Hypergeometric Galois Actions

A. Muhammed Uludağ

Galatasaray University, (Istanbul)

March 18, 2020

Monodromy and Hypergeometric Functions International Conference
17-21 February 2020 Galatasaray University, İstanbul, Turkey

15. year of the conference
“Geometry and Arithmetic around Hypergeometric Functions”, Galatasaray University, June 2005

The poster

CIMPA-UNESCO SUMMER SCHOOL AGAHF 2005
ECOLE D'ETE CIMPA-UNESCO AGAHF 2005
Dates-Place / Date et lieu :
June 13-25, Galatasaray University, Istanbul, Turkey
Arithmetic and Geometry Around Hypergeometric Functions
Arithmétique et Géométrie Autour des Fonctions Hypergéométriques
Organizing Committee
Comité d'organisation
Scientific Advisory Board
Comité scientifique
Fellowships / Bourses
Web page / Site internet
Sponsors
CIMPA, UNESCO, TUBITAK, IMU, ICTP, AUF
Lecturers / Conférenciers :
Daniell Allcock (Texas)
Igor Dolgachev (Michigan)
Rolf-Peter Holzapfel (Berlin)
Michel Jambu (Nice)
Anatoly N. Kochubei (Kiev)
Shigeyuki Kondo (Nagoya)
Eduard Looijenga (Utrecht)
Keiji Matsumoto (Hokkaido)
Hironori Shiga (Chiba)
Jan Stienstra (Utrecht)
Toshiaki Terada (Shiga)
A. Muhammed Uludağ (Istanbul)
Alexander Varchenko (Chapel Hill)
Jürgen Wolfart (Frankfurt)
Masaaki Yoshida (Kyushu)
IMU

Talks of the 2005 meeting

D. Allcock: Real hyperbolic geometry in moduli problems
I. Dolgachev: Moduli spaces of K3 surfaces and complex ball quotients
R. P. Holzapfel: Orbital Varieties and Invariants
M. Jambu: Arrangements of Hyperplanes
A. Kasparian: On Holzapfel's Conjecture on Ball-quotient surfaces
A. Kochubei: Hypergeometric functions and Carlitz differential equations over function fields
S. Kondo: Complex ball uniformizations of the moduli spaces of del Pezzo surfaces
E. Looijenga: Hypergeometric functions associated to arrangements
K. Matsumoto: Invariant functions with respect to the Whiteland link
H. Shiga: Hypergeometric functions and arithmetic geometric means
J. Stienstra: Gel'fand-Kapranov-Zelevinsky hypergeometric systems and their role in mirror symmetry and in string theory
T. Terada: Hypergeometric representation of the group of pure braids
A. M. Uludağ: Introduction to coverings of the plane branched along line arrangements and ball-quotient surfaces
A. Varchenko: Special functions, KZ type equations, and representation theory
J. Wolfart: Arithmetic of Schwarz maps (2-4 h)
M. Yoshida: Schwarz maps

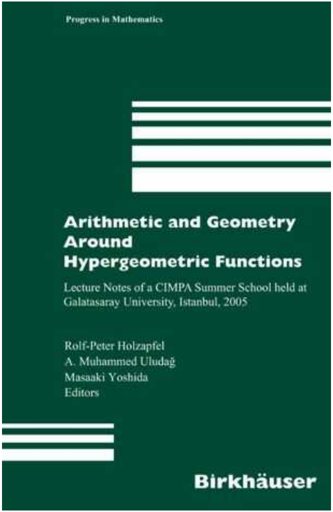
Ortaköy excursion


;

Boat trip


;

Proceedings



;

Why not publish a sequel?

Why not organize a follow-up conference?

My talk is an account of the project “Hypergeometric Galois Actions”, published in 2015:

Hypergeometric Galois Actions

Muhammed Uludağ * *and İsmail Sağlam* **

Galatasaray University
Çırağan Cad. No: 36, 34349, Ortaköy, İstanbul, Turkey
email: muhammed.uludag@gmail.com

Galatasaray University
Çırağan Cad. No: 36, 34349, Ortaköy, İstanbul, Turkey
email: isaglamtrfr@gmail.com

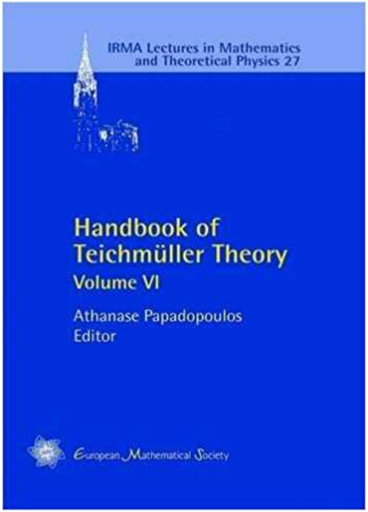
Abstract. We outline a project to study the Galois action on a class of modular graphs (special type of dessins) which arise as the dual graphs of the sphere triangulations of non-negative curvature, classified by Thurston. Because of their connections to hypergeometric functions, there is a hope that these graphs will render themselves to explicit calculation for a study of Galois action on them, unlike the case of a general dessin.

Keywords: Sphere triangulation, hypergeometric functions, dessins, Belyi maps, modular graphs, trivalent ribbon graphs, Galois actions, cone metric, flat structure, euclidean structure, ball quotient, branched covering of the sphere, complex hyperbolic space.

Contents

*Work supported by TÜBİTAK Grant No. 110T690 and a GSU research grant.
**Work partially supported by TÜBİTAK Grant No. 110T690.

;



;

Synopsis:

- Thurston gave in the '80s gave a very concrete and explicit classification of **sphere triangulations of non-negative curvature**. (related to the works of Picard, Terada, Deligne and Mostow on Lauricella's higher-dimensional hypergeometric functions)
- The dual graph of every sphere triangulation is a kind of dessin and determines a covering of the sphere branched at three points. This covering is defined over $\overline{\mathbb{Q}}$. Grothendieck initiated a program to study the action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on these covers to understand the structure of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.
- This program have largely failed, because a general dessin is a combinatorial object and it is hard to study them the point of view of algebra and arithmetic.
- Dessins originating from Thurston's sphere triangulations are special and there is a hope that they are amenable to study from the point of view of the Galois action. This project turns out to be simpler then expected.
- There are various questions pertaining to these covers. Our aim is to expose these questions and also suggest some ways to go beyond these hypergeometric triangulations.

Synopsis:

- Thurston gave in the '80s gave a very concrete and explicit classification of **sphere triangulations of non-negative curvature**. (related to the works of Picard, Terada, Deligne and Mostow on Lauricella's higher-dimensional hypergeometric functions)
- The dual graph of every sphere triangulation is a kind of dessin and determines a covering of the sphere branched at three points. This covering is defined over $\overline{\mathbf{Q}}$. Grothendieck initiated a program to study the action of the Galois group $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ on these covers to understand the structure of $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$.
- This program have largely failed, because a general dessin is a combinatorial object and it is hard to study them the point of view of algebra and arithmetic.
- Dessins originating from Thurston's sphere triangulations are special and there is a hope that they are amenable to study from the point of view of the Galois action. This project turns out to be simpler then expected.
- There are various questions pertaining to these covers. Our aim is to expose these questions and also suggest some ways to go beyond these hypergeometric triangulations.

Synopsis:

- Thurston gave in the '80s gave a very concrete and explicit classification of **sphere triangulations of non-negative curvature**. (related to the works of Picard, Terada, Deligne and Mostow on Lauricella's higher-dimensional hypergeometric functions)
- The dual graph of every sphere triangulation is a kind of dessin and determines a covering of the sphere branched at three points. This covering is defined over $\overline{\mathbf{Q}}$. Grothendieck initiated a program to study the action of the Galois group $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ on these covers to understand the structure of $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$.
- This program have largely failed, because a general dessin is a combinatorial object and it is hard to study them the point of view of algebra and arithmetic.
- Dessins originating from Thurston's sphere triangulations are special and there is a hope that they are amenable to study from the point of view of the Galois action. This project turns out to be simpler then expected.
- There are various questions pertaining to these covers. Our aim is to expose these questions and also suggest some ways to go beyond these hypergeometric triangulations.

Synopsis:

- Thurston gave in the '80s gave a very concrete and explicit classification of **sphere triangulations of non-negative curvature**. (related to the works of Picard, Terada, Deligne and Mostow on Lauricella's higher-dimensional hypergeometric functions)
- The dual graph of every sphere triangulation is a kind of dessin and determines a covering of the sphere branched at three points. This covering is defined over $\overline{\mathbf{Q}}$. Grothendieck initiated a program to study the action of the Galois group $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ on these covers to understand the structure of $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$.
- This program have largely failed, because a general dessin is a combinatorial object and it is hard to study them the point of view of algebra and arithmetic.
- Dessins originating from Thurston's sphere triangulations are special and there is a hope that they are amenable to study from the point of view of the Galois action. This project turns out to be simpler then expected.
- There are various questions pertaining to these covers. Our aim is to expose these questions and also suggest some ways to go beyond these hypergeometric triangulations.

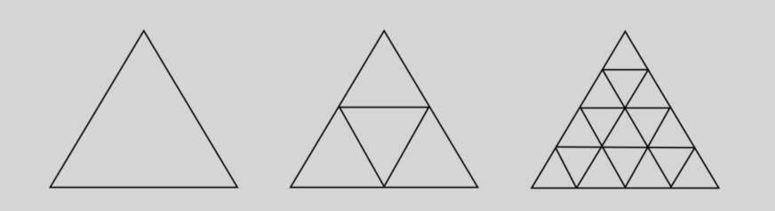
Synopsis:

- Thurston gave in the '80s gave a very concrete and explicit classification of **sphere triangulations of non-negative curvature**. (related to the works of Picard, Terada, Deligne and Mostow on Lauricella's higher-dimensional hypergeometric functions)
- The dual graph of every sphere triangulation is a kind of dessin and determines a covering of the sphere branched at three points. This covering is defined over $\overline{\mathbf{Q}}$. Grothendieck initiated a program to study the action of the Galois group $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ on these covers to understand the structure of $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$.
- This program have largely failed, because a general dessin is a combinatorial object and it is hard to study them the point of view of algebra and arithmetic.
- Dessins originating from Thurston's sphere triangulations are special and there is a hope that they are amenable to study from the point of view of the Galois action. This project turns out to be simpler then expected.
- There are various questions pertaining to these covers. Our aim is to expose these questions and also suggest some ways to go beyond these hypergeometric triangulations.

Sphere triangulations

Given a triangulation of the sphere (or of any surface), we may identify each triangle by the Euclidean equilateral triangle (Shabat-Voevodsky), thereby obtaining a flat metric with some singular points on the sphere.

Simplest triangulations looks like this:



(imagine that two copies of the figures are glued along their boundaries)

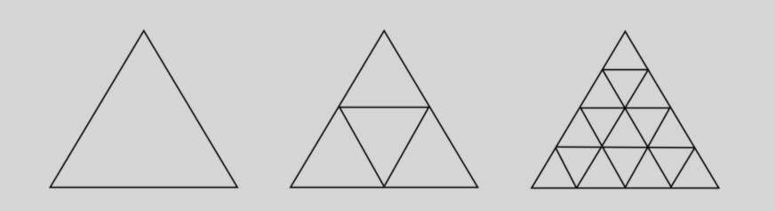
Where are the singular points?

Note that these are obtained from the first triangulation by a subdivison operation, which does not change the singular points.

Sphere triangulations

Given a triangulation of the sphere (or of any surface), we may identify each triangle by the Euclidean equilateral triangle (Shabat-Voevodsky), thereby obtaining a flat metric with some singular points on the sphere.

Simplest triangulations looks like this:



(imagine that two copies of the figures are glued along their boundaries)

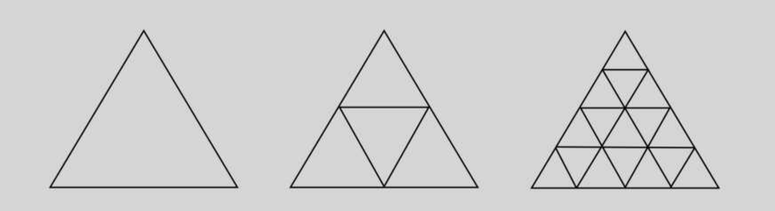
Where are the singular points?

Note that these are obtained from the first triangulation by a subdivison operation, which does not change the singular points.

Sphere triangulations

Given a triangulation of the sphere (or of any surface), we may identify each triangle by the Euclidean equilateral triangle (Shabat-Voevodsky), thereby obtaining a flat metric with some singular points on the sphere.

Simplest triangulations looks like this:



(imagine that two copies of the figures are glued along their boundaries)

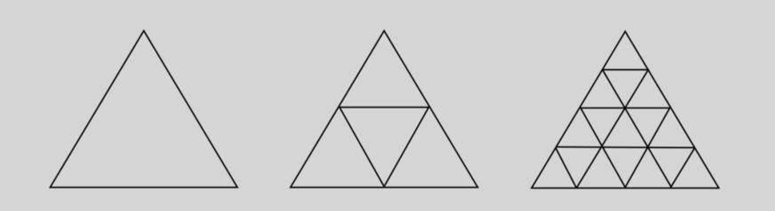
Where are the singular points?

Note that these are obtained from the first triangulation by a subdivison operation, which does not change the singular points.

Sphere triangulations

Given a triangulation of the sphere (or of any surface), we may identify each triangle by the Euclidean equilateral triangle (Shabat-Voevodsky), thereby obtaining a flat metric with some singular points on the sphere.

Simplest triangulations looks like this:



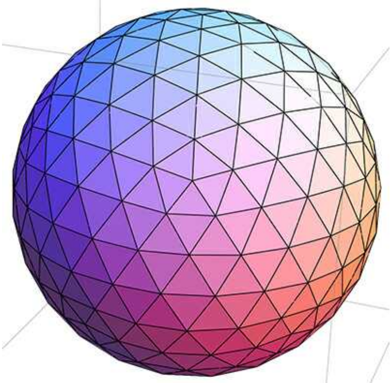
(imagine that two copies of the figures are glued along their boundaries)

Where are the singular points?

Note that these are obtained from the first triangulation by a subdivison operation, which does not change the singular points.

Sphere triangulations

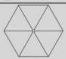
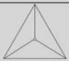
Here is a slightly more complicated triangulation:



Where are the singular points?

(there must be 12 singular points of the type above)

Cone singularities of the metric

cone picture	d (vertex degree)	$6-d$	κ (curvature)	$\theta = 2\pi - \kappa$ (cone angle)
	6	0	$\kappa = 0$	2π
	5	1	$\kappa = \frac{2\pi}{6} = \frac{\pi}{3}$	$\frac{5\pi}{3}$
	4	2	$\kappa = \frac{4\pi}{6} = \frac{2\pi}{3}$	$\frac{4\pi}{3}$
	3	3	$\kappa = \frac{6\pi}{6} = \pi$	π
	2	4	$\kappa = \frac{8\pi}{6} = \frac{4\pi}{3}$	$\frac{2\pi}{3}$
	1	5	$\kappa = \frac{10\pi}{6} = \frac{5\pi}{3}$	$\frac{\pi}{3}$

(If more then 6 triangles meet at a vertex, then the curvature is negative)

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{Aut}(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$, where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{A}ut(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$,where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{Aut}(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$, where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{Aut}(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$,where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{Aut}(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$,where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{Aut}(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$,where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{Aut}(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$,where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

Let $\mathbf{C}^{(1,9)}$ be the complex Lorenz space, i.e. $\mathbf{C}^{10}$ with a Hermitian form of signature (1,9).

Theorem (Thurston) *There is a lattice $\mathcal{L}$ in $\mathbf{C}^{(1,9)}$ and a group $\Gamma_{DM} \subset \mathrm{Aut}(\mathcal{L})$, such that sphere triangulations of curvature ≥ 0 are elements of $\mathcal{L}_+/\Gamma_{DM}$,where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The square norm of a lattice point is the number of triangles in the triangulation. The action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume.*

Denote by

$$\Phi : \mathbf{C}\mathbb{H}^9 \to \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

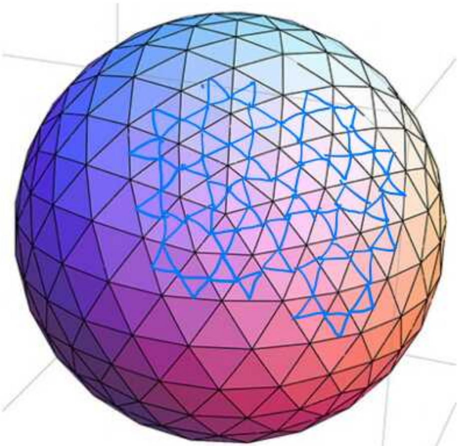
the quotient map. Its inverse is given by Lauricella hypergeometric functions.

("DM" stands for "Deligne-Mostow")

We shall call these triangulations "HG triangulations"

HG triangulations are lattice points

HG triangulations lying on the same line through the origin are simultaneous subdivisions of a "primitive" triangulation on the line and therefore define isometric polyhedra.



HG triangulations are lattice points

Hence the projectivization

$$\mathbb{P}\mathcal{L}_+/\Gamma_{DM} \subset \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

classifies the isometry classes of polyhedra, where $\mathcal{M}_{DM}$ is the ball-quotient space $\mathbf{C}\mathbb{H}^9/\Gamma_{DM}$.

We shall call these "HG points" of the moduli space.

Thurston also describes a very explicit method to construct these triangulations and gives the estimation $O(n^{10})$ for the number of triangulations in $\mathcal{L}_+$ with up to $2n$ triangles.

HG triangulations are lattice points

Hence the projectivization

$$\mathbb{P}\mathcal{L}_+/\Gamma_{DM} \subset \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

classifies the isometry classes of polyhedra, where $\mathcal{M}_{DM}$ is the ball-quotient space $\mathbf{C}\mathbb{H}^9/\Gamma_{DM}$.

We shall call these "HG points" of the moduli space.

Thurston also describes a very explicit method to construct these triangulations and gives the estimation $O(n^{10})$ for the number of triangulations in $\mathcal{L}_+$ with up to $2n$ triangles.

HG triangulations are lattice points

Hence the projectivization

$$\mathbb{P}\mathcal{L}_+/\Gamma_{DM} \subset \mathcal{M}_{DM} := \mathbf{C}\mathbb{H}^9/\Gamma_{DM}$$

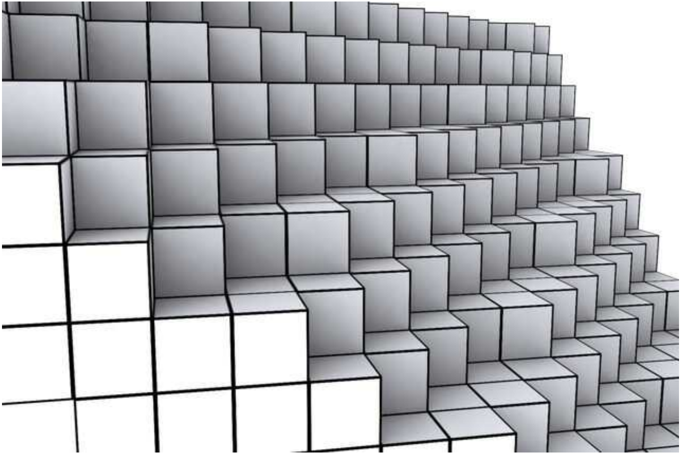
classifies the isometry classes of polyhedra, where $\mathcal{M}_{DM}$ is the ball-quotient space $\mathbf{C}\mathbb{H}^9/\Gamma_{DM}$.

We shall call these "HG points" of the moduli space.

Thurston also describes a very explicit method to construct these triangulations and gives the estimation $O(n^{10})$ for the number of triangulations in $\mathcal{L}_+$ with up to $2n$ triangles.

Remark: Sphere quadrangulations

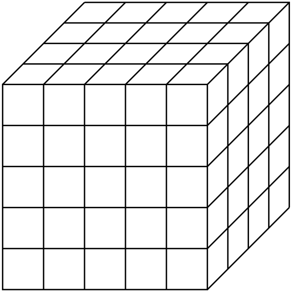
One may also consider sphere quadrangulations, assuming that each quadrangle is an Euclidean square..



What are the singular points? Which ones are of positive, negative and zero curvature?

HG Quadrangulations are lattice points

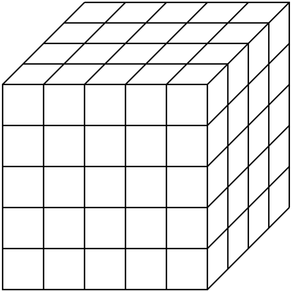
Theorem (Ayberk Zeytin) *(Quadrangulations are lattice points) There is a lattice $\mathcal{L}$ in complex Lorenz space $\mathbf{C}^{(1,8)}$ and a group Γ_{DM} of automorphisms,* *such that quadrangulations of non-negative combinatorial curvature are elements of $\mathcal{L}_+/\Gamma_{DM}$, where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The projective action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume. The square of the norm of a lattice point is the number of quadrangles in the quadrangulation.*



A HG sphere quadrangulation

HG Quadrangulations are lattice points

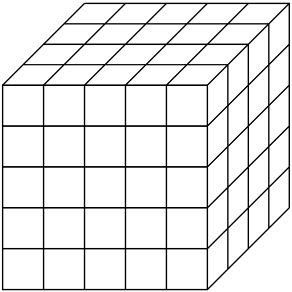
Theorem (Ayberk Zeytin) *(Quadrangulations are lattice points) There is a lattice $\mathcal{L}$ in complex Lorenz space $\mathbf{C}^{(1,8)}$ and a group Γ_{DM} of automorphisms, such that quadrangulations of non-negative combinatorial curvature are elements of $\mathcal{L}_+/\Gamma_{DM}$, where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The projective action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume. The square of the norm of a lattice point is the number of quadrangles in the quadrangulation.*



A HG sphere quadrangulation

HG Quadrangulations are lattice points

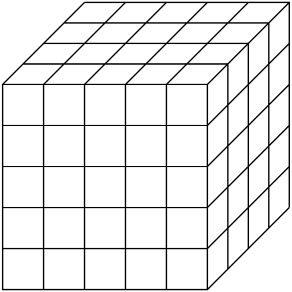
Theorem (Ayberk Zeytin) *(Quadrangulations are lattice points) There is a lattice $\mathcal{L}$ in complex Lorenz space $\mathbf{C}^{(1,8)}$ and a group Γ_{DM} of automorphisms, such that quadrangulations of non-negative combinatorial curvature are elements of $\mathcal{L}_+/\Gamma_{DM}$, where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The projective action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume. The square of the norm of a lattice point is the number of quadrangles in the quadrangulation.*



A HG sphere quadrangulation

HG Quadrangulations are lattice points

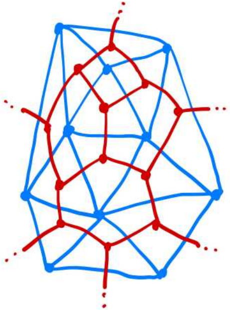
Theorem (Ayberk Zeytin) *(Quadrangulations are lattice points) There is a lattice $\mathcal{L}$ in complex Lorenz space $\mathbf{C}^{(1,8)}$ and a group Γ_{DM} of automorphisms, such that quadrangulations of non-negative combinatorial curvature are elements of $\mathcal{L}_+/\Gamma_{DM}$, where $\mathcal{L}_+$ is the set of lattice points of positive square-norm. The projective action of Γ_{DM} on complex projective hyperbolic space $\mathbf{C}\mathbb{H}^9$ (the unit ball in $\mathbf{C}^9 \subset \mathbf{C}\mathbb{P}^9$) has quotient of finite volume. The square of the norm of a lattice point is the number of quadrangles in the quadrangulation.*



A HG sphere quadrangulation

Triangulations are dessins

From a triangulation (blue) we produce a modular graph (red) (kind of dessin) as follows:



This modular graph determines a covering of the sphere branched at $0, 1, \infty$, branched with index 1 or 2 above 0, with index 1 or 3 above 1.

For example, if the triangulation is just , then covering determined by it, is Galois of degree 6.

Triangulations are branched covers

If the triangulation is HG, then the branching above ∞ can have index 1,2,3,4,5 or 6.

Problem: Classify all covers $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that f has ramification index 2 at each fiber above $0 \in \mathbb{P}^1$, ramification index 3 at each fiber above $1 \in \mathbb{P}^1$ and has $k_i \geq 0$ points of ramification index i above $\infty \in \mathbb{P}^1$ for $i = 1, 2, 3, \ldots$.

Thurston's classification of HG triangulations completely solves this problem, under the assumption that $k_i = 0$ for $i \geq 7$.

This amounts to the classification of subgroups of the modular group $\mathrm{PSL}_2(\mathbf{Z})$ satisfying a certain regularity condition.

Triangulations are branched covers

If the triangulation is HG, then the branching above ∞ can have index 1,2,3,4,5 or 6.

Problem: Classify all covers $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that f has ramification index 2 at each fiber above $0 \in \mathbb{P}^1$, ramification index 3 at each fiber above $1 \in \mathbb{P}^1$ and has $k_i \geq 0$ points of ramification index i above $\infty \in \mathbb{P}^1$ for $i = 1, 2, 3, \ldots$.

Thurston's classification of HG triangulations completely solves this problem, under the assumption that $k_i = 0$ for $i \geq 7$.

This amounts to the classification of subgroups of the modular group $\mathrm{PSL}_2(\mathbf{Z})$ satisfying a certain regularity condition.

Triangulations are branched covers

If the triangulation is HG, then the branching above ∞ can have index 1,2,3,4,5 or 6.

Problem: Classify all covers $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that f has ramification index 2 at each fiber above $0 \in \mathbb{P}^1$, ramification index 3 at each fiber above $1 \in \mathbb{P}^1$ and has $k_i \geq 0$ points of ramification index i above $\infty \in \mathbb{P}^1$ for $i = 1, 2, 3, \ldots$.

Thurston's classification of HG triangulations completely solves this problem, under the assumption that $k_i = 0$ for $i \geq 7$.

This amounts to the classification of subgroups of the modular group $\mathrm{PSL}_2(\mathbf{Z})$ satisfying a certain regularity condition.

Triangulations are branched covers

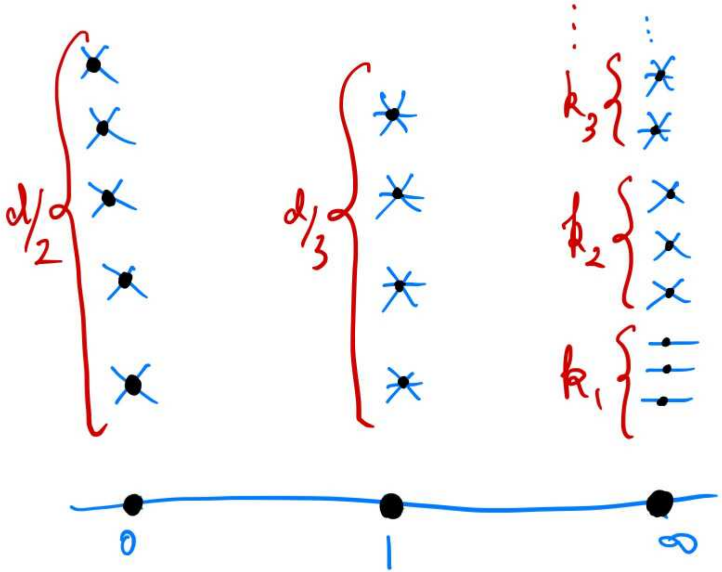
If the triangulation is HG, then the branching above ∞ can have index 1,2,3,4,5 or 6.

Problem: Classify all covers $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that f has ramification index 2 at each fiber above $0 \in \mathbb{P}^1$, ramification index 3 at each fiber above $1 \in \mathbb{P}^1$ and has $k_i \geq 0$ points of ramification index i above $\infty \in \mathbb{P}^1$ for $i = 1, 2, 3, \dots$.

Thurston's classification of HG triangulations completely solves this problem, under the assumption that $k_i = 0$ for $i \geq 7$.

This amounts to the classification of subgroups of the modular group $\mathrm{PSL}_2(\mathbf{Z})$ satisfying a certain regularity condition.

Triangulations are branched covers



Triangulations are branched covers

Suppose f is of degree d. The Riemann-Hurwitz formula yields

$$2 = e(\mathbb{P}^1) = d \cdot e(\mathbb{P}^1 \backslash \{0,1,\infty\}) + \frac{d}{2} + \frac{d}{3} + \sum_{i=1}^{\infty} k_i = -\frac{d}{6} + \sum_{i=1}^{\infty} k_i \quad (1)$$

where $e(\mathbb{P}^1 \backslash \{0,1,\infty\}) = -1$ is the Euler characteristic. Since $\sum_{i=1}^{\infty} ik_i = d$, one has

$$\sum_{i=1}^{\infty} (6-i)k_i = 12 \quad (2)$$

Suppose $k_i = 0$ for $i \geq 7$ and note that the number k_6 does not have any effect in the above formula. The set of tuples $(k_1, \ldots k_n)$ satisfying the formula is precisely the Deligne-Mostow list.

Triangulations are branched covers

Suppose f is of degree d. The Riemann-Hurwitz formula yields

$$2 = e(\mathbb{P}^1) = d \cdot e(\mathbb{P}^1 \backslash \{0, 1, \infty\}) + \frac{d}{2} + \frac{d}{3} + \sum_{i=1}^{\infty} k_i = -\frac{d}{6} + \sum_{i=1}^{\infty} k_i \quad (1)$$

where $e(\mathbb{P}^1 \backslash \{0, 1, \infty\}) = -1$ is the Euler characteristic. Since $\sum_{i=1}^{\infty} ik_i = d$, one has

$$\sum_{i=1}^{\infty} (6 - i) k_i = 12 \quad (2)$$

Suppose $k_i = 0$ for $i \geq 7$ and note that the number k_6 does not have any effect in the above formula. The set of tuples $(k_1, \ldots k_n)$ satisfying the formula is precisely the Deligne-Mostow list.

Triangulations are branched covers

dim	k_1	k_2	k_3	k_4	k_5	deg	Compact?	Number	Pure?	ar?
9	0	0	0	0	12	2	N	10	I	AR
8	0	0	0	1	10	2	N	11	I	AR
7	0	0	1	0	9	2	N	12	I	AR
7	0	0	0	2	8	2	N	13	I	AR
6	0	1	0	0	8	2	N	14	I	AR
6	0	0	1	1	7	2	N	15	I	AR
5	1	0	0	0	7	2	N	16	I	AR
6	0	0	0	3	6	2	N	17	I	AR
5	0	1	0	1	6	2	N	18	I	AR
5	0	0	2	0	6	2	N	19	I	AR
5	0	0	1	2	5	2	N	20	I	AR
4	1	0	0	1	5	2	N	22	I	AR
4	0	1	1	0	5	2	N	23	I	AR
5	0	0	0	4	4	2	N	24	I	AR
4	0	0	2	1	4	2	N	25	I	AR
3	1	0	1	0	4	2	N	26	I	AR
3	0	2	0	0	4	2	N	27	I	AR
4	0	0	1	3	3	2	N	28	I	AR
3	1	0	0	2	3	2	N	29	I	AR
3	0	1	1	1	3	2	N	30	I	AR
3	0	0	3	0	3	2	N	31	I	AR
3	0	0	0	6	0	2	N	1	P	AR
2	0	1	0	4	0	2	N	2	P	AR

A conjecture

We conjectured the following in 2015:

Conjecture. The set of "shapes" of triangulations $\mathcal{L}_+/\Gamma_{\mathrm{DM}} \subset \mathcal{M}_{\mathrm{DM}}$ is defined over $\overline{\mathbb{Q}}$, and the Galois actions on the shapes and the triangulations of the same shape, viewed as dessins, are compatible.

This conjecture have been proved in 2019 by Engels, who moreover determined the fields of definition of the corresponding dessins

Theorem 1.1. *Let $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ be a Belyi map with branch profile $(3^{2d}, 2^{3d}, \mu)$ and all $\mu_i \leq 6$. Then f is defined over the maximal abelian extension of $\mathbb{Q}[\zeta_6]$. Similarly if f has profile $(2^{2d}, 4^d, \mu)$ with all $\mu_i \leq 4$, then f is defined over the maximal abelian extension of $\mathbb{Q}[i]$.*

It turns out that they are not very interesting from the point of view of Galois theory. So, one may declare the "hypergeometric Galois actions" project a failure as well.

A conjecture

We conjectured the following in 2015:

Conjecture. The set of "shapes" of triangulations $\mathcal{L}_+/\Gamma_{\mathrm{DM}} \subset \mathcal{M}_{\mathrm{DM}}$ is defined over $\overline{\mathbb{Q}}$, and the Galois actions on the shapes and the triangulations of the same shape, viewed as dessins, are compatible.

This conjecture have been proved in 2019 by Engels, who moreover determined the fields of definition of the corresponding dessins

Theorem 1.1. *Let $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ be a Belyi map with branch profile $(3^{2d}, 2^{3d}, \mu)$ and all $\mu_i \leq 6$. Then f is defined over the maximal abelian extension of $\mathbb{Q}[\zeta_6]$. Similarly if f has profile $(2^{2d}, 4^d, \mu)$ with all $\mu_i \leq 4$, then f is defined over the maximal abelian extension of $\mathbb{Q}[i]$.*

It turns out that they are not very interesting from the point of view of Galois theory. So, one may declare the "hypergeometric Galois actions" project a failure as well.

A conjecture

We conjectured the following in 2015:

Conjecture. The set of "shapes" of triangulations $\mathcal{L}_+/\Gamma_{\mathrm{DM}} \subset \mathcal{M}_{\mathrm{DM}}$ is defined over $\overline{\mathbb{Q}}$, and the Galois actions on the shapes and the triangulations of the same shape, viewed as dessins, are compatible.

This conjecture have been proved in 2019 by Engels, who moreover determined the fields of definition of the corresponding dessins

Theorem 1.1. *Let $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ be a Belyi map with branch profile $(3^{2d}, 2^{3d}, \mu)$ and all $\mu_i \leq 6$. Then f is defined over the maximal abelian extension of $\mathbb{Q}[\zeta_6]$. Similarly if f has profile $(2^{2d}, 4^d, \mu)$ with all $\mu_i \leq 4$, then f is defined over the maximal abelian extension of $\mathbb{Q}[i]$.*

It turns out that they are not very interesting from the point of view of Galois theory. So, one may declare the "hypergeometric Galois actions" project a failure as well.

Complex multiplication

Note the analogy with a result from class field theory: The modular function $j(\tau) : \mathcal{H} \to \mathbf{C}$ is algebraic on imaginary quadratic numbers τ (complex multiplication points). In our case, τ is a lattice point, so it is defined over $\mathbf{Q}[\zeta_6]$, and its image is defined over a maximal abelian extension of $\mathbf{Q}[\zeta_6]$.

Shiga and Wohlfart studied the algebraic values of the map Φ and obtained some results parallel with the class field theory.

Complex multiplication

Note the analogy with a result from class field theory: The modular function $j(\tau) : \mathcal{H} \to \mathbf{C}$ is algebraic on imaginary quadratic numbers τ (complex multiplication points). In our case, τ is a lattice point, so it is defined over $\mathbf{Q}[\zeta_6]$, and its image is defined over a maximal abelian extension of $\mathbf{Q}[\zeta_6]$.

Shiga and Wohlfart studied the algebraic values of the map Φ and obtained some results parallel with the class field theory.

Complex multiplication

Note the analogy with a result from class field theory: The modular function $j(\tau) : \mathcal{H} \to \mathbf{C}$ is algebraic on imaginary quadratic numbers τ (complex multiplication points). In our case, τ is a lattice point, so it is defined over $\mathbf{Q}[\zeta_6]$, and its image is defined over a maximal abelian extension of $\mathbf{Q}[\zeta_6]$.

Shiga and Wohlfart studied the algebraic values of the map Φ and obtained some results parallel with the class field theory.

Beyond hypergeometric

It must be possible to extend the classification of results of triangulations of non-negative curvature to more general triangulations (and quadrangulations). To achieve this, we need the right conditions to control the curvature. Some suggestions:

- "just one point of negative curvature above infinity"
- "just one point of negative curvature above infinity, whose curvature is bounded below by κ",
- "just one point of fixed curvature κ above infinity" (in each case, the points of non-negative curvature are arbitrary).
- A fixed number of points with controlled negative curvature.

These relaxed conditions may bring in non-discrete groups into the picture, the signatures of the Hermitian forms will change, complex hyperbolic structure will decay, and there is a possibility that the parameter spaces will brake up into disconnected components.

One may also consider hyperbolic triangulations with cone points

Beyond hypergeometric

It must be possible to extend the classification of results of triangulations of non-negative curvature to more general triangulations (and quadrangulations). To achieve this, we need the right conditions to control the curvature. Some suggestions:

- "just one point of negative curvature above infinity"
- "just one point of negative curvature above infinity, whose curvature is bounded below by κ ",
- "just one point of fixed curvature κ above infinity" (in each case, the points of non-negative curvature are arbitrary).
- A fixed number of points with controlled negative curvature.

These relaxed conditions may bring in non-discrete groups into the picture, the signatures of the Hermitian forms will change, complex hyperbolic structure will decay, and there is a possibility that the parameter spaces will brake up into disconnected components.

One may also consider hyperbolic triangulations with cone points

Beyond hypergeometric

It must be possible to extend the classification of results of triangulations of non-negative curvature to more general triangulations (and quadrangulations). To achieve this, we need the right conditions to control the curvature. Some suggestions:

- "just one point of negative curvature above infinity"
- "just one point of negative curvature above infinity, whose curvature is bounded below by κ ",
- "just one point of fixed curvature κ above infinity" (in each case, the points of non-negative curvature are arbitrary).
- A fixed number of points with controlled negative curvature.

These relaxed conditions may bring in non-discrete groups into the picture, the signatures of the Hermitian forms will change, complex hyperbolic structure will decay, and there is a possibility that the parameter spaces will brake up into disconnected components.

One may also consider hyperbolic triangulations with cone points

Beyond hypergeometric

It must be possible to extend the classification of results of triangulations of non-negative curvature to more general triangulations (and quadrangulations). To achieve this, we need the right conditions to control the curvature. Some suggestions:

- “just one point of negative curvature above infinity”
- “just one point of negative curvature above infinity, whose curvature is bounded below by κ ”,
- “just one point of fixed curvature κ above infinity” (in each case, the points of non-negative curvature are arbitrary).
- A fixed number of points with controlled negative curvature.

These relaxed conditions may bring in non-discrete groups into the picture, the signatures of the Hermitian forms will change, complex hyperbolic structure will decay, and there is a possibility that the parameter spaces will brake up into disconnected components.

One may also consider hyperbolic triangulations with cone points

Beyond hypergeometric

It must be possible to extend the classification of results of triangulations of non-negative curvature to more general triangulations (and quadrangulations). To achieve this, we need the right conditions to control the curvature. Some suggestions:

- "just one point of negative curvature above infinity"
- "just one point of negative curvature above infinity, whose curvature is bounded below by κ ",
- "just one point of fixed curvature κ above infinity" (in each case, the points of non-negative curvature are arbitrary).
- A fixed number of points with controlled negative curvature.

These relaxed conditions may bring in non-discrete groups into the picture, the signatures of the Hermitian forms will change, complex hyperbolic structure will decay, and there is a possibility that the parameter spaces will brake up into disconnected components.

One may also consider hyperbolic triangulations with cone points

Beyond hypergeometric

It must be possible to extend the classification of results of triangulations of non-negative curvature to more general triangulations (and quadrangulations). To achieve this, we need the right conditions to control the curvature. Some suggestions:

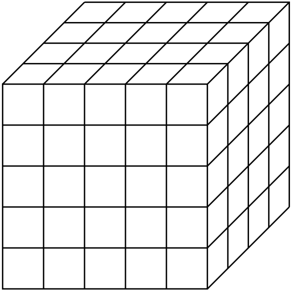
- "just one point of negative curvature above infinity"
- "just one point of negative curvature above infinity, whose curvature is bounded below by κ ",
- "just one point of fixed curvature κ above infinity" (in each case, the points of non-negative curvature are arbitrary).
- A fixed number of points with controlled negative curvature.

These relaxed conditions may bring in non-discrete groups into the picture, the signatures of the Hermitian forms will change, complex hyperbolic structure will decay, and there is a possibility that the parameter spaces will brake up into disconnected components.

One may also consider hyperbolic triangulations with cone points

Beyond hypergeometric

If we further relax the control of the points of negative curvature by simply requiring that it be bounded globally from below, then things will totally go out of control.



THANKS!