Friday - Problem session

G \to W \quad W \xrightarrow{\pi} W/\mathcal{G} \xrightarrow{f \#} \mathbb{R}

\text{covering invariants}

\begin{array}{rl}
\text{CCW} & \text{CCW}^a \\
\text{CCW}^a : \text{module over semi-simple} & \\
\text{CCW}^a & \text{module over semi-simple}
\end{array}

\text{Graded G-linear reductive}
- any rep. W is a sum of irreducible
- \mathcal{W} = \text{sum of all irreducible as a G-module}

\text{CCW} = \text{CCW}^a \oplus \mathcal{W}_G

\text{Proposition on CCW}^a \text{ Reynolds operator}

\text{Lemma: } f^* \text{ is a CCW}^a \text{-module homomorphism on } \mathcal{W}^a

\text{Prop: } f^* \text{ is a } \mathcal{W}^a \text{-module homomorphism}

f^* \text{ is } \mathcal{W}^a \text{-module homomorphism}

\text{Proof: } f^* \text{ is } \mathcal{W}^a \text{-module homomorphism}

\begin{array}{rll}
\text{Prop: } & \text{If } X, Y \text{ are } G \text{-stable closed, disjoint} & \\
& \text{subsets of } W & \text{then there is a } f^* \text{ for } \text{CCW}^a \text{ s.t.} \ & \\
& f^*_X = 0 & f^*_Y = 1
\end{array}

\text{Proof: } X \cap Y \text{ closed, disjoint by Nullstellensatz}

f^*_X = 0 \quad f^*_Y = 1

\text{Lemma: } f^* \text{ is a } \mathcal{W}^a \text{-module homomorphism}

f^*_X \mid Y = 0 \quad f^*_Y \mid X = 1

\text{Prop: } X \cap Y \text{ closed, disjoint by Nullstellensatz}

f^*_X \mid Y = 0 \quad f^*_Y \mid X = 1

\text{Def: } \text{An algebraic group } G \text{ is called reductive if no non-trivial, closed, normal, unipotent subgroup}

\text{Ex: } \begin{array}{rl}
\text{If } X, Y \text{ are disjoint closed } K \text{-stable subsets, then } & \\
\text{there is a } & \\
\text{polynomial } & \text{ s.t. } f^*_X < 0, f^*_Y > 0
\end{array}

\text{Hint: } \text{Stone-Weierstrass}

\text{Ex: } (Kah) \text{ If } V \text{ is a rep. of } K, X, Y \text{ are disjoint closed } K \text{-stable subsets, then } \text{ there is a } K \text{-invariant polynomial } \phi \text{ s.t. } f^*_X < 0, f^*_Y > 0

\text{A matrix } A \text{ is unipotent if its characteristic polynomial is } (X-1)^n \quad \text{all eigenvalues are 1.}
- Jordan Normal Form: $A = A_1 \oplus \cdots \oplus A_k$, $A_k$ is unipotent. 
  \[ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \]
  \[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
  
  Fact: This decomposition is intrinsic.

Ex: $G_2$ is reductive, $G_2$ is unipotent.

Then: Over $\mathbb{C}$, TFAE

1) $G$ is reductive.
2) $G$ has no closed normal subgroup isomorphic to $(G_2)^n = G_2^n$.
3) $G$ has a compact real subgroup $K$ s.t. $T^*G = T^*G \otimes \mathbb{C}$.
4) $G$ has a compact real subgroup $K$ that is Zariski dense.

5) $G$ is linearly reductive.

Tangent vectors $\mathcal{V}$ to $X$ is a map $\mathcal{V} \to \mathcal{X}$ where $\mathcal{V} = \frac{\text{Spec}(\mathbb{C}[E]/(e^2))}{\mathbb{C}[X]}$.

Ex: $G_2 = \mathbb{C}$ at $p \in E$.

- $C[t] \to \mathbb{C}[E]/(e^2)$
  \[ t \mapsto t(p) + a \in E \]

- $G_2 = \mathbb{C}^*$ at $p$.

Ex: $G_2 = \mathbb{C}^*$ at $p$.

- $C[t, t^{-1}] \to \mathbb{C}[E]/(e^2)$
  \[ t \mapsto t(p) + a \in E \]
  \[ t^{-1} \mapsto \frac{1}{t(p) + a} \]

$$= \frac{t(p)(1 + aE)}{t(p)} $$

image of $t^{-1} = 0$ in $\mathbb{C}[E]/(e^2)$.

- $C^*, K$: compact subgroup $S^1$, $R$

  $R[K] = R[x, y]/(x^2 + y^2 - 1)$

  $= R[z]/(z^2 - 1)$

  at $p = 1$, $z \mapsto 1 + aE$ a pure imaginary

  $z \in (1 + aE)(1 + bE) = 1 + (a + b)E$

  $\delta + i \phi \in (-i\phi) + 1(\delta i)$

Ex: $\text{SL}(n), G(n)$

$\text{SL}(n), G(n) = \{ A \mid A^* A = I_n \}$ is compact.

The $E$ term of a tangent vector to $\text{U}(n)$ at $I_n$ is any $B$ s.t. $B + B^* = 0$.

Ex: $\text{U}(n), \text{G}(n)$

- $C[X] \to \frac{\text{Spec}(\mathbb{C}[E]/(e^2))}{\mathbb{C}[X]}$

  is a tangent vector at $p$.

$\phi(f) = f(p) + aE$ where $\phi(fg) = \phi(f)g(p) + f(p)\phi(g)$.

Derivation

2) The tangent vectors at $p$ form a $C$-vector space.
Example 4: \( G \cong X, Y \in X, H \text{ acts on } Y \)

- \( S = \{ x_i \} \) invariant poly. on \( X \)
- \( \text{CEVJ}^H \) is generated by \( p_{1, y} \)

Then, \( S \) generates \( \text{CEVJ}^G \).

2) \( Sl(n) \to M_{nxn}(\mathbb{C}) \) by conjugation:

\[
M_{nxn}(\mathbb{C}) \xrightarrow{\text{CEVJ}^{Sl(n)}} \text{CEVJ}^{M_{nxn}(\mathbb{C})} \\
A \xrightarrow{\text{CEVJ}^{M_{nxn}(\mathbb{C})}} (\text{det}(A) \cdot \text{CEVJ}^A)
\]

Claim: \( \text{CEVJ}^{M_{nxn}(\mathbb{C})} \) generated by coef. of characteristic polynomial \( \lambda \cdots \lambda \), \( \text{det} \)

(b) These coefficients are algebraically independent

Hint: \( H = S_n \) symmetric group of permutation matrices.

\[
M_{nxn}(\mathbb{C}) \to M_{nxn}(\mathbb{C}) / Sl(n) = \mathbb{C}^n \\
A \to \det(tI - A)
\]

What are the fibers of the quotient map?

What orbits lie in each fiber?

What similarity classes are given by each characteristic polynomial?

4) \[
\begin{pmatrix}
t & a \\ c & d \\
\end{pmatrix}
\begin{pmatrix}
t & 0 \\ 0 & t \\
\end{pmatrix}
\begin{pmatrix}
t & a \\ c & d \\
\end{pmatrix}
= \begin{pmatrix}
t^2 & t^2a \\ t^2c & t^2d \\
\end{pmatrix}
\]

- Ask what this says as we send \( t \to 0 \).

- In \( 2 \times 2 \), use this to understand:

\( Sl(2)A \) for any \( A \)

- Conjecture a generalization to \( M_{nxn} \).
\[ T : N_{\mathbb{H}}(C) \to \mathbb{C}^n = T N_{\mathbb{H}}(C)/S_1 \mathbb{C} \]

\[ A \mapsto B, A \mapsto B(A) \]

\[ A \mapsto B, C \mapsto C' \]

\[ G \times W \to W/G \quad \text{by "values of invariants"} \]

\[ \mathbb{C}[W] \to \mathbb{C}[W]^G \]

Reynolds operator \( Q : \mathbb{C}[W] \to \mathbb{C}[W]^G \)

\[ \text{G-linearly reducible} \iff \text{G-reductive} \]

\[ \text{not closed, normal unipotent subgroup except the trivial} \]

\[ \implies \text{There is a compact subgroup } K \text{ of } G \]

\[ \implies \text{There is a dense compact subgroup } K \]

\[ \implies G \text{ is linearly reductive} \]

Ex: Show \( T_e \text{SL}(n) \cdot \{ B | \text{Tr}(B) = 0 \} \)

1) Given \( B \in T_e K \) pick \( 1 - PS, \alpha(t) \in K \)

\[ \alpha(0) = I, \quad \alpha'(0) = B \quad \mathbb{C} \text{-inv} \]

\[ \mathcal{U} \oplus \mathcal{U}' = \mathcal{W} \quad \mathcal{U}' \text{ is } T_e K \text{ - weak } \]

2) \( \mathcal{U}' \text{ is } T_e G \text{ - invariant} \)

3) \( \alpha(t) \in G \)

\[ \alpha'(0) = B \in T_e G \]

So \( \mathcal{U}' \) is inv. under a neighborhood of \( e \) in \( G \).
Ex. 1. Any neighborhood of $e$ in a connected topological group generates the group.

2. For an algebraic group $G$, the connected component of $G$ in $G$ is a closed, normal subgroup of finite index.

3. $\Delta \subset \text{GL}(V)$. Then $W$ is $G$-semisimple if $W$ is $H$-semisimple.

Ex: $\mathbb{Z}/2\mathbb{Z}$ only $\mathbb{F}_2$ irreducible is trivial rep.

$p$ is linearly reductive if $G$ is $W$-stable, $\text{Lie}(W) = 0$.

$\text{SL}(2) \cong \text{M}_{22}^1$ by conjugation.

$W^G \cong \text{ker}([\text{tr}, \text{det}])$.

- fixed points $(a \in G)$, $\text{tr}(a^g) = 0$ if $\det(a) = 0$.
- non-vanishing homogeneous invariant.

Geometric reductivity:


Haboush: reductive groups are geometrically reductive in all characteristics.

4. $\overline{\pi(X \cap X')} = \overline{\pi(X)} \cap \overline{\pi(X')}$. 

5. Every fiber of $\pi$ contains a unique closed orbit.

Take $X, X'$ take $G \cdot X \cap G \cdot X' = \emptyset$ closed.

$\pi(X) \cap \pi(X') = \emptyset \iff X, X'$ are in different fibers.

Ex. 2. $\overline{\pi(X)}$ is a categorical quotient.

Given a $G$-equivariant $\psi : W \to Y$ affine, there is a unique $\hat{\psi}$.

$\pi : W \to W//G$.

$p^*: \text{CC}(Y) \to \text{CC}(W)$.

Let $\psi : W \to Y$.

If $x \subset X$ is $G$-stable then $X//G \to W//G$ i.e. $\pi(W)(x) = x//G$.

$\text{CC}(W) \xrightarrow{\psi^*} \text{CC}(X) \xrightarrow{\text{CC}(X)^G} \text{CC}(X//G) \xrightarrow{\pi(W)} W//G$.
Closures of orbits

Prop: 1. For any alg. $G$, any $w \in W$,
      $G_w$ is locally closed, smooth.
5. $G_w$ is the union of all other orbits of all lower dimension.
6. $\dim(G) = \dim(G_w) + \dim(G_w^w)$, stabilize.
7. Any orbit in $G_w$ of minimal dimension,
is closed, so any $G_w$ contains a closed orbit.

Pf.
1. $G_w$ is a finite union of $G_w^0$ orbits.
2. $(G_w^0)_C \subset (G_w)_C \subset G_w$
   all of same dim.

WLOG. Assume that $G$ is connected.

$G \subset W$ $G_w$ isomorphic, so is constructible.

$G$ acts transitively on $G_w$.

So contains $G \cap G_w$ $W$ open in $W$.

Facts about fibers of affine maps

1. If $X \to Y$ is a surjective map of closed,
   affine varieties

then $\dim(f^{-1}(y))$ $y \in Y$ is upper-semicontinuous.

This dimension has a generic value $d_f$

$\dim(X) = \dim(Y) + df$. 

1. $\pi: X \to X/\alpha$.

$X/\alpha$ is regular if $X$ is and if so, is normal if $X$ is.

$C(X) \subset C(X)/\alpha$.

Given element of $X$ integral over $C[X]$.

I'd get an element of $C(X)$ integral over $C[X]$.

$\implies$ Problem section

ex. 1: $f \subset C[Y]^G$, let $g = f^3 \subset C[Y]^G$,
$g = 0$ on $X$.

Let $Q(P_1, \ldots, P_k) = 0$ on $X$.

Let $P_1, P_2$ are $G$-inv.

$f = Q(P_1, \ldots, P_k)$ on $G \cdot Y$ so on $G \cdot Y = X$.

$f = Q(P_1, \ldots, P_k)$ in $C[X]$.

ex. 2: $H = \text{permutation matrices}/S_n$ as permutation matrices

$Y = \text{diagonal matrices}$.

$\sigma : (\lambda_1, \ldots, \lambda_n) = P_0 (\lambda_1, \ldots, \lambda_n) P_0^{-1}$.

$\det(X - \lambda I) = e_1(\lambda) + e_2(\lambda)t^2 + \cdots + e_n(\lambda)$

$\det(X - \lambda I) = e_{\ell}(\lambda)$

$X^{\ell}$

$e_k = \ell \text{th } \text{sym. functions of } X$. 

$C[Y] = C[\lambda_1, \ldots, \lambda_n]$.

$\text{aff}$.
For polynomials of any degree $x^n + a_1 x^{n-1} + \ldots + a_n$, there is a polynomial $\Delta(a_1, \ldots, a_n)$ that vanishes if $f(x)$ has repeated roots.

\[ \Delta = \prod (x_i - x_j)^2 \]

**First $\Delta$ for 3x3 matrices**

$SL_2(C) \to \text{Sym}^2(C^2) \cong W = C[a_0, \ldots, a_7]$ homog poly. of deg $d$ in $x, y$

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}
\]

\[ A = \sum a_i x^i y^j \quad \Delta = \sum a_i (ax + by)^i (cx + dy)^j, a_i \neq 0 \]

**Example:** write this down for $d = 2$

\[ G_w \cong O_w \quad P(W) \cong W^{\vee} \quad \text{if } w \text{ is semi-stable} \]

**$G_w \not\cong O_w$**

$w$ is unstable if there is a non-constant and non-vanishing homog. inversive.

\[ P(x, y) = \sum a_i x^i y^j \Delta(P) = 0 \Rightarrow P \text{ has repeated roots in } P^1. \]

$\Delta = \text{Res}_z \left( \frac{z^2}{z_{13}}, \ldots, \frac{z^2}{z_{1n}} \right)$

Polynomials with no repeated roots are semi-stable $\iff \Delta(p) \neq 0$

$\text{SL}_2(n) \cong \text{Sym}^2(C^2) / C_g \cong \text{Sym}^2(C^2) / \text{SL}_2(n)$

\[ \text{P}(\text{Sym}^2(C^2)) = \mathbb{P}^4 \quad 9 \text{ points on } \mathbb{P}^2 \text{ by Chevalier} \]

$G \rightarrow \text{Lin. on ample amp}$

For every $x$, there is a section $s$ of $L^m$ not vanishing at $x$

$H^0(X, \mathcal{O}_m)$

$X \rightarrow \mathbb{P}^N$

$X \rightarrow \text{closed immersion} \rightarrow \text{ample}$

$X / \mathcal{G} = \text{Proj } H^0(X, \mathcal{O}_m)$

**Riemann--Roch**

$H^0(C, \mathcal{O}_C(D)) \cong H^1(C, \mathcal{O}_C(D) \otimes K_C^{-1})$ for $g \geq 2$

$g(D) \text{ very ample}$

**Algebraic Geometry**

- Joe Harris
- William Fulton
What are the fibers of the quotient $\mathbb{P}^3/(x \mapsto t^a x)$?

What orbits lie in each fiber?

What similarity classes have a given characteristic polynomial?

\[
\begin{pmatrix}
t & 0 \\
0 & t^{-1}
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
t^{-1} & 0 \\
0 & t
\end{pmatrix}
= 
\begin{pmatrix}
a t^2 & b \\
c t & d
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda t & 0 \\
0 & \lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]

All conjugate matrices are conjugate to each other.

What JNF's correspond to this \(\mathcal{X}(A)\)?

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]

Matrices with distinct eigenvalues have closed orbits.

Ex. Show that \(\mathrm{SL}_2(\mathbb{Q}) \cong \mathbb{Q}_x^3\).

\(N, N'\) nilpotent matrices is that JNF of \(N'\) is obtained from JNF of \(N\) by replacing some 1's with 0's.

Use only matrices of form \(
\begin{pmatrix}
t & 0 & 0 \\
0 & t & 0 \\
0 & 0 & t
\end{pmatrix}
\)

\(t \in \mathbb{Q}_x\)