d.

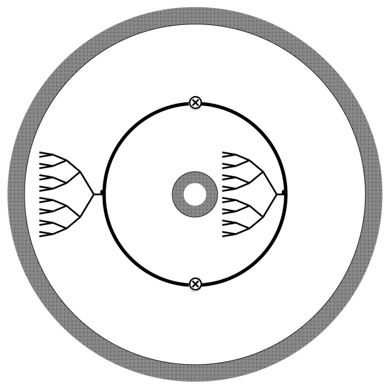


FIGURE 1. A çark

2. Çarks, Binary Quadratic Forms and Class Groups

This section is devoted to introducing foundational material on çarks, quadratic number fields and binary quadratic forms. We refer to [15] and [23] for precise definitions and facts concerning finite graphs on surfaces, dessins and related issues, and to [40, §10] and [4] for statements of theorems related to quadratic number fields and binary quadratic forms.

2.1. **Çarks.** Let us fix the generators of the modular group, $\mathrm{PSL}_2(\mathbf{Z})$, as:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

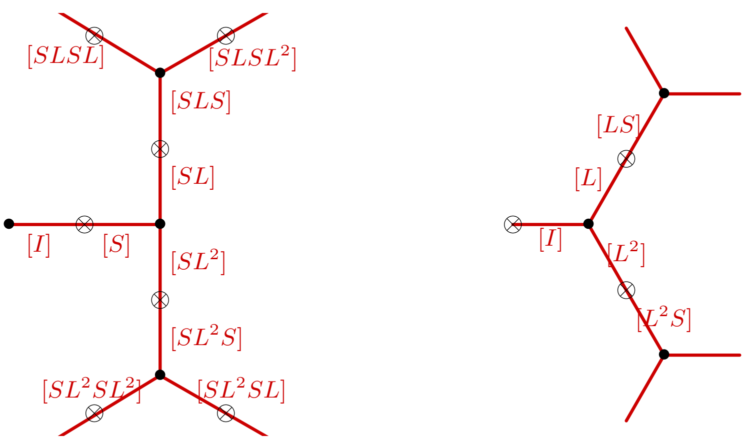
S is of order 2 and L is of order 3 and these elements generate the modular group freely, i.e. the modular group is isomorphic to $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. One can define edges of *bipartite Farey tree*, denoted $\mathcal{F}$, as the set of elements of $\mathrm{PSL}_2(\mathbf{Z})$ and the set of vertices as the orbits of the subgroups $\langle S\rangle$ and $\langle L\rangle$. The incidence relation is defined as intersection, namely there is an edge between two vertices if and only if the intersection is non-empty. As there are no relations between S and L, there are no loops. By construction $\mathcal{F}$ is a tree and bipartite. Vertices of degree 2, which are orbits of $\langle S\rangle$, are denoted by $\otimes$ and vertices of degree 3, which are orbits of $\langle L\rangle$, are denoted by $\bullet$.

For an element $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{Z})$ the map $M \cdot z := \frac{pz+q}{rz+s}$ defines an action of the modular group acts on the upper half plane, $\mathbb{H} := \{z \in \mathbf{C} \colon \mathrm{im}(()z) > 0\}$. In particular, S maps each z to $-1/z$ and L maps z to $1 - 1/z$. If we denote the fixed point of S by $\otimes$ and the fixed point of L by $\bullet$ then the orbit of the geodesic, γ, in $\mathbb{H}$, connecting $\otimes$ to $\bullet$ then the topological realization of $\mathcal{F}$ is the orbit of γ under the mentioned action. Note that $\mathcal{F}$ is a ribbon graph on which the orientation around each vertex of degree 3 is induced by the ordering of $(1, L, L^2)$. The associated surface of $\mathcal{F}$ is homeomorphic to $\mathbb{H}$. For details we refer to [35].

One can apply the same construction to any subgroup, Γ, of the modular group using cosets of the subgroup in question and obtain a bipartite ribbon graph, $\Gamma\backslash\mathcal{F}$. The genus, puncture(s) and boundary component(s) are determined completely by the graph $\Gamma\backslash\mathcal{F}$.

If Γ is of finite index then $\Gamma\backslash\mathcal{F}$ is finite and is called a dessin d'enfant, [18]. In particular, there are no boundary components of the surface $\Gamma\backslash\mathbb{H}$ apart from the punctures. After "filling" the punctures, one obtains a compact Riemann surface or equivalently an algebraic curve, in a unique fashion. This curve admits equations whose coefficients are from a number field as a result of a well-known theorem of Belyi, [3]. Conjugate subgroups give rise to isomorphic ribbon graphs. The graphs corresponding to two conjugate subgroups differ only in their base edges. Indeed, given $\Gamma = \langle\gamma\rangle$, the corresponding graph has a distinguished edge, namely the edge which is represented by the coset $I \cdot \Gamma$. For the subgroup $M^{-1}\langle\gamma\rangle M$, the corresponding base edge is the edge $M \cdot \Gamma$. However, the same algebraic curve admit infinitely many dessins, a situation which depends mostly on the graph, as investigated in [16].

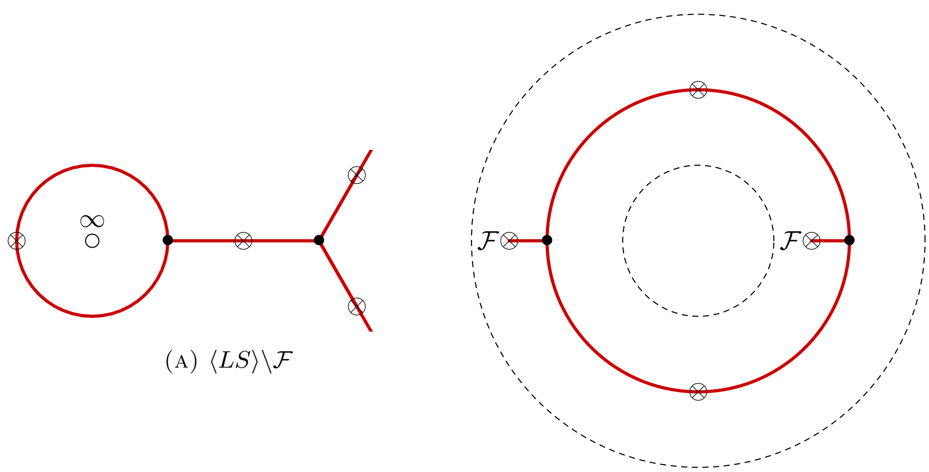
The absolute Galois group acts on the set of algebraic curves defined over $\overline{\mathbf{Q}}$ and thus to finite bipartite ribbon graphs. This action is known to be faithful in genus 0, [28]. It is also faithful in genus one. Indeed, if E is an elliptic curve defined over $\overline{\mathbf{Q}}$ with j-invariant $j(E)$, then $j(E)$ is also algebraic. Choose a σ in the absolute Galois group which does not fix $j(E)$. Then as the dessin corresponding to E is defined over an extension of $\mathbf{Q}(j(E))$, σ cannot fix the dessin.



(A) by L. (B) by S.

FIGURE 2. Elliptic quotients of bipartite Farey tree.

If one leaves aside the action of the absolute Galois group the above construction associating a subgroup a bipartite ribbon graph can still be considered, even for subgroups of infinite index. In particular, if we consider Γ a subgroup generated by only one element, denoted $\gamma \in \mathrm{PSL}_2(\mathbf{Z})$ then the typical properties of the resulting graph is determined by the classification of elements of $\mathrm{PSL}_2(\mathbf{Z})$. Recall that elements of $\mathrm{PSL}_2(\mathbf{Z})$ are divided into three types according to the absolute value of their traces(absolute trace, in short). Namely, elements whose absolute trace is less than 2 are called *elliptic*, exactly 2 are called *parabolic* and greater than two are called *hyperbolic*. As such groups are of infinite index, the corresponding surfaces associated to these ribbon graphs has boundary. If γ is elliptic the graph $\Gamma\backslash\mathcal{F}$ is a rooted tree, see Firgure 2. The Riemann surface is a disk with an orbifold point at its *center*, of stabilizer of order 2 or 3 depending on the order of γ. If γ is parabolic or hyperbolic then the graph $\Gamma\backslash\mathcal{F}$ is not a tree. It has one loop to which we will refer as the *spine* of $\Gamma\backslash\mathcal{F}$. The associated Riemann surface has two boundary components in either case. If γ is parabolic, then there is on puncture and on boundary component homeomorphic to circle, S^1. If γ is hyperbolic, then the boundary has two components each homeomorphic to S^1. We refer to [34] for details. In the elliptic case there is a unique bipartite Farey tree attached to the root, expanding in the direction of the boundary of the disk. In parabolic and hyperbolic cases there are a finite number of bipartite Farey tree components attached to the spine from the degree 3 vertices pointing outward to the boundary components, see Figure 3a and Figure 3b.



(A) $\langle LS\rangle\backslash\mathcal{F}$

(B) $\langle LSL^2S\rangle\backslash\mathcal{F}$

FIGURE 3. Examples of parabolic and hyperbolic subgroups.

Definition 2.1. For a subgroup, Γ, of the modular group generated by one *hyperbolic* element the ribbon graph $\Gamma\backslash\mathcal{F}$ is called a *çark*. If we mark the base edge of the çark then it is called a *base pointed* çark.

2.2. **Binary quadratic forms.** An element $f(X,Y) \in \mathbf{Z}[X,Y]$ which is homogeneous of degree two will be referred to as a binary quadratic form. We also use the matrix form

$$(X\,Y)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}(X\ Y)^t;$$

where a, b and c are integers and write $f = (a, b, c)$ for short. The matrix $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ is denoted by M_f. The discriminant of a binary quadratic form is defined as $\Delta(f) = b^2 - 4ac$. If $\Delta(f)$ is negative and $a > 0$(resp. $a < 0$) then f is called positive(resp. negative) definite. Forms with positive discriminant are called indefinite. We will not consider forms with 0 discriminant.

Modular group acts on the set of all binary quadratic forms as:

$$W \cdot M_f \mapsto W^t\, M_f\, W$$

The orbit under $\mathrm{PSL}_2(\mathbf{Z})$ of a binary quadratic form f is called its class and denoted by $[f]$. The discriminant of a form is invariant under this action. The stabilizer of a form under this action is called its automorphism group, which can be determined explicitly:

Theorem 2.1 ([4,]). *Let f be a binary quadratic form of non-zero discriminant. Then $Aut(f)$ is isomorphic to*

$\mathbf{Z}/4\mathbf{Z}$ *if* $\Delta(f) = -4$
$\mathbf{Z}/6\mathbf{Z}$ *if* $\Delta(f) = -3$
$\mathbf{Z}$ *if* $\Delta(f) > 0$

and trivial otherwise.

2.3. **Narrow Ideal classes in quadratic number fields.** Let K be a quadratic number field. Then there is an integer d so that $K = \mathbf{Q}(\sqrt{d})$. The discriminant Δ of $K = \mathbf{Q}(\sqrt{d})$ is then equal to $\Delta = 4d$ whenever $d \equiv 2, 3$ modulo 4 and $\Delta = d$ whenever $d \equiv 1$ modulo 4. We assume d to be square-free. Let $\mathcal{O}_d$ denote the ring of integers of this number field, i.e. the set of elements of K which are roots of monic polynomials with integer coefficients. $\mathcal{O}_d$ is isomorphic to $\mathbf{Z} + \sqrt{d}\,\mathbf{Z}$ if d is congruent to 2 or 3 modulo 4 and is isomorphic to $\mathbf{Z} + \frac{1+\sqrt{d}}{2}\,\mathbf{Z}$ if d is congruent to 1 modulo 4. A fractional ideal, $\mathfrak{a}$, of K is a finitely generated subgroup of K which is closed under multiplication by elements of $\mathcal{O}_d$, i.e. for which $\lambda \in \mathcal{O}_d$ and $a \in \mathfrak{a}$ implies $\lambda \cdot a \in \mathfrak{a}$. For any fractional ideal there is a non-unique natural number $n = n_{\mathfrak{a}}$ so that $n \cdot \mathfrak{a} \subset \mathcal{O}_d$. The norm, $N(\mathfrak{a})$ of a fractional ideal, $\mathfrak{a}$, is defined as $\frac{1}{n^2} N(n_{\mathfrak{a}}\mathfrak{a})$; where the norm of an ideal of $\mathcal{O}_d$ is defined as its index in $\mathcal{O}_d$. Norm of any fractional ideal is independent of the choice of $n_{\mathfrak{a}}$. Recall that ring of integers of a number field is a Dedekind domain, hence every ideal is generated by at most 2 elements. As a result, any fractional ideal is a two dimensional $\mathbf{Z}$-module, i.e. for any fractional ideal $\mathfrak{a}$ there are elements α and β in K so that $\mathfrak{a} = \alpha\mathbf{Z} + \beta\mathbf{Z}$. The discriminant of $\mathfrak{a}$ is then defined as the square of the determinant of the matrix $\begin{pmatrix} \alpha & \beta \\ \overline{\alpha} & \overline{\beta} \end{pmatrix}$.

The product of two fractional ideals is again a fractional ideal, and in particular, the product of the ideal (1) with any other fractional ideal $\mathfrak{a}$ is equal to $\mathfrak{a}$. The inverse of a fractional ideal $\mathfrak{a}$ is given by $\frac{1}{N(\mathfrak{a})}\mathfrak{a}$. So the set of fractional ideals is an abelian group, denoted $I(d)$. The subset of principal ideals in $I(d)$ forms a subgroup of $I(d)$, denoted by $P(d)$. The quotient $H(d) := I(d)/P(d)$ is called the *class group* of $\mathbf{Q}(\sqrt{d})$ (or $\mathcal{O}_d$). Every element in this group will be referred to as an ideal class. The set $P^+(d) := \{(\xi) \in P(d) \colon N((\xi)) > 0\}$ is also subgroup of $I(d)$ contained in $P(d)$. The cardinality of the quotient $P(d)/P^+(d)$ is two whenever $\mathcal{O}_d$ contains a unit of norm -1, and is one whenever $\mathcal{O}_d$ does not have a unit of norm

-1. In particular, if $d<0$, i.e. if $\mathbf{Q}(\sqrt{d})$ is imaginary quadratic, then $\mathcal{O}_d$ cannot contain such a unit. The group $H^+(d) := I(d)/P^+(d)$ is called the *narrow class group* of $\mathcal{O}_d$. Every element in this group will be referred to as a *narrow ideal class.*

2.4. **Çarks and Indefinite Binary Quadratic Forms.** For a given çark on traversing the spine of the çark in counterclockwise direction, one meets successive Farey branches expanding in the same (inner or outer) boundary component. Each such successive Farey branch will be referred to as a Farey bunch. We introduce the following notation: a Farey bunch of size n (i.e. a bunch containing n successive Farey branches) pointing in the inner boundary circle is denoted by a $+n$ and similarly a Farey bunch of size n pointing in the outer boundary circle is denoted by a $-n$. For instance, çark appeared in Figure 3b will be denoted by $[+1,-1]$.

Given a base pointed çark, Ç, starting from the base edge one goes in the direction of the spine, traverses once around the spine in the counterclockwise direction then comes back to the same edge. In doing so one records the following word in $\mathrm{PSL}_2(\mathbf{Z})$: Once we arrive at the spine, on visiting a Farey bunch of size n, if the bunch expands in the direction of outer boundary then we write $(LS)^n$ and similarly on visiting a Farey bunch of size n expanding in the direction of inner boundary then we write $(L^2S)^n$ and outside the spine, on visiting a vertex of degree 3 we write LS if the turn is to left and write L^2S if the turn is to right. After completing the full turn, we obtain a word in S, L and L^2 which is an element of $\mathrm{PSL}_2(\mathbf{Z})$, say $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then we define the corresponding binary quadratic form as

$$f_{\text{Ç}}(X,Y) = \frac{r}{\delta}X^2 + \frac{s-p}{\delta}XY + \frac{-q}{\delta}Y^2;$$

where $\delta = \gcd(c, d-a, b)$.

In conclusion, we have:

Theorem 2.2 ([35, Corollary 3.2])**.** *There is a one to one correspondence between base pointed çarks and indefinite binary quadratic forms.*

Remark first that, W is an automorphism of the corresponding binary quadratic form. The correspondence in Theorem 2.2 can be stated using the language of subgroups of the modular group. More precisely, there is a one-to-one correspondence between conjugacy classes of subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ generated by one hyperbolic element and çarks. Moreover, the conjugation action of $\mathrm{PSL}_2(\mathbf{Z})$ on itself amounts to translation of the base edge in the language of çarks.

Conversely, given an indefinite binary quadratic form, f, the group $\mathrm{Aut}(f)$ is a one parameter subgroup of the modular group generated by one hyperbolic element, see Theorem 2.1. The generator of this subgroup, call W, can be written as a word in S and L. Among all elements that are conjugate to W, choose word of minimal length[5]. This word is of even length and can be written as $W = (LS)^{n_1}(L^2S)^{m_1}\cdot \ldots \cdot (LS)^{n_k}(L^2S)^{m_k}$; where none of $n_i, m_j,\ i,j \in \{1,2,\ldots,k\}$ is zero. Then the corresponding çark is the one given by $(n_1,-m_1,n_2,-m_2,\ldots,n_k,-m_k)$. Not having chosen a base edge results in an action of cyclic group of order $2k$ on such tuples. We denote the corresponding class as $[n_1,-m_1,n_2,-m_2,\ldots,n_k,-m_k]$. In this case, the çark is said to have *length* $2k$.

[5]The length of an element is defined as the number of letters it has. An element of minimal length exists because the set of lengths is a subset of $\mathbf{N}$.

2.5. **Indefinite Binary Quadratic Forms and Narrow Ideal Classes.** Given a binary quadratic form $f = (a, b, c)$ of fundamental discriminant, the class containing the fractional ideal $1 \cdot \mathbf{Z} + \omega \cdot \mathbf{Z})$; where $\omega = \frac{b+\sqrt{\Delta(f)}}{2a}$ is associated to f. This map respects the action of the modular group on binary quadratic forms, i.e. equivalent forms are mapped onto equivalent narrow ideal classes.

Conversely, for a fractional ideal $\mathfrak{a}$ in a real quadratic number field, we consider the function $f_{\mathfrak{a}} : \mathfrak{a} \longrightarrow \mathbf{Z}$ sending each element $a \in \mathfrak{a}$ to $\frac{a\overline{a}}{N(a)}$. If one chooses a $\mathbf{Z}$-basis α, β for $\mathfrak{a}$ and write $a = X\alpha + Y\beta$, then $f_{\mathfrak{a}} = (\frac{\alpha\,\overline{\alpha}}{N(\mathfrak{a})}, \frac{\alpha\,\overline{\beta}+\overline{\alpha}\,\beta}{N(\mathfrak{a})}, \frac{\beta\,\overline{\beta}}{N(\mathfrak{a})})$ is a binary quadratic form. The discriminant of this form is equal to the discriminant of the corresponding real quadratic number field.

However, the binary quadratic form depends not only on the choice of basis, $\mathfrak{a} = (\alpha, \beta)$, but also on the ordering of the generating elements, for instance if for the basis (α, β) one gets $f = (a, b, c)$ then for the basis (β, α) one gets the form $f' = (c, b, a)$. For some forms this is indeed the case f is equivalent to f', for example $f = (2, 8, -5)$ is equivalent to $(-5, 8, 2)$. Such forms are called ambiguous by Gauß. There are however forms, $f = (a, b, c)$ which are not equivalent under the action of $\mathrm{PSL}_2(\mathbf{Z})$ to $f' = (c, b, a)$. The equivalence of f and f' is equivalent to the existence of a unit of negative norm, i.e. a unit of norm -1, in the group $\mathcal{O}_d^{\times}$. This is true if and only if the equation $X^2 - dY^2 = -1$ has integral solution. For instance, the form $f = (2, 8, -5)$ is of discriminant 104, i.e. it represents an element in $H(26)$. The integer $5 + \sqrt{26}$ is of norm -1, and the pair $(5, 1)$ is a solution of the equation $X^2 - 26Y^2 = -1$. In terms of çarks this translates as the corresponding çark being symmetric with respect to its spine, i.e. using the notation introduced in (2.4) this amounts to

$$\begin{aligned}[n_1, -m_1, n_2, -m_2, \ldots, n_k, -m_k] &= -[n_1, -m_1, n_2, -m_2, \ldots, n_k, -m_k] \\ &:= [-n_1, m_1, -n_2, m_2, \ldots, -n_k, m_k].\end{aligned}$$

2.6. **Narrow Ideal Classes and Çarks.** The modular group acts on the narrow class group of any quadratic number field $K = \mathbf{Q}(\sqrt{d})$ of fundamental discriminant in the following manner:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot (\alpha, \beta) := (p\alpha + r\beta, q\alpha + s\beta).$$

As mentioned in the previous section, even though $(\alpha, \beta) = (\beta, \alpha)$ the corresponding binary quadratic forms may not be equivalent. To remedy this situation, one identifies the set of narrow ideal classes with the set of ordered pairs (α, β) having positive norm, i.e. those pairs (α, β) satisfying $\overline{\alpha}\beta - \alpha\overline{\beta} > 0$.

Example 2.1. For some real quadratic number fields, e.g. 2, 5, 10, 13, etc., the orbit of a fractional ideal (α, β) is equal to the orbit of (β, α). On the other hand there are cases where the orbits of (α, β) and (β, α) are disjoint, e.g. 3, 6, 7, etc.

In fact, a characterization of this is provided by the continued fraction expansion of $\sqrt{d}$.

Theorem 2.3 ([27])**.** *The two ideal classes (α, β) and (β, α) are equivalent under the action of the modular group if and only if the length of periodic part of the continued fraction expansion of $\sqrt{d}$ is odd.*

Under the mentioned action, a narrow ideal class determined by $\mathfrak{a} = (\alpha, \beta)$ has stabilizer, Stab($\mathfrak{a}$). As a result of the considerations on the automorphisms of binary quadratic forms, this group is known to be isomorphic to $\mathbf{Z}$ and has one generator, call $W_{\mathfrak{a}}$. Then writing $W_{\mathfrak{a}}$ in terms of the generators S and L of the modular group and choosing the word $W_{\mathfrak{a},o}$ in the conjugacy class of $W_{\mathfrak{a}}$ which has the shortest length and writing $W_{\mathfrak{a},o}$ in terms of the generators gives the corresponding çark.

Conversely, to a çark given as $\text{Ç} = [n_1, -m_1, \dots, n_k, -m_k]$ we associate the element $W_{\text{Ç}} = (LS)^{n_1}(L^2S)^{m_1}\dots(LS)^{n_k}(L^2S)^{m_k} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ in $\mathrm{PSL}_2(\mathbf{Z})$. This element of the modular group is then mapped to the narrow ideal class, $I_{\text{Ç}}$, containing the two dimensional $\mathbf{Z}$ module $1\cdot\mathbf{Z}+\omega\cdot\mathbf{Z}$ where $\omega = \dfrac{p-s+\sqrt{\mathrm{Tr}(W_{\text{Ç}})-4}}{2r}$. It is merely a result of the construction that the composition of two maps is identity and hence we get a one to one correspondence.

3. Lang conjectures

Our aim in this section is to explain some terminology around in order to be able to state Lang conjectures which are relevant to our approach. For a comprehensive treatment of the subject we invite the reader to consult [7], and to [38].

We first define the Kobayashi pseudo-distance and list some geometric properties. We will then mention on of the conjectures of Lang on the arithmetic properties of Kobayashi hyperbolic algebraic varieties. We will exemplify the situation in the case of curves and then state results concerning hyperbolicity of projective hypersurfaces.

3.1. **The Kobayashi Pesudo-distance.** By $\mathrm{Hol}(X, Y)$ let us denote the set of all holomorphic maps from a complex space X to a complex space Y. In particular, an element of the set $\mathrm{Hol}(\mathbb{D}, X)$ will be referred to as a holomorphic disk in X; where $\mathbb{D}$ denotes the unit disk in $\mathbf{C}$. A holomorphic chain in X joining $x \in X$ to $y \in X$ is a finite set of holomorphic disks, $\{f_i : \mathbb{D} \longrightarrow X\}$ in X together with a finite set of distinguished points, $\{(a_i, b_i)\}$, of $\mathbb{D} \times \mathbb{D}$ satisfying

- $f_1(a_1) = x$ and $f_n(b_n) = y$
- $f_i(b_i) = f_{i+1}(a_{i+1})$ for all $i = 1, 2, \dots, n-1$

A chain will be denoted by $\alpha = ((f_1, a_1, b_1), \dots, (f_n, a_n, b_n))$, see Figure 4.

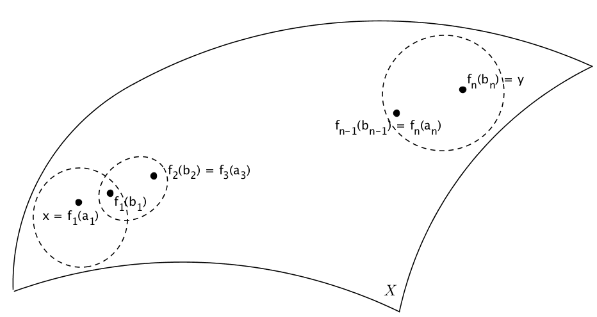


FIGURE 4. A holomorphic chain in X.

We equip $\mathbb{D}$ with the Poincaré metric, which is given in the form $\frac{4}{(1-|z|^2)^2}\mathrm{d}z\mathrm{d}\overline{z}$ as a metric tensor, and denote it by ρ. Then the length of a holomorphic chain α is defined as

$$\ell(\alpha) = \sum_{i=1}^{n} \rho(a_i, b_i).$$

Definition 3.1. Given a complex space X, the Kobayashi pseudo-distance, denoted d_X, on X between any two points $x, y \in X$ is:

$$d_X(x, y) := \inf\{\ell(\alpha)\};$$

where the infimum is taken over all holomorphic chains α joining x to y.

Any holomorphic map $\varphi\colon X \longrightarrow Y$ between two complex spaces is distance decreasing, that is $d_X(x, y) \geq d_Y(\varphi(x), \varphi(y))$ as any holomorphic chain joining x to y in X can be used to obtain a holomorphic chain joining $\varphi(x)$ to $\varphi(y)$ in Y. In particular, d_X is invariant under the holomorphic automorphisms of X. However it is not a birational invariant, see the paragraph after Definition 3.2.

Example 3.1. Let $x, y \in \mathbf{C}$ be two distinct complex numbers. Composing a translation and a rotation, we can map x to 0 and y to $\alpha \in \mathbf{R}_{>0} \subset \mathbf{C}$. For any integer $n > 1$, the map $\varphi_n : \mathbb{D} \longrightarrow \mathbf{C}$ defined as $\varphi_n(z) = \alpha\, n\, z$ gives a holomorphic disk in $\mathbf{C}$ satisfying $\varphi(0) = 0$ and $\varphi(1/n) = \alpha$. Recalling the fact that the Poincaré distance in the unit disk between origin and a point of Euclidean distance r to the origin is given by the formula $\log(\frac{1+r}{1-r})$, we get that $d_{\mathbf{C}}(x, y) = 0$. That is, the Poincaré pseudo-distance vanishes on $\mathbf{C}$.

Example 3.2. For $x, y \in \mathbb{D}$, the unit disk in $\mathbf{C}$ since $d_{\mathbb{D}}(x, y) \leq \inf_{\alpha, |\alpha|=1} \ell(\alpha)$; where the infimum is taken over all holomorphic disks in $\mathbb{D}$ containing only one map $f : \mathbb{D} \longrightarrow \mathbb{D}$. Taking f as the identity map gives us the fact that the distance between x and y is bounded above by the Poincaré distance $\rho(x, y)$. Conversely, for any α a holomorphic chain in $\mathbb{D}$,

$$\begin{aligned} \ell(\alpha) &= \sum_{i=1}^{n} \rho(a_i, b_i) \\ &\geq \sum_{i=1}^{n} \rho(f_i(a_i), f_i(b_i)) \end{aligned}$$

where the last inequality is provided by the Schwarz lemma. We conclude by triangle inequality that $\ell(\alpha) \geq \rho(x, y)$ for any x and y, and thus $d_{\mathbb{D}}(x, y) = \rho(x, y)$.

Definition 3.2. We say that a complex space is Kobayashi hyperbolic whenever its Kobayashi distance is non-degenerate.

For instance, $\mathbb{D}$ is Kobayashi hyperbolic, by Example 3.2 and therefore the Kobayashi distance restricted to $\varphi(\mathbb{D}) \subset X$ of any non-constant holomorphic map $\varphi\colon \mathbb{D} \longrightarrow X$ is non-degenerate. On the other hand, if X is a complex space admitting a non-constant holomorphic map $\varphi\colon \mathbf{C} \longrightarrow X$, then the Kobayashi pseudo-distance restricted to $\varphi(\mathbf{C})$ vanishes. In particular, $\mathbf{P}^1$ is not Kobayashi hyperbolic, which in turn implies that hyperbolicity is not a birational invariant.

In fact, this provides a criterion for Kobayashi hyperbolicity. Namely:

Theorem 3.1 (Brody's Theorem, [25, Theorem 2.1]). *Let X be a compact complex space. Then X is Kobayashi hyperbolic if and only if X does not admit any non-constant holomorphic map from $\mathbf{C}$ whenever X is compact.*

In addition, Kobayashi hyperbolicity behaves as expected under topological coverings, that is:

Theorem 3.2 ([20, Theorem 2.5]). *Let $\widetilde{X}$ be a hyperbolic complex space and $\pi\colon \widetilde{X} \longrightarrow X$ a topological covering (universal or not) of X. X is hyperbolic if and only if $\widetilde{X}$ is hyperbolic.*

Proof. Let α be a chain connecting two points x and y in X. Each map f_i can be lifted to produce a holomorphic chain in $\widetilde{X}$ and hence has positive length whenever $x \neq y$. Conversely, if $\widetilde{x}$ and $\widetilde{y}$ are two points joined by a holomorphic chain $\widetilde{\alpha}$, composing with π gives a holomorphic chain in X and hence the length is positive, unless $x = y$. □

3.2. **Hyperbolicity and Curvature of Differential Metrics.** The answer to the question whether the Kobayashi pseudo-metric can be interpreted as a differential metric tensor turns out to be negative. Nevertheless, one can define Kobayashi pseudo-metric using infinitesimal norms. One this is done, it is possible to define lengths of curves and passing to infimums endows the space in question with the Kobayashi pseudo-metric. In general, however, the Kobayashi pseudo-metric is smooth (may not even be continuous), yet there is a result due to Lempert, [26], guaranteeing the smoothness of the Kobayashi metric in case when X is a strongly convex open connected subset in $\mathbf{C}^n$.

Nevertheless, this point of view provides a passage from the Kobayashi metric to smooth metrics. That is, one can relate the non-vanishing of the Kobayashi pseudo-metric on a complex space X to the existence of negatively curved metrics on X:

Theorem 3.3 ([21, Theorem 4.11]). *If X is a complex manifold admitting a metric of holomorphic sectional curvature bounded by a negative constant, then X is Kobayashi hyperbolic.*

Describing metrics of prescribed curvature on complex manifolds is equivalent to solving the complex Monge-Ampére equation on the manifold which is in general very complicated.

In [24], Lang proposes a series of conjectures relating arithmetic properties (mordellicity) of an algebraic variety to

- algebro-geometric(all subvarieties (including the variety X itself) being of general type),
- complex analytic(every map from an abelian variety or $\mathbf{C}$ or $\mathbf{P}^1$ to X being constant), and
- differential geometric(metric of negatively bounded curvature)

properties. The relation between these different categories are provided by the hyperbolicity of X. Of the many conjectures around these topics, for purposes of this work, the relevant conjecture of Lang is the following:

Conjecture 3.1 ([24]). if an algebraic variety is hyperbolic then it is mordellic[6].

[6]Recall that an algebraic variety X is called *mordellic* if for any finitely generated field $K \subset \mathbf{C}$ the K rational points of X, denoted by $X(K)$, is not Zariski dense in $X(\mathbf{C})$

3.3. **Hyperbolicity in Dimension One.** Let X be a non-singular compact Riemann surface. The genus g of X determines both the universal covering space of X (an analytic property), the structure of $\mathrm{Hol}(\mathbf{C}, X)$ (existence of non-constant maps), the sign of the curvature of the Hermitian metric that X admits (a differential geometric property) and the mordellicity of X. We summarize this in the following table:

Genus(g)	Metric	Curvature	Universal cover	$X(\mathbf{Q})$	$\mathrm{Hol}(\mathbf{C}, X)$
$g = 0$	spherical	+1	$\mathbb{P}^1$	$\emptyset$ or infinite	there are non-const. maps
$g = 1$	flat	0	$\mathbb{C}$	finitely generated abelian group	there are non-const. maps
$g \geq 2$	Poincaré	-1	$\mathbb{H}$	finite set	only const. maps

For instance, $X = \mathbb{P}^1$ is equipped with its Fubini-Study metric, which is of constant curvature 1. Genus one surfaces are elliptic curves. The classical flat metric $ds^2 = dz \wedge d\bar{z}$ on $\mathbf{C}$ induces a flat metric on such curves whose K-rational points form a finitely generated abelian group(Mordell-Weil Theorem).

If X is of genus $g > 1$, then by the celebrated theorem of Faltings, $X(K)$ is finite. Such curves are uniformized by the upper half plane (or the unit disk), i.e. there is a discrete subgroup, Γ, of $\mathrm{PSL}_2(\mathbf{R})$ so that $X(\mathbf{C}) \cong \Gamma\backslash\mathbb{H}$. Poincaré metric on $\mathbb{H}$ induces an Hermitian metric of constant curvature -1 on X, which we also call the Poincaré metric on X. As a consequence of Liouville's theorem the only holomorphic maps from $\mathbf{C}$ to X are constant maps.

For an example of non-compact case, we consider $X = \mathbb{P}^1 - \{3pts.\} \cong \Gamma(2)\backslash\mathbb{H}$; where $\Gamma(2)$ is the congruence subgroup of the modular group of level 2. By Picard's theorem, we know that X does not admit any non-constant holomorphic map from $\mathbf{C}$. One can also write explicit metrics of bounded negative curvature on X, for instance ([17])

$$ds^2 = \left[\frac{(1+|z|^{1/3})^{1/2}}{|z|^{5/6}}\right]\left[\frac{(1+|z-1|^{1/3})^{1/2}}{|z-1|^{5/6}}\right] dz \wedge d\bar{z}.$$

3.4. **Hyperbolicity of Hypersurfaces of High Degree.** Let X be a complex manifold of dimension n. The real tangent space to X at and point $x \in X$ can be defined as the set of equivalence classes of tangent vectors of differentiable curves passing through x. This can be generalized in the following sense. For $f \colon \mathbb{D}_r \longrightarrow X$ the germ of a holomorphic mapping with $f(0) = x$; where $\mathbb{D}_r$ denotes the disk in $\mathbf{C}$ with center 0 and radius r one can write f as a power series

$$f(z) = (f_1(z), \ldots, f_n(z))$$

where $f_i(z) := (z_i \circ f)(z)$ with respect to any holomorphic set of coordinates $z_1, \ldots z_n$ around x. Each $f_i(z)$ can be expanded as a power series around 0 and be written as:

$$f(z) = (\sum_{k=0}^{\infty} \frac{f_1^{(k)}(0)}{k!} z^k, \dots, \sum_{k=0}^{\infty} \frac{f_n^{(k)}(0)}{k!} z^k)$$

which will be written, by abuse of notation, as

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \dots \tag{1}$$

and in particular by $f^{(k)}(z)$ we mean $(f_1^{(k)}(z), \dots, f_n^{(k)}(z))$.

Definition 3.3. Two holomorphic germs $f \colon \mathbb{D}_r \longrightarrow X$ and $g \colon \mathbb{D}_r \longrightarrow X$ with $f(0) = x$ and $g(0) = x$ are called k-equivalent if

$$f^{(k)}(0) = g^{(k)}(0)$$

for any $k = 1, 2, \dots, n$.

Note that this is an equivalence relation and the classes are independent of the chosen coordinate system. An equivalence class will be referred to as a k-jet and the set of k-equivalence classes of germs of holomorphic functions $f \colon \mathbb{D}_r \longrightarrow X$ is denoted by $J_x^k X$ and $J^k X$ is defined as the union $\cup_{x \in X} J_x^k X$. We will write $J_x^{*,k} X$ for non-constant k-jets at x. For $k = 1$ $J^1 X$ is the complex holomorphic tangent bundle and hence is a vector bundle, however for $k \geq 2$ $J^k X$ is an affine bundle over the base X with the distinguished point being the constant map. Each fiber is of dimension kn, hence the total space is of dimension $(k+1)n$.

Given a non-constant holomorphic map $f \colon \mathbb{D}_r \longrightarrow X$, for any $t \in \mathbf{C}^*$, we define

$$\begin{array}{rcl} (t \cdot f) \colon \mathbb{D}_{r/|t|} & \longrightarrow & X \\ z & \mapsto & tz \end{array}$$

For $k = 1$, the quotient of the set $J^{*,1} X$ under this action gives the classical projectivized tangent bundle. However, for $k \geq 2$, the fibers of the quotient are weighted projective spaces. Indeed, for $x \in X$, the action on the fiber of $J_x^k X$ is

$$t \cdot (f^{(1)}(z), \dots, f^{(k)}(z)) = (t f^{(1)}(z), \dots, t^k f^{(k)}(z)),$$

that is, in the notation of [11] each fiber is $\mathbf{P}(1, \dots, 1, 2 \dots, 2, \dots, k, \dots, k)$.

Fixing a weight $w \in \mathbf{Z}_{>0}$. Then a jet differential of weight w is given as a weighted polynomial with coefficients being holomorphic functions of the local coordinates $z_1, \dots, z_n$ of weight w in kn coordinates in variables dz_i^j, for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$; where the weight of dz_i^j is j. Such differentials, denoted by $\mathcal{J}_{k,w}$, form a sheaf. For instance, a 1-jet differential of weight w is a section of the w^{th} symmetric power of the sheaf of 1-forms, i.e. $\mathcal{J}_{k,w} = \text{Sym}^w \Omega_X^1$.

Given a vector sub-bundle, V of the holomorphic tangent bundle, TX, of X it is also possible to define the corresponding jet bundle. In this construction jets whose tangent vectors lie in the sub-bundle V are taken into consideration. In this case, some care must bu taken in defining higher jets. We invite the reader to consult [7] for details of the construction.

Jet-differentials have proven themselves as useful tools in determining the hyperbolicity of hypersurfaces in projective space. In fact, certain sub-bundles of the jet bundle plays an important role. For instance, Demailly and ElGoul, [9], have used Semple jet bundles to prove that a very general surface in $\mathbf{P}^3$ of degree greater than or equal to 21 is Kobayashi hyperbolic. The technique of slanted vector fields and jet differentials on the universal family of hypersurfaces, due to Siu [30], is closely related to the works of Clemens, [5, 6], Ein, [12, 13], and Voisin, [36, 37], has led to the proof of the following:

Theorem 3.4. [30, Theorem 1.1] *Any generic hypersurface in* $\mathbf{P}^n$ *of sufficiently high degree is Kobayashi hyperbolic.*

A result in the same vein as the above is due to Demailly, [8]:

Theorem 3.5 ([8, Theorem 4.2])**.** *A very general hypersurface in* $\mathbf{P}^{n+1}$ *of degree greater than or equal to* $2n+2$ *is Kobayashi hyperbolic.*

Let us close this section by noting that there are a number of related research directions have been left aside. In addition to log-versions of all the discussion made in this part, for instance among many others there are results/conjectures concerning *algebraic hyperbolicity* of algebraic varieties.

4. Hyperbolicity of Çark Hypersurfaces

In this section, we will first define çark hypersurfaces. After a discussion of some geometric properties of these surfaces, we will end with giving some examples.

Throughout we fix $K = \mathbf{Q}(\sqrt{d})$ be the unique real quadratic number field; where $d > 0$ is a square-free integer and we let δ be the square-free part of the discriminant, Δ, of K.

4.1. **Çark Hypersurfaces.**

Finiteness of the narrow class group implies the following:

Lemma 4.1. *There is a positive integer* k *(which is not unique) with the property that all narrow ideal classes of discriminant* Δ *can be represented by a çark of length* $2k$*.*

Remark. If d is not square-free, then the correspondences between indefinite binary quadratic forms, narrow ideal classes and çarks (see Sections 2.4 2.5 2.6) fail to exist. For instance, both the narrow class number and the class number of the field $\mathbf{Q}(\sqrt{13})$ are 1. The square-free part of the discriminant of the forms $(1, 3, -1)$ and $(1, 393, -1)$ are 13, whereas they are not equivalent under the action of the modular group, merely because their çarks are different, being $[3, -3]$ and $[393, -393]$, respectively. Although it is not easy to produce such examples fixing the length of the çark, it is quite easy to show that the discriminant of the form $(1, 8, 3)$ is $52 = 2^2 \cdot 13$ but whose çark is $[1, -6, 1, -1, 1, -1, 6, -1, 1, -1]$.

Now, we fix the smallest such k provided by Lemma 4.1. If $\mathcal{O}_d$ admits a unit of negative norm, we let $G_d = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/k\mathbf{Z}$ and if there are no units of negative norm in $\mathcal{O}_d$, then we set $G_d = \mathbf{Z}/k\mathbf{Z}$. In [41] the following theorem is proven:

Theorem 4.2. *For a fixed fundamental discriminant* Δ*, there is an affine hypersurface in* $\mathbf{C}^{2k+1}$*, called çark hypersurface, denoted by* $\mathcal{C}_d$ *admitting an action of* G_d *so that there is a two to one correspondence between the* G_d *invariant points on* $\mathcal{C}_d$ *and binary quadratic forms whose discriminant's square-free part is* δ*.*

As a result of Remark 4.1 the above correspondence fails to be *injective* in the following sense: the correspondence between the G_d invariant integral points of $\mathcal{C}_d$ and narrow ideal classes fails to be two to one. In order to exemplify the situation let us note first that one can explicitly determine equations of these hypersurfaces.

It turns out that çark hypersurfaces are not irreducible. They have two irreducible components one being in a sense a translate of the other. More precisely, the equation of the hypersurface $\mathcal{C}_d$ turns out to be of the form $f_d(f_d+4) = z^2\delta$; where the polynomial $f_d \in \mathbf{Z}[x_1, y_1, \ldots, x_k, y_k] = \mathbf{Z}[X, Y]$. One can prove that all the coefficients are equal to one in f_d.

The action of the group $G_d = \mathbf{Z}/k\mathbf{Z}$ on $\mathbf{C}[X, Y]$is defined as:

$$\begin{array}{rcl} \mathbf{Z}/k\mathbf{Z} \times \mathbf{C}[X,Y] & \longrightarrow & \mathbf{C}[X,Y] \\ p(x_1, y_1 \ldots, x_k, y_k) & \mapsto & p(x_k, y_k, x_1, y_1, \ldots x_{k-1}, y_{k-1}) \end{array}$$

in case $\mathcal{O}_d$ does not have any unit of negative norm; and the action of the group $G_d = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/k\mathbf{Z}$ is defined as

$$\begin{array}{rcl} \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/k\mathbf{Z} \times \mathbf{C}[X,Y] & \longrightarrow & \mathbf{C}[X,Y] \\ ((0,1), p(x_1, y_1, \ldots, x_k, y_k)) & \mapsto & p(x_k, y_k, x_1, y_1, \ldots, x_{k-1}, y_{k-1}) \\ ((1,0), p(x_1, y_1, \ldots, x_k, y_k)) & \mapsto & p(y_1, x_1, \ldots, y_k, x_k) \end{array}$$

Example 4.1. The equation defining the çark hypersurface when $d = 13$ is $x_1y_1(x_1y_1 + 4) = 13z^2$, i.e. $f_{13} = x_1y_1$, and the group acting on f_{13} is $G_{13} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/1\mathbf{Z}$ interchanging y_1 and x_1. Every integral point on the variety defined by the zero set of the equation $X^2(X^2+4) = Z^2 13$ gives rise to an indefinite binary quadratic form of discriminant $z^2 \cdot 13$. An immediate solution is $(3, 3, \pm 3) \in \mathbf{Z}^3$. A simple calculation modulo 13 tells us that both X should be congruent to ± 3 modulo 13. Then, the problem reduces to determining integers k for which the polynomial $13k^2 + 6k + 1$ is a perfect square (or $13k^2 - 6k + 1$). Let us be content with providing two other integral points: $(393, 393, \pm 109)$, $(46764, 46764, \pm 129700)$.

Let us also note that each f_d is closely related to Lucas polynomials:

Proposition 4.3. [41] *Let Δ be a fundamental discriminant and f_d be the corresponding polynomial, in particular $f_d \in \mathbf{Z}[x_1, y_1, \ldots, x_k, y_k]$. Then:*

i. *for any $1 \leq l \leq k$ the polynomial f_d does not contain any monomial of degree $2l - 1$, and*
ii. *for any $1 \leq l \leq k$ the polynomial f_d contains exactly m_{2l}-many monomials of degree $2l$; where m_{2l} is the coefficient of x^{2l} in $2k^{th}$ Lucas polynomial.*
iii. *the polynomial f_d is of degree $2k$.*

Table 1 is a list of polynomials f_d; where narrow ideal classes in $\mathbf{Q}(\sqrt{d})$ can be represented by çarks of length at most $2k$.

The degree of the çark hypersurface is $4k$ in $\mathbf{C}^{2k+1}$. Its intersection with the hypersurface $z = 1$ gives us a hypersurface in $\mathbf{P}^{2k}$ of degree $4k$. Let us name this intersection C_d. Remark that the integral points of this intersection corresponds to those matrices which induce under the map described in Section 2.4 a binary quadratic form whose discriminant is exactly equal to d. In particular, these points is a subset of the narrow class group of $\mathbf{Q}(\sqrt{d})$, in particular can at most be finite.

In the light of Theorem 3.5 we expect the intersection to be Kobayashi hyperbolic. Assuming Lang conjectures on the arithmetic, C_d is expected to have finitely many rational points which is indeed the case according to the previous paragraph.

d	k	f_d	h_d	h_d^+	çarks
2	2	x_1y_1	1	1	$[2,-2]$
3	2	x_1y_1	1	2	$[2,-1]$ and $[1,-2]$
7	4	$x_1x_2y_1y_2+(x_1+x_2)(y_1+y_2)$	1	2	$[1,-1,1,-4]$ and $[1-1,4,-1]$
17	6	$x_1x_2x_3y_1y_2y_3+x_1x_2y_1y_2+x_2x_3y_1y_2+x_1x_2y_1y_3+x_1x_3y_1y_3+x_1x_3y_2y_3+x_2x_3y_2y_3+(x_1+x_2+x_3)(y_1+y_2+y_3)$	1	1	$[1,-3,1,-1,3,-1]$
19	6	$x_1x_2x_3y_1y_2y_3+x_1x_2y_1y_2+x_2x_3y_1y_2+x_1x_2y_1y_3+x_1x_3y_1y_3+x_1x_3y_2y_3+x_2x_3y_2y_3+(x_1+x_2+x_3)(y_1+y_2+y_3)$	1	2	$[2,-1,3,-1,2,-8]$ and $[1,-3,1,-2,8,-2]$

TABLE 1

Example 4.2. Let us end with the case $d=7$. The narrow class number in this case is equal to 2. The corresponding ideal classes are represented by the forms $(1,4,-3)$ and $(-1,4,3)$. The polynomial f_d is equal to $x_1x_2y_1y_2+(x_1+x_2)(y_1+y_2)$ and the group $G_d=\mathbf{Z}/2\mathbf{Z}$. The four 5-tuples $(1,1,1,4,\pm 2)$ and $(1,1,4,1,\pm 2)$ gives the integral points on the çark hypersurface $\mathcal{C}_7$, respectively.

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