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Basic Invariant Theory and its applications to moduli

Refs: Kač – Invariant Theory
Dolgachev – Lectures on Invariant Theory
⋮
GIT – Mumford

—, GIT constructions of moduli spaces of curves
Alper, Hyem, GIT constructions of Log minimal models
Fedorchuk – Smyth, Alternate Compactifications of moduli spaces of curves

$\boxed{\text{Setup}}$: $G \curvearrowright W$ representation
$\qquad\qquad X$ affine variety

G: affine algebraic groups

$\mu: G \times G \longrightarrow G$ morphism
$i: G \longrightarrow G$

linear algebraic groups $G \subset Gl_n(\mathbb{C}) = \mathbb{A}^{n^2}$

G: $G_m = \mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[t, t^{-1}])$ (multiplicative)

$G_a = \mathbb{C} = \mathrm{Spec}(\mathbb{C}[t])$ (additive)

$Sl_n(\mathbb{C}) = \mathrm{Spec}\left(\mathbb{C}[a_{ij}] / \underbrace{\det - 1}_{\text{generated by one elt.}}\right)$

W: $SL(V)$ $\quad V = \mathbb{C}^n$

W is a "multilinear relative" of V $\quad M_{n \times n}(V)$

$\mathrm{Sym}^d V$, $V \otimes V$, $\Lambda^p \mathrm{Sym}^d(V)$

GOAL of GIT: produce and study the geometry of a "good" quotient

$$W \longrightarrow W/\!/G$$

usually with an orbit space

Strategy: $\mathbb{C}[W] \supset^{\pi^\#} \mathbb{C}[W]^G$ (π invariant)

$$W \xrightarrow{\pi} W/\!/_G = \mathrm{Spec}(\mathbb{C}[W]^G)$$

Goldilocks and 3 bears

"not too many" Is $\mathbb{C}[W]^G$ f.g.? Not in general

Hilbert's 14^{th} Problem - Solved by Nagata $(G_a)^N$ char$k$) > 0 exact cases

Mukai, Totaro: $(G_a)^N$, $N \geq 5$ any k.

YES: if G is reductive

"not too few"

Ex: $G_m : \mathbb{C}^n$ by homothety $t.w = \overline{tw} \ni \vec{0}$
only constants are invariant
$\mathbb{C}^n/\!/_{\mathbb{C}^*} = \{pt\}$

If G is reductive, then G-invariants separated, disjoint, closed G-stable subvarieties.

"not just right"

Ex: $G_a : \mathbb{C}^2$ $t[a,b] = [a, ta+b]$ 'skew'

$\mathbb{C}[\mathbb{C}^2]^{G_a} = \mathbb{C}[a]$ $\mathbb{C}^2/\!/_{G_a} = \mathbb{C}$

$[a,b]$ is a closed map, all map to 0.

Ex: $G_a : M_{2\times 2}(\mathbb{C})$ $\quad t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$

$$\mathbb{C}[M_{2\times 2}(\mathbb{C})]^{G_a} = \mathbb{C}[c, d, ad-bc]$$

$$M_{2\times 2}(\mathbb{C})/\!/G_a = \mathbb{C}^3 \qquad [0, 0, z], z \neq 0$$

not in image

Prop: If R is a noetherian ring, $S \subset R$ and $\ell : R \to S$ is a surjective S-module homo., then S is noetherian.

$$J \subset S, \quad I = JR \quad \ell(I) = \ell(JR) = J_\ell(R) = J$$

$G \curvearrowright X \qquad G \times X \xrightarrow{\alpha} X$ $\quad$ s.t $\quad h(g.x) = (hg).x$

$$(g, x) \longmapsto g \cdot x$$

induced action of G on $\mathbb{C}[X]$

$$g \cdot f = (x \longmapsto f(g^{-1}x))$$

$$\mathbb{C}[X] \xrightarrow{\alpha^*} \mathbb{C}[G \times X] = \mathbb{C}[G] \otimes \mathbb{C}[X]$$

$$f \longmapsto ((g, x) \longmapsto f(g.x)) = \sum_{i=1}^{k} \Phi_i \otimes \Psi_i$$

$$f(g.x) = \sum_{i=1}^{k} \Phi_i(g)\Psi_i(x)$$

$$(g.f)\cancel{x} = \sum_{i=1}^{k} \Phi_i(g^{-1})\Psi_i(\cancel{x})$$

Lemma: ① For any $f \in \mathbb{C}[x]$, span$\{g.f \mid g \in G\}$ is f.d.

② $\mathbb{C}[X]$ is the union of its f.d submodules.

Exercise: ① if $f_1, \dots, f_n$ generate $\mathbb{C}[X]$

W is the span of their translates.

$p_1, \dots, p_N$ is a basis of W.

$$X \xrightarrow{\Psi} \mathbb{C}^N = W$$
$$x \longmapsto (\dots, p_i, \dots)$$

- check that Ψ is an isom. onto its image $\tilde{X}$ and that Ψ lifts to a diagram

$$\begin{tikzcd} G \times X \arrow[r, "\bar{\Psi}"] \arrow[d, "\alpha_X"'] & G \times \tilde{X} \arrow[r, "\subseteq"] \arrow[d, "\alpha_{\tilde{X}}"] & G \times W \arrow[d] \\ X \arrow[r, "\Psi"] & \tilde{X} \arrow[r, "\subseteq"] & W \end{tikzcd}$$

that commutes

② Deduce that any affine algebraic group is isomorph. to a linear alg group!

Def - Thm: A finite dimensional G module V is called semisimple (completely reducible) if the 3 equivalent conditions below hold:

(i) V is the sum of its irreduc. proper subreps.

(ii) V is the direct sum of ~~sub~~ irreducible subrepresentations

(iii) If U is a G-submodule, then U has an G-stable complement U', $V = U \oplus U'$

(if action is non trivial then U' is canonical)

Ex: $G_a : \mathbb{C}^2 \quad t.(a,b) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (a + tb, b)$

only irred. . is the x-axis

Pf: (i) ⇒ (ii) $\mathcal{S} = \left\{ \begin{array}{l} \text{sums of irreduc.} \\ \text{that are direct} \end{array} \right\}$

partially ordered by inclusion
pick a maximal elt.

Claim: sum equals V

(ii) ⇒ (iii) As above but additional sum is disjoint from $\mathcal{U}$.

(iii) ⇒ (i)

ex: show that every V contains an irred.
(ii) use (i) to show that V is the sum of all its irreducibles.

ex: The only G-submodule V_G complementary to V^G = invariant subspace of V is the sum of all irreducibles on which G acts non-trivially.

- $\mathbb{G}_m : M_{2\times 2}(\mathbb{C})$ conjugate

$$t . \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

- $GL_2(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$

$$G . A = G A G^{-1} \qquad \mathbb{C}[M_{2\times 2}(\mathbb{C})]^{GL_2(\mathbb{C})} = \mathbb{C}[tr, det]$$