Friday - Problem session 30.04.2014

$G \curvearrowright W \supset X$

$W \xrightarrow{\pi} W/\!/G$ ↗ f.g.?

$\mathbb{C}[W] \xleftarrow[\ \ell\ ]{\pi^*} \mathbb{C}[W]^G$ ↪ enough invariants

$\mathbb{C}[W]^G$-module homo

G linearly reductive:
- any rep. W is a sum of irred. (semi-simple)
- $\mathcal{U} \subset W \quad W = \mathcal{U} \oplus \mathcal{U}'$
- $W = W^G \oplus W_G \quad W_G$ = sum of all non-trivial irreduc.

$\mathbb{C}[W] = \mathbb{C}[W]^G \oplus W_G$

ℓ: projection on $\mathbb{C}[W]^G$: Reynolds operator

__Lemma__: ℓ is a $\mathbb{C}[W]^G$-module homomap. __Cor__: $\mathbb{C}[W]^G$ f.g.

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi} & V \\ \downarrow{\scriptstyle \ell_{\mathcal{U}}} & & \downarrow{\scriptstyle \ell_V} \\ \mathcal{U}^G & \xrightarrow{\Phi^G} & V^G \end{array}$$

if Φ is G-map then this commutes

$\mathcal{U} = V = \mathbb{C}[W]$

Φ = mult. by $f \in \mathbb{C}[W]^G$

$\ell(fh) = \ell \circ \Phi(h) = \Phi^G \circ \ell(h) = f\ell(h)$

__Prop__: If X, Y are G-stable closed, disjoint subsets of W then there is $f' \in \mathbb{C}[W]^G$ s.t.

$$f'|_X \equiv 0 \qquad f'|_Y \equiv 1$$

__Pf__: $X \cap Y$ closed disjoint by Nullstellensatz

$1_W = f + g \qquad f \in I(X),\ g \in I(Y)$

$1_W = \ell(1_W) = \underset{I(X)}{\ell(f)} + \underset{I(Y)}{\ell(g)}$

$I(X)$, $I(Y)$ are G-stable $\qquad f' = \ell(f)$.

__Ex__: ① If G is finite $\ell(f) = \frac{1}{|G|} \sum_{g \in G} g.f$ is a Reynolds operator.

② If K is compact, produce a Reynolds operator by using Haar measure to average over K.

__Lemma__: If K is compact, acts on a f.d real vector space B, Ω is closed, convex K-stable subset, then Ω contains a K-fixed point.

__Pf__: $u_0 \in \Omega$, $X = \overline{\mathrm{Conv}}(\overline{K.x_0})$ ← conv. compact, K-stable.

$u = \int_X x \, dV$ ← euclidean volume form

__Prop__: If K is compact, K acts real $\varnothing$s V / complex $\varnothing$s W, then there is K-invariant { bilinear, pos definite / sesquilinear, hermitian form.

__Pf__: B = space of forms. Ω = subsets of pos. definite, hermitian forms.

__Cor__: representations of compact K are semisimple.

Given rep. $V \supset \mathcal{U}$, take $V = \mathcal{U} \oplus \mathcal{U}'$ w.r.t invariant inner product.

__Ex__: (Kac) If V is a rep. of K, X, Y are disjoint closed K-stable subsets, there is K-invariant polynomial f s.t. $f|_X < 0$, $f|_Y > 0$

Hint: Stone - Weierstrass

__Def__: An algebraic group G is called reductive if it has no non-trivial, closed, normal, unipotent subgroup.

- $A_{n \times n}$ matrix A is unipotent if its char poly is $(X-1)^n$ ← all eigenvalues are 1.

- Jordan Normal Form $A = \underset{\text{diagonalizable}}{A_{ss}} A_u$ ← unipotent

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Fact: This decomposition is intrinsic

Ex: G_m is reductive, G_a is unipotent. $t \longmapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

Thm: Over $\mathbb{C}$, TFAE

1) G is reductive.
2) G has no closed normal subgroup isomorphic to $(G_a)^n = \mathbb{C}^n$
3) G has a compact real subgroup K s.t. $T_e G = T_e^{\mathbb{R}} K \otimes \mathbb{C}$
4) G has a compact real subgroup K that is Zariski dense
5) G is linearly reductive.

Tangent vectors to X is a map $\mathbb{I} = \operatorname{Spec}\left(\mathbb{C}[\epsilon]/\langle \epsilon^2 \rangle\right)$ ← dual numbers

$\mathbb{I} \xrightarrow{\Phi} X$

$\operatorname{Spec}(\mathbb{C}) \longrightarrow X$

$\operatorname{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \xleftarrow{\Phi^*} \mathbb{C}[X]$

Ex: i $G_a = \mathbb{C}$ at $p \in \mathbb{C}$

$\mathbb{C}[t] \longrightarrow \mathbb{C}[\epsilon]/(\epsilon^2)$

$t \longmapsto t(p) + a\epsilon$

ii $G_m = \mathbb{C}^*$ at p

$\mathbb{C}[t, t^{-1}] \longrightarrow \mathbb{C}[\epsilon]/(\epsilon^2)$

$t \longmapsto t(p) + a\epsilon$

$t^{-1} \longmapsto \dfrac{1}{t(p) + a\epsilon} = \dots$

$$= \frac{1}{t(p)\left(1 + \frac{a\epsilon}{t(p)}\right)} = \frac{1}{t(p)}\left(1 - \frac{a\epsilon}{t(p)}\right) = t^{-1}(p) - \frac{a\epsilon}{t(p)^2}$$

image of $tt^{-1} - 1 = 0$ in $\mathbb{C}[\epsilon]/(\epsilon^2)$

- $\mathbb{C}^*$, K: compact subgroup S^1, $\mathbb{R}$

$\mathbb{R}[K] = \mathbb{R}[x,y]/(x^2+y^2-1)$

$= \mathbb{R}[z]/(z\bar{z}-1)$

at $p=1$ $\quad z \longmapsto 1 + a\epsilon$ a pure imaginary

$z\bar{z} = (1+a\epsilon)(1+\bar{a}\epsilon) = 1 + (a+\bar{a})\epsilon$

$b + ai = i(-ib) + 1(ai)$

Ex: $SL(n)$, $GL(n)$

$SU(n)$ $\quad U(n) = \{ A \mid AA^* = I_n \}$, $A^* = \bar{A}^t$, $U(n) \subset GL(n)$

compacts

The ϵ term of a tangent vector to $U(n)$ at I_n is any B s.t. $B + B^* = 0$

Ex: ① $\Phi: \mathbb{C}[X] \longrightarrow \operatorname{Spec}\left(\mathbb{C}[\epsilon]/(\epsilon^2)\right)$ is a tangent vector at p.

$\Updownarrow$

$\Phi(f) = f(p) + \delta(f)\epsilon$ where $\delta(fg) = \delta(f)g(p) + f(p)\delta(g)$

derivation

② The tangent vectors at p form a $\mathbb{C}$-vector space.

<u>EX</u>: ① $G \curvearrowright X$, $Y \subset X$, H fixes Y
$\quad \cup$
$\quad H$

$S = \{P_i\}$ invariant poly. on X

i) $\overline{G.Y} = X$
ii) $\mathbb{C}[Y]^H$ is generated by $P_i|_Y$

Then S generates $\mathbb{C}[X]^G$.

② $SL(n) \curvearrowright M_{n\times n}(\mathbb{C})$ by conjugation.

$$M_{n\times n}(\mathbb{C}) = W$$

<u>Claim</u>: a) $\mathbb{C}[W]^{SL(n)}$ generated by coef. of characteristic polynomial tr, ..., det

(b) These coefficients are algebraically independent

<u>HINT</u>: $H = \sigma_n$ symmetric group of permutation matrices.

$$M_{n\times n}(\mathbb{C}) \longrightarrow M_{n\times n}(\mathbb{C})/\!/_{SL(n)} = \mathbb{C}^n$$
$$A \longmapsto \det(tI - A)$$

③ What are the fibers of the quotient map? What orbits lie in each fiber?

What similarity classes are given by each characteristic polynomial?

④ $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}$

G_m

- Ask what this says as we send $t \to 0$?
- In 2×2, use this to understand $\overline{SL(2).A}$ for any A
- Conjecture a generalization to $n\times n$ case.

2) By using ① $\quad X = M_{n\times n}(\mathbb{C})$

H = Symmetric group of [illegible] $\quad Y$ = [illegible]

$G = SL_n(\mathbb{C})$

i) $\overline{SL_n(\mathbb{C}).Y} = X$

$G \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} G^{-1}$ distinct eigenvalues

$\varphi: M_{n\times n}(\mathbb{C}) \longrightarrow \mathbb{C}^n$

$A \longmapsto (\varphi_1(A), \dots, \varphi_n(A))$ [illegible]

$\exists X \in M_{n\times n}(\mathbb{C})$ $X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\varphi(X) = (\varphi_1, \dots)$ λ_i roots

$A \rightsquigarrow (\varphi_1(A), \dots, \varphi_n(A))$

with distinct eigenvalues

(ii) $P_i \in \mathbb{C}[M_{n\times n}(\mathbb{C})]^{SL_n(\mathbb{C})}$

$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$ $\quad \det(X - tI) = \prod_{i=1}^{n} (t - \lambda_i)$

$= t^n + \sigma_1(\lambda) t^{n-1} + \dots + \sigma_n(\lambda)$

[illegible]

$P_i|_Y = \sigma_i$ $\quad G.P_i(X)$ [illegible] $G^{-1}XG$

$= P_i(X)$

$\mathbb{C}[Y]^{\sigma_n} = \mathbb{C}[\sigma_1, \dots, \sigma_n]$

③ $\pi: M_{n\times n}(\mathbb{C}) \longrightarrow \mathbb{C}^n = M_{n\times n}(\mathbb{C}) /\!\!/ SL_n(\mathbb{C})$

$A \longmapsto p_1(A), \ldots, p_n(A)$

$X \mid p_i(X) = c_i \rightsquigarrow (c_1, \ldots, c_n)$

02.05.2014

$G \curvearrowright W$ $\quad \pi: W \longrightarrow W/\!\!/_G$ by "values of invariants"

$\mathbb{C}[W] \longleftarrow \mathbb{C}[W]^G$

Reynolds operator $\quad \varrho: \mathbb{C}[W] \to \mathbb{C}[W]^G$
G-equivariant
$\mathbb{C}[W]^G$-module homomorphism.

$\Uparrow$ (over $\mathbb{C}$)

G linearly reducible $\Leftarrow$ G reductive
no closed, normal, unipotent subgroup except the trivial

$\Leftrightarrow$ There is a compact subgroup $K \subset G$ s.t. $T_e G = T_e K \otimes \mathbb{C}$

Hermann Weyl Unitarian Trick

$\Leftrightarrow$ There is a dense compact subgroup K

$\Leftrightarrow$ G is linearly reductive

Ex: Show $T_e SL(n) = \{ B \mid Tr(B) = 0 \}$

1) Given $B \in T_e K$ pick 1-PS, $\alpha(t) \in K$, $\alpha(0) = e$, $\alpha'(0) = B$ $\quad \underset{G\text{-inv.}}{\underset{\uparrow}{U}} \subset W$

$\underset{K\text{-inv.}}{U \oplus \underset{\uparrow}{U'}} = W$ $\quad U'$ is $T_e K$ ~~inv~~

2) U' is $T_e G$-invariant

3) $\hat{\alpha}(t) \in G$, $\hat{\alpha}(0) = e$, $\hat{\alpha}'(0) = B \alpha T_e G$ $\quad \alpha(t) = \exp(tB)$

So U' is inv. under a neighborhood of e in G

Ex: ① Any neighborhood of e in a connected topological generates the group

② For an algebraic group, the connected component of e in G — G^0 — is a closed, normal subgroup of finite index.

③ $H \triangleleft G$, $[G:H] < \infty$. Then W is G-semisimple $\Leftrightarrow$ W is H-semisimple.

Ex: $\mathbb{Z}/p\mathbb{Z}$ only $\mathbb{F}_p$ irreducible is trivial repr.

G is linearly reductive $\rightsquigarrow$ $G \curvearrowright W$ $w \in W^G$, $\exists f \in W^G$ s.t $f(w) = 0$

Ex: $SL(2) \curvearrowright M_{2\times 2}(\mathbb{C})$ by conjugation

$W^G = k[tr, det]$

- fixed points $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ $tr\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = 0$ if $char(k) = 0$
- non-vanishing homogeneous invariant

geometric reductivity

Nagata: geometric reductive then $\mathbb{C}[W]^G$ is f.g.

Habush: reductive groups are geometrically reductive in all characteristics.

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① π is surjective.

maximal $I \subset \mathbb{C}[W]^G$, $J = I.\mathbb{C}[W]$ $I \subset e(J) \subset I \Rightarrow e(J) = I$

Pick maximal $J' \supset J$, $e(J') = I$

② π is a categorical quotient.

Given a G-equivariant $\varphi: W \to Y$ ← affine var

there is a unique ψ s.t

$$\begin{tikzcd} W \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & W/\!/G \arrow[d, "\psi"] \\ & Y \end{tikzcd}$$

$$\begin{tikzcd} \varphi^{\#}: \mathbb{C}[Y] \arrow[r] \arrow[dr, "\psi^{\#}"'] & \mathbb{C}[W] \\ & \mathbb{C}[W]^G \arrow[u, phantom, "\cup"] \end{tikzcd}$$

③ If $X \overset{i}{\hookrightarrow} W$ is closed immersion (X - G-stable)

then $X/\!/G \xhookrightarrow{i^G} W/\!/G$ i.e. $\pi_W(X) \cong X/\!/G$

$\mathbb{C}[W] \overset{i^{\#}}{\twoheadrightarrow} \mathbb{C}[X] \xrightarrow{\text{by using } e} \mathbb{C}[W]^G \xrightarrow{i^{G\#}} \mathbb{C}[X]^G$

$X/\!/G \xrightarrow{i^G} W/\!/G$

④ $\pi(X \cap X') = \underline{\pi(X)} \cap \underline{\pi(X')}$

$I \quad I' \quad I^G \quad I'^G$

$I^G + I'^G = e(I) + e(I') = e(I + I') = (I + I')^G$

⑤ Every fiber of π contains a unique closed orbit.

Take X, X' take $\underbrace{G.X}_{\text{closed}}$ $G.X'$ $G.X \cap G.X' = \emptyset$

$\pi(X) \cap \pi(X') = \emptyset \Leftrightarrow X, X'$ are in different fibers.

Closures of orbits

Prop: ① For any alg. G, any $w \in W$, $G.w$ is locally closed, smooth.

② $\overline{G.w}$ is the union of $G.w$ and of other orbits of all lower dimensions

③ $\dim(G) = \dim(G.w) + \dim(G_w)$ → stabilizer

④ Any orbit in $\overline{Gw}$ of minimal dimension is closed, so any $\overline{Gw}$ contains a closed orbit.

Pf: ① $G.w$ is a finite union of G^0 orbits

$(G^0_w) \subset (G^0_w)^0_w \subset G_w$ — all of same dim —

WLOG Assume that G is connected.

$G \xrightarrow{\Phi} W$, $g \mapsto g.w$ $\quad G.w = \mathrm{im}(\Phi)$ so is constructible. G acts transitively on Gw.

So contains set $\mathcal{U} \cap G.w$ $\mathcal{U}$ open in W

Facts about fibers of affine maps

① $f: X \to Y$ is a surjective map of irred. affine varieties

then $\dim(f^{-1}(y))$ $y \in Y$ is upper-semicontinuous

This dimension has a generic value d_f

$\dim(X) = \dim(Y) + d_f$.

- $\pi: X \to X/\!/G$, $G \curvearrowright W$

$X/\!/G$ irred. if X is and if so, is normal if X is

$\mathbb{C}(X) \supset \mathbb{C}(X)^G \supset$ quotient field of $\mathbb{C}[X]^G$ $\;(= K)$

given element of K integral over $\mathbb{C}[X]^G$

I'd get an element of $\mathbb{C}(X)$ integral over $\mathbb{C}[X]$.

→ Problem session

ex1: $f \in \mathbb{C}[X]^G$, let $g = f|_Y \in \mathbb{C}[Y]^H$ $\; g = Q(P_1|_Y, \dots, P_k|_Y)$

- $g - Q(P_1|_Y, \dots, P_k|_Y) = 0$ on Y
- $f - Q(P_1, \dots, P_k) = 0$ on Y

f, P_i are G-inv.

$f - Q(P_1, \dots, P_k) = 0$ on $G.Y$ so on $\overline{GY} = X$

$f = Q(P_1, \dots, P_k)$ in $\mathbb{C}[X]$

ex2: H = permutation matrices / S_n as permutation matrices
Y = diagonal matrices

$$\sigma \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = P_\sigma \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{= X} P_\sigma^{-1} = \begin{pmatrix} \lambda_{\sigma(1)} & & 0 \\ & \ddots & \\ 0 & & \lambda_{\sigma(n)} \end{pmatrix}$$

$$\det(X - \lambda I) = t^n + \underbrace{e_1(\lambda)}_{= \sum \lambda_i} t + e_2(\lambda) t^2 + \dots + \underbrace{e_n(\lambda)}_{= \prod \lambda_i}$$

$e_i = i^{th}$ elem. sym functions of λ's.

- $\mathbb{C}[Y] = \mathbb{C}[\lambda_1, \dots, \lambda_n] \quad \mathbb{C}[e_1, \dots, e_n]$ they are the only ones. S_n ~~[illegible]~~ thm. on elementary symm. poly.

For polynomials of any degree $x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n$ there is a polynomial $\Delta(a_1, \dots, a_n)$ that vanishes iff $f(x)$ has repeated roots.

$$\Delta = \prod_{i<j} (\lambda_i - \lambda_j)^2$$

ex: Find Δ for 3×3 matrices

$$SL_2(\mathbb{C}) \curvearrowright \overset{W}{\overset{\shortparallel}{Sym^d(\mathbb{C}^2)}} \leftarrow W = \mathbb{C}[a_0, \dots, a_d]$$

homog poly. of deg d in x, y

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$A. \sum a_i x^i y^{d-i} = \sum a_i (ax+by)^i (cx+dy)^{d-i}$$
$$= \sum \hat{a}_i x^i y^{d-i}$$

ex: write this down for $d=2$

$$\overline{G w} \ni 0_w \qquad \underset{W \text{ unstable}}{\subset} \mathbb{P}(W) \overset{\pi}{\dashrightarrow} \mathbb{P}(W)/\!/G$$

W is unstable.

π is given by values of homog. invariants.

$$\overline{G w} \not\ni 0_w \qquad \mathbb{P}(W^{ss}) \longrightarrow \mathbb{P}(W^{ss})/\!/G$$

w is semi-stable

if there is a non-constant and non-vanishing homog. inver.

ex: $P(x,y) = \sum a_i x^i y^{d-i} \longleftarrow \Delta(P) = 0 \iff P$ has repeated roots in $\mathbb{P}^1$

$(x,y) \in \mathbb{P}^1 \longleftarrow \Delta$ is an invariant

$$\Delta = Res\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

Polynomials with no repeated roots are semi-stable $\longleftrightarrow \Delta(p) \neq 0$

$$SL_2(n) \curvearrowright \underset{\mathbb{C}^5}{\underset{\shortparallel\wr}{Sym^4(\mathbb{C}^2)}} \cong \mathbb{C}^2 = Sym^4(\mathbb{C})/\!/SL_2(n)$$

$$\mathbb{P}(Sym^4(\mathbb{C}^2)) = \mathbb{P}^1$$

- 4 fixes points on $\mathbb{C} \cup \{\infty\}$ cross ration

$G \hookrightarrow L \leftarrow$ linearization on ample

$G \curvearrowright X \qquad X/\!/G$

for every x, there is a section s of L^m not vanishing at x

for every x, y there is a section s.t. $s(x) = 0$, $s(y) \neq 0$ + no pinching.

$H^0(X, L^{\otimes m})$

$X \longrightarrow \mathbb{P}^N$

$x \longmapsto (s_0(x), \dots, s_N(x))$

closed immersion $\rightsquigarrow$ ample

$$X/\!/G = \mathrm{Proj}\left(\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})\right)$$

$X = \mathbb{P}^N$

$G = SL_n(\mathbb{C})$

$X/\!/G = \emptyset$

$G = \mathbb{C}^* \qquad X/\!/G = \emptyset$

L on smooth proj. curve C = sections which are meromorphic functions on C with poles bounded by $\sum_{\substack{p \in C \\ \text{poles}}} a_p p = D$ divisor on C, $a_p > 0$

Riemann-Roch $\qquad H^0(C, \mathcal{O}_C(D)) - H^1(C, \mathcal{O}_C(D)) = d - g + 1$

$H^0(C, \mathcal{O}_C(K_C - D))$

$d \geq 2g \qquad \mathcal{O}(D)$ very ample

Algebraic Geometry - Joe Harris

Algebraic curves - William Fulton

- What are the fibers of the quotient? What orbits lie in each fiber?

  i $(X-\lambda_1)(X-\lambda_2)$

  ii $(X-\lambda)^2$

What similarity classes have a given characteristic poly?

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2}c & d \end{pmatrix}$$

i all conjugate matrices are conjugate $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ closed by todays lecture

ii what JNF's correspond to this $\chi(A)$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$$

$$-\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}- = \begin{pmatrix} \lambda & t^2 \\ 0 & \lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\overline{SL_2\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}} \supset \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

- matrices distinct eigenvalues have closed orbits

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ & & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ex: show that $\overline{GL(3)N} \supset N'$

N, N' nilpotent matrices $\chi(N) = X^3$

is that JNF of N' is obtained from JNF on N by replacing some 1's with 0's.

use only matrices of form

$$\begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \quad a+b+c=0$$