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Abstract

Zeta functions have been introduced to the world of mathematics by Riemann, see [3], almost 150 years ago. Yet many "properties" and many relations concerning zeta functions remains conjectural. In this work we have studied a combinatorial zeta function introduced to the literature by Yasutaka Ihara in the 1960s.

In the first chapter, we study some preliminary in Graph Theory. After that, we give the definition of the Ihara Zeta Function. In the next sections, we define the Riemann Hypothesis for regular and irregular graphs and relate with the original Riemann Hypothesis. Next, in this chapter, we calculate with Mathematica the poles of the Ihara Zeta Functions of some irregular graphs and examined the locations of the poles.

In the second chapter, we consider a special family of covering graphs. We describe them and calculate their Ihara Zeta Functions. Also, we calculate the derivatives of their Ihara Zeta Functions to investigate their ramification points. Furthermore, we examine the multiplicative inverses of these Ihara Zeta Functions. Finally, we propose a conjecture which we work for its proof.

Keywords: Graph Theory, Riemann Hypothesis, Zeta Function, Riemann, Ihara, Ramanujan Graph.

I dedicate this thesis
in honor of my grandfather
FİKRET ACAR ÇALIŞAL
and
in memory of my grandfather
YAŞAR PORTAKAL

Contents

Chapter 1

The Ihara Zeta Function and the Graph Theory Prime Number Theorem

Usual Hypotheses: Our graphs will be finite, connected and undirected. It will usually be assumed that they contain no degree 1 vertices (called "leaves" or "hair" or "danglers"). We will also usually assume the graphs are not cycles or cycles with hair. However, we will allow our graphs to have loops and multiple edges.

1.1 The Ihara Zeta Function of a Weighted Graph

Definition 1.1.1 *A graph G is a pair $G = (V, E)$ comprising a nonempty set V of vertices and together with a set E of edges which are 2-element subsets of V.*

Definition 1.1.2 *A walk is a sequence of vertices and edges such that the vertices and edges are adjacent. A trail is a walk in which all edges are distinct and a path is a trail in which all vertices are distinct.*

Definition 1.1.3 *For a graph X with oriented edge set $\overrightarrow{E}$, consisting of $2|E|$ oriented edges, suppose we have a weighting function $L : \overrightarrow{E} \to \mathbb{R}_+$. Then define the weighted length of a closed path $C = a_1a_2 \ldots a_s$ where a_j $in \overrightarrow{E}$, by*

$$v(C, L) = v_X(C, L) = \sum_{i=0}^{s} L(a_i)$$

Definition 1.1.4 *For a closed path $C = a_1 \dots a_s$, the equivalence class $[C]$ is*

$$[C] = \{a_1 \dots a_s, a_2 \dots a_s a_1, \dots, a_s a_1 \dots a_{s-1}\}$$

That is, we call two paths equivalent if we get one from the other by changing the starting point.

Definition 1.1.5 *For $|u|$ small and $u \notin (-\infty, 0)$ the Ihara zeta function of a weighted (undirected) graph X as*

$$\zeta_X(u, L) = \prod_{[P]} (1 - u^{v(P,L)})^{-1}$$

where [P] denotes the equivalence class of a closed prime path P in X.

In particular, if $v : \vec{E} \longrightarrow \mathbb{R}_+$ is chosen to assign the value 1 to every oriented edge in $\vec{E}$, then

$$\zeta_X(u, 1) = \zeta_X(u)$$

is called the Ihara zeta function.

A path or walk $C = a_1 \dots a_s$ where a_j is an oriented or directed edge of X, is said to have a *backtrack* if $a_{j+1} = a_j^{-1}$ for some $j = 1, \dots, s-1$. A path C is said to have a tail if $a_s = a_1^{-1}$. The length of $C = a_1 \dots a_s$ is $s = v_X(C, L)$. A closed path is a path whose starting vertex is the same as its terminal vertex. A closed path $C = a_1 \dots as$ is called *primitive* or *prime* if it has no backtrack or tail and $C \neq D^f$ for $f > 1$; where D is an arbitrary path in X and D^f is a path which repeats D f times. A prime in the graph X is an equivalence class $[C]$ of prime paths.

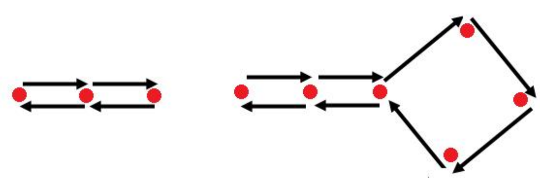

Figure 1.1: *Example of a backtrack and a tail*

Definition 1.1.6 *Primes in graphs are equivalence classes [C] of closed bactrackless tailless primitive paths C.*

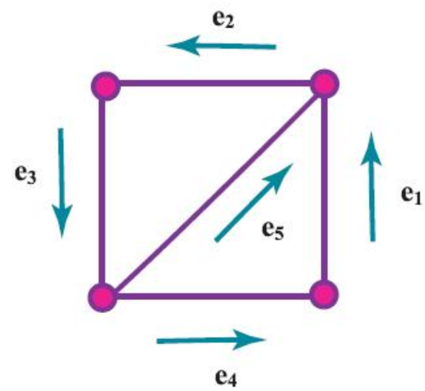

Figure 1.2: *We choose an arbitrary orientation of the edges of a graph. Then we label the inverse edges via $e_j + |E| = e_j^{-1}$, for j=1...5*

Example 1 *For the graph in Figure 1.2, we have primes $[C] = [e_2, e_3, e_5]$, $[D] = [e_1, e_2, e_3, e_4]$, $[E] = [e_1, e_2, e_3, e_4, e_1, e_{10}, e_4]$. Here $e_{10} = e_5^{-1}$ and the lengths of these primes are: $v(C) = 3$, $v(D) = 4$, $v(E) = 7$.*
We have infinitely many primes since $E_n = [(e_1, e_2, e_3, e_4)^n e_1, e_{10}, e_4]$ is prime for all $n \geq 1$. We don't have unique factorization into primes. The only nonprimes are powers of primes.

Definition 1.1.7 *Given a graph X with positive integer-valued weight function L, define the inflated graph X_L in which each edge e is replaced by an edge with L(e)-1 new degree 2 vertices.*

Then clearly, $v_X(C, L) = v_{X_L}(C, 1)$, where the 1 means again that $1(e) = 1$, for all edges e. It follows that for positive integer-valued weights L, we have the identity relating the weighted zeta and the ordinary Ihara zeta:

$$\zeta_X(u, L) = \zeta_{X_L}(u)$$

It follows that $\zeta_X(u, L)^{-1}$ is a polynomial for integer valued weights L.

Example 2 *(Inflation of K_5) Suppose $Y = K_5$, the complete graph on 5 vertices. Let L(e)=5 for each of the 10 edges of X. Then $X = Y_L$ is the graph on the left in Figure 1.3. The new graph X has 45 vertices (4 new vertices on the 10 edges of K_5). On sees that*

$$\zeta_X(u)^{-1} = \zeta_{K_5}(u^5)^{-1} = (1 - u^{10})^5(1 - 3u^5)(1 - u^5)(1 + u^5 + 3u^{10})$$

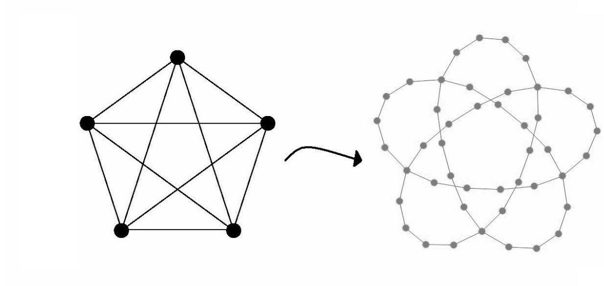

Figure 1.3: *Inflation of* K_5

1.2 Regular Graphs, Locations of Poles of Zeta, Functional Equations

Now, we want to consider the Ihara zeta function of regular graphs which are unweighted and satisfy our usual hypotheses for the most part. We need some definitions from the graph theory first.

Definition 1.2.1 *A graph is called a bipartite graph whenever the set of vertices can be partitioned into 2 disjoint sets* S, T *such that no vertex in* S *is adjacent to any other vertex in* S *and no vertex in* T *is adjacent to any other vertex in* T.

Example 3 *The 3-regular cube is an example of a bipartite graph.*

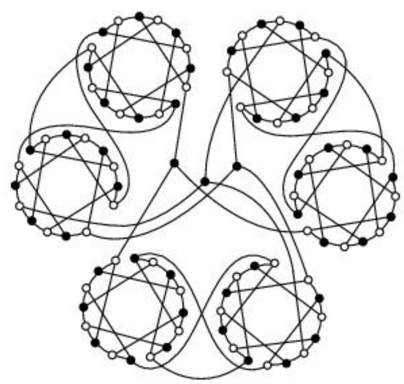

Figure 1.4: *Horton Graph: 3-regular cubic and bipartite graph with 96 vertices and 144 edges.*

Definition 1.2.2 *The adjacency matrix of a finite graph G on n vertices is the $n \times n$ matrix where the non-diagonal entry a_{ij} is the number of edges from vertex i to vertex j, and the diagonal entry a_{ii}, depending on the convention, is either once or twice the number of edges (loops) from vertex i to itself.*

An adjacency matrix helps to represente which vertices of a graph are adjacent to which other vertices.

Definition 1.2.3 *The spectrum of a (finite-dimensional) matrix is the set of its eigenvalues.*

This notion can be extended to the spectrum of an operator in the infinite-dimensional case.

Definition 1.2.4 *A regular graph is a graph where each vertex has the same degree. A regular graph in which each vertex is of degree (q+1) is called (q+1)-regular.*

Example 4 *Complete graph K_n is an (n-1)-regular graph.*

Proposition 1.2.5 *Assume that X is a connected (q+1)-regular graph and let A be its adjacency matrix.*
1) $\lambda \in$Spectrum(A) implies $|\lambda| \leq q+1$.
2) q+1$\in$Spectrum(A) and it has multiplicity 1.
3) -(q+1)$\in$Spectrum(A) if and only if the graph X is bipartite.

Proof. 1) Note that $(q+1)$ is clearly an eigenvalue of A corresponding to the constant vector. Suppose $Av = \lambda v$, for some vector $v = {}^t(v_1 \dots v_n) \in \mathbb{R}^n$. And suppose that the maximum of the $|v_i|$ occurs at $i = a$. Then, using the notation $b \sim a$, to mean the b^{th} vertex is adjacent to the a^{th}, we have

$$|\lambda||v_a| = |(Av)_a| = |\sum_{b\sim\ a} v_b| \leq (q+1)|v_a|$$

Fact 1 follows.
2) Suppose $Av = (q+1)v$, for some non-0 vector $v = {}^t(v_1 \dots v_n) \in \mathbb{R}^n$. Again suppose that the maximum of the $|v_i|$ occurs at $i = a$. We can assume $v_a > 0$, by multiplication of the vector v by -1. As in the proof of Fact 1.

$$(q+1)|v_a| = (Av)_a = \sum_{b\sim\ a} v_b \leq (q+1)v_a$$

To have equality, there can be no cancellation in this sum and $v_b = v_a$, for each b adjacent to a. Since we assume that X is connected, we can iterate this argument and conclude that v must be the constant vector. □

Riemann Hypothesis is a conjecture proposed by Bernhard Riemann (1859) about the distribution of the zeros of the Riemann zeta function (a function of a complex variable s: $\sum_{n=1}^{\infty} \frac{1}{n^s}$) which states that all non-trivial zeros of the Riemann zeta function have real part 1/2.

Definition 1.2.6 *Suppose that X is a connected (q+1)-regular graph (without degree 1 vertices). We say that the Ihara zeta function $\zeta_X(q^{-s})$ satisfies the **Riemann hypothesis** if and only if when $0 < Re(s) < 1$;*

$$\zeta_X(q^{-s})^{-1} = 0 \Longrightarrow Res = \frac{1}{2}$$

Note that if $u = q^{-s}$, Re(s)=$\frac{1}{2}$ corresponds to $|u|$=$\frac{1}{\sqrt{q}}$

The following is Ihara's determinant formula generalized by Bass, Hashimoto et al.

Theorem 1.2.7 *Let A be the adjacency matrix of X and Q the diagonal matrix with j^{th} diagonal entry q_j such that $q_j + 1$ is the degree of the j^{th} vertex of X. Suppose that r is the rank of the fundamental group of X; r-1= $|E|$-$|V|$. Then we have the Ihara determinant formula*

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1}\ det(1 - Au + Qu^2)$$

Definition 1.2.8 *A connected (q+1)-regular graph X is called Ramanujan whenever* $\mu \leq 2\sqrt{q}$ *where* $\mu = max\{|\lambda| \mid \lambda \in Spectrum(A),\ |\lambda| \neq q+1\}$.

Example 5 $K_{n,n}$ *(complete bipartite graph) is Ramanujan. For example the eigenvalues of the adjaceny matrix of* $K_{3,3}$ *are -3, 0 and 3. By definition,* $\mu = 0 < 2\sqrt{2}$. *Also, Petersen Graph (3-regular undirected graph with 10 vertices and 15 edges) is Ramanujan. The eigenvalues of the adjacency matrix of Petersen Graph are 1, -2 and 3. By definition,* $\mu = 2$ *and we have clearly* $2 \leq 2\sqrt{2}$

Theorem 1.2.9 *For a connected (q+1)-regular graph X,* $\zeta_X(u)$ *satisfies the Riemann hypothesis if and only if the graph X is Ramanujan.*

Example 6 *Think for the graph* $K_{2,2}$ *(2-regular graph).*

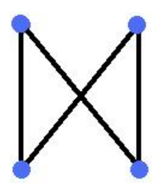

Figure 1.5: $K_{2,2}$. *Complete bipartite graph.*

The adjacency matrix of $K_{2,2}$ *is*

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and we have the eigenvalues -2,0,2. Thus $K_{2,2}$ *is Ramanujan.*

$$\zeta_{K_{2,2}}(1^{-s})^{-1} = (1-(1^{-s})^2)^0 \prod_{\lambda \in Spectrum(K_{2,2})} (1 - \lambda 1^{-s} + q1^{-2s}) = 0$$

$$1 + \frac{1}{1^{6s}} - \frac{1}{1^{4s}} - \frac{1}{1^{2s}} = 0$$

$$Res(s) = 1/gcd(2,4,6) = 2 = 1/2$$

Proof. Use Theorem 1.2.7 to see that

$$\zeta_X(q^{-s})^{-1} = (1-u^2)^{r-1} \prod_{\lambda \in Spectrum(A)} (1-\lambda u + qu^2)$$

Write $(1-\lambda u+qu^2) = (1-\alpha u)(1-\beta u)$, where $\alpha\beta = q$ and $\alpha+\beta = \lambda$. Note that α, β are the reciprocals of poles of $\zeta_X(u)$. Using the facts in Proposition above, we have 3 cases:
Case 1: $\lambda = \pm(q+1)$ implies $\alpha = \pm q$ and $\beta = \pm 1$
Case 2: $\lambda \le 2\sqrt{q}$ implies $|\alpha| = |\beta| = \sqrt{q}$
Case 3: $2\sqrt{q} < |\lambda| < q+1$ implies $\alpha, \beta \in \mathbb{R}$ and $1 < |\alpha| = |\beta| < q$, $|\alpha| = |\beta| \neq \sqrt{q}$
To see these things, let u be either α^{-1} or β^{-1}. Then by the quadratic formula, we have α or $\beta = u^{-1}$ where

$$u = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q}$$

Cases 1 and 2 easily seen. To understand case 3, first assume $\lambda > 0$ and note that $u = \frac{\lambda+\sqrt{\lambda^2-4q}}{2q}$ is a monotone increasing function of λ. This implies that the larger root u is in the interval $(\frac{1}{\sqrt{q}}, 1)$. Where is the smaller root $u' = \frac{\lambda-\sqrt{\lambda^2-4q}}{2q}$? Answer: $|u'| \in (\frac{1}{q}, \frac{1}{\sqrt{q}})$. Here we use the fact that u.u'=1/q. A similar argument works for negative λ.

The proof of the theorem is finished by noting that when $u = q^{-s}$, case 2 is Re(s) $= \frac{1}{2}$. $\square$

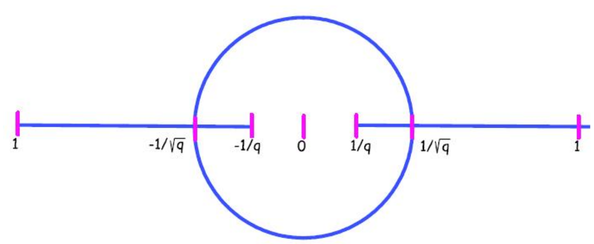

Figure 1.6: *Possible locations for poles of $\zeta_X(u)$ for a regular graph are marked in blue. The circle corresponds to the part of the spectrum of the adjacency matrix satisfying the Ramanujan inequality. The real poles correspond to the non-Ramanujan eigenvalues of A, except for the two poles on the circle itself and the endpoints of the intervals*

Figure 1.6 shows the possible locations of poles of the Ihara zeta function of a $(q+1)$-regular graph. The poles satisfying the Riemann hypothesis are those on the circle. The circle basically corresponds to Case 2 in the preceding proof. The real axis corresponds to Case 1 and Case 3.

The following proposition gives some functional equations of the Ihara zeta function for a regular graph. If we set $u = q^{-s}$, the functional equations relate the value at s with that at $1-s$; just as is the case for the Riemann zeta function.

Proposition 1.2.10 *Suppose that X is a (q+1)-regular connected graph without degree 1 vertices with n=$|V|$. Then we have the following functional equations among others.*
1) $\Lambda_X(u) = (1-u^2)^{r-1+\frac{n}{2}}(1-q^2u^2)^{\frac{n}{2}}\zeta_X(u) = (-1)^n\Lambda_X(\frac{1}{qu})$
2) $\xi_X(u) = (1+u)^{r-1}(1-u)^{r-1+n}(1-qu)^n\zeta_X(u) = \xi_X(\frac{1}{qu})$
3) $\Xi_X(u) = (1-u^2)^{r-1}(1+qu)^n\zeta_X(u) = \Xi_X(\frac{1}{qu})$

Proof. We will prove part 1). 2) and 3) can be proven similarly. To see part 1), write by Theorem 1.2.7

$$\begin{aligned}\Lambda_X(u) &= (1-u^2)^{\frac{n}{2}}(1-q^2u^2)^{\frac{n}{2}}det(1-Au+qu^2I)^{-1}\\ &= (\frac{q^2}{q^2u^2}-1)^{\frac{n}{2}}(\frac{1}{q^2u^2}-1)^{\frac{n}{2}}det(1-A\frac{1}{qu}+\frac{q}{(qu)^2}I)^{-1}\\ &= (-1)^n\Lambda(\frac{1}{qu})\end{aligned}$$

□

To produce examples of regular graphs, the easiest method is to start with a generating set S of your favorite finite group G. Assume that S is symmetric, meaning that $s \in S$ implies $s^{-1} \in S$. Then, the Cayley graph, denoted X(G,S), has as its vertices the elements of G and has edges between vertex g and gs for all $g \in G$ and $s \in S$.

Cayley graphs are always regular with degree $|S|$. Connectedness of X(G,S) is guaranteed by S being a generating set. If one chooses S to be symmetric, then the resulting graph has to be undirected.
The Cube is $X(\mathbb{F}_2^3, S)$, where $\mathbb{F}_2$ denotes the field with 2 elements, $\mathbb{F}_2^3$ is the additive group of 3-vectors with entries in this field, and
$S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}.$

1.3 The Riemann Hypothesis for Irregular Graphs

Next let us speak about irregular graphs which are unweighted and satisfying our usual hypotheses.
For a graph X let R_X denote the radius of the largest circle of convergence of the Ihara Zeta Function attached to X and let Δ_X denote the greatest common divisor of the lengths of prime paths in X, i.e.

$$\Delta_X = gcd\{v(P) \mid [P] \ prime \ of \ X\}$$

With the above notation, Kotani and Sunada Theorem:

Theorem 1.3.1 *Suppose the graph X satisfies our usual hypotheses and has vertices with maximum degree q+1 and minumum degree p+1.*
1) Every pole u of $\zeta_X(u)$ satisfies $R_X \leq |u| \leq 1$ and

$$q^{-1} \leq R_X \leq p^{-1}$$

2) Every non-real pole u of $\zeta_X(u)$ satisfies the inequality

$$q^{-1/2} \leq |u| \leq p^{-1/2}$$

3) The poles of ζ_X on the circle $|u|=R_X$ have the form $R_X e^{2\pi i a/\Delta_X}$, where a=1,...,$\Delta_X$.

Now let us define two constants associated to the graph X. First we should describe the universal covering tree.

To construct the universal covering graph T of a connected graph G, first of all, choose an arbitrary vertex r of G as a starting point. Each vertex of T is a non-backtracking walk that begins from r, that is, a sequence $w = (r, v_1, v_2, \ldots, v_n)$ of vertices of G such that

- v_i and v_{i+1} are adjacent in G for all i, i.e., w is a walk
- $v_{i-1} \neq v_{i+1}$ for all i, i.e., w is non-backtracking.

Definition 1.3.2 *The graph T constructed above is called the universal covering graph of the connected graph G.*

Observe that, two vertices of T are adjacent if one is a simple extension of another. For instance, the vertex $(r, v_1, v_2, \dots, v_n)$ is adjacent to the vertex $(r, v_1, v_2, \dots, v_{n-1})$. Remark also that changing the base point results in an isomorphic T.

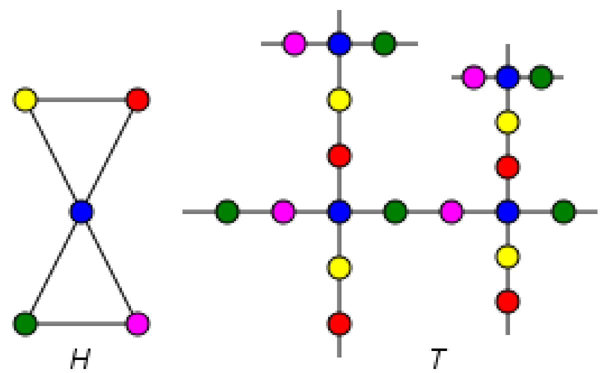

Figure 1.7: *Example of a universal covering tree*

Definition 1.3.3

$$\rho_X = max\{|\lambda| \mid \lambda \in spectrum(A_X)\}$$

$$\rho'_X = max\{|\lambda| \mid \lambda \in spectrum(A_X), |\lambda| \neq \rho_X\}$$

We will say that the naive Ramanujan inequality is

$$\rho'_X \leq 2\sqrt{\rho_X - 1} \tag{1.3.1}$$

Lubotzy has defined X to be Ramanujan if,

$$\rho'_X \leq \sigma_X \tag{1.3.2}$$

where σ_X is the spectral radius of the adjacency operator on the universahel; where we define t covering tree of X. The spectral radius of the operator A is the supremum of $|\lambda|$ such that $A - |\lambda| I$ has no inverse (we should search for the eigenvalues of A).

Remark 1.3.4 *If X is a (q+1)-regular graph, then we have $\rho - 1 = q$ in naive Ramanujan inequality which is equivalent to the definition of the Ramanujan graph (Definition 1.2.8).*

Notation: $\overline{d_X}$ denotes the average degree of the vertices of X

Theorem 1.3.5

$$2\sqrt{\overline{d_X}-1} \leq \sigma_X$$

From this theorem one has a criterion for a graph X to be Ramanujan in Lubotzky's sense. It need only to satisfy the Hoory inequality

$$\rho'_X \leq 2\sqrt{\overline{d_X}-1} \tag{1.3.3}$$

To develop the Riemann Hypothesis for irregular graphs, the natural change of variable is $u = R_X^s$. All poles of $\zeta_X(u)$ are then located in the "critical strip", $0 \leq Re(s) \leq 1$ with poles at $s = 0$ $(u = 1)$ and $s = 1$ $(u = R_X)$.The examples below show that, for irregular graphs, one cannot expect a functional equation relating $f(s) = \zeta(R_X^s, X)$ and $f(1-s)$. Therefore it is natural to say that the Riemann Hypothesis for X should require that $\zeta_X(u)$ has no poles in the open strip $1/2 < Re(s)/ < 1$. This is the graph theory Riemann Hypothesis below. After looking at examples, it seems that one rarely sees an Ihara zeta satisfying this Riemann Hypothesis (although random graphs do seem to approximately satisfy the Riemann Hypothesis). Thus we also consider the weak graph theory Riemann hypothesis below.

Graph theory Riemann Hypothesis $\zeta_X(u)$ is pole free for

$$R_X < |u| < \sqrt{R_X} \tag{1.3.4}$$

Weak graph theory Riemann Hypothesis $\zeta_X(u)$ is pole free for

$$R_X < |u| < 1/\sqrt{q} \tag{1.3.5}$$

If the graph is regular, then $R_X = 1/q$ and (1.3.4) and (1.3.5) are the same. We have examples (such as the first example below) for which $R_X > q^{-1/2}$ and in such cases the weak graph theory Riemann Hypothesis is true but vacuous. Sometimes number theorists state a modified Generalized Riemann Hypothesis for the Dedekind zeta function and this just ignores all possible real zeros while only requiring the non-real zeros to be on the line $Re(s) = 1/2$. The graph theory analog of the modified weak Generalized Riemann Hypothesis would just ignore the real poles and require that there are no non-real poles of $\zeta_X(u)$ in $R_X < |u| < q^{-1/2}$ But this is true for all graphs by Theorem 1.3.1: if μ is a pole of $\zeta_X(u)$ and $|u| < q^{-1/2}$ then μ is real.

Remark 1.3.6 $\rho_X \geq \overline{d_X}$

Next we give some examples including answers to the questions: Do the spectra of the adjacency matrices satisfy the naive Ramanujan inequality (1.3.1) or the Hoory inequality (1.3.3)? Do the Ihara zeta functions for the graphs have the pole-free region (1.3.5) of the weak graph theory Riemann Hypothesis or the pole-free region (1.3.4) of the full graph theory Riemann Hypothesis?

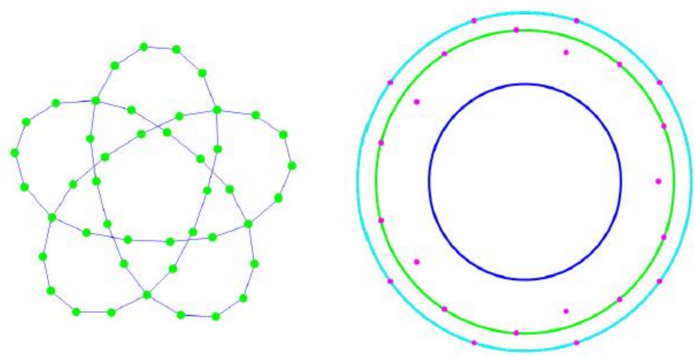

Figure 1.8: *On the left is the graph $X = Y_5$ obtained by adding 4 vertices to each edge of $Y = K_5$, the complete graph on 5 vertices. On the right the poles $\neq -1$ of the Ihara zeta function of X are the magenta points. The circles have centers at the origin and radii $\{q^{-1/2}, R^{1/2}, p^{-1/2}\}$*

Example 7

Let X be the graph obtained from the complete graph on 5 vertices by adding 4 vertices to each edge as shown on the left in Figure 1.3.
For the graph X, we find that $\rho' \approx 2.32771$ and

$$\{\rho, 1 + 1/R, \overline{d_X}\} \approx \{2.39138, 2.24573, 2.22222\}$$

This graph satisfies the naive Ramanujan inequality but not the Hoory inequality. The magenta points in the picture on the right in Figure 1.3 are the poles not equal to -1 of $\zeta_X(u)$. Here

$$\zeta_X(u)^{-1} = \zeta_{K_5}(u^5)^{-1} = (1 - u^{10})^5(1 - 3u^5)(1 - u^5)(1 + u^5 + 3u^{10})$$

The circles in the picture on the right in Figure 1.3 are centered at the origin with radii.

$$\{q^{-1/'}, R, R^{1/2}, p^{-1/2}\} \approx \{0.57735, 0.802742, 0.895958, 1\}$$

The zeta function satisfies the Riemann Hypothesis and thus the weak Riemann Hypothesis. However the weak Riemann Hypothesis is vacuous.

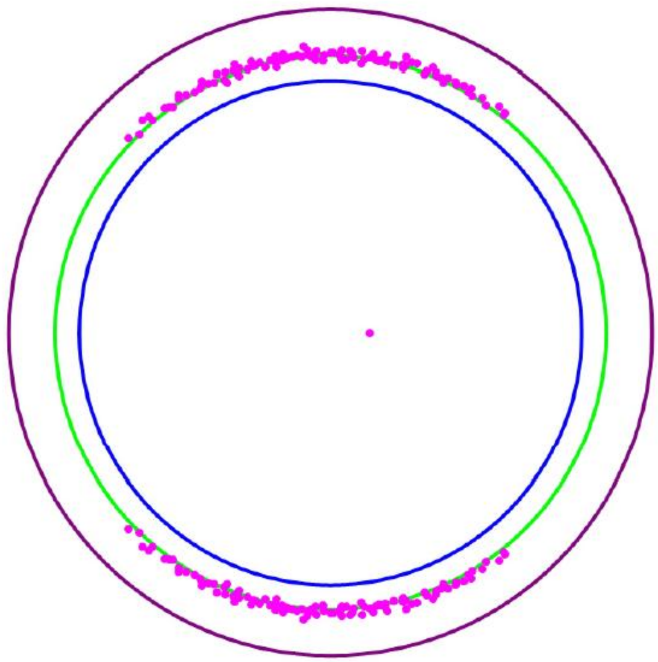

Figure 1.9: *The magenta points are poles ($\neq -1$) of the Ihara zeta function for a random graph produced by Mathematica with the command RandomGraph[100,1/2]. The circles have centers at the origin and radii $\{q^{-1/2}, R^{1/2}, p^{-1/2}\}$ The Riemann Hypothesis looks approximately true but is not exactly true. The weak Riemann Hypothesis is true.*

Example 8

Random graph with probability 1/2 of an edge. The magenta points in figure 1.9 are the poles not equal to $\underline{+}1$ of the Ihara zeta function of a random graph. There are 100 vertices and the probability of an edge between any 2 vertices is 1/2. The graph satisfies the Hoory inequality and thus it is Ramanujan in Lubotzky's sense. It also satisfies the naive Ramanujan inequality. We find $\rho' \approx 10.0106$ and $\{\rho, 1+1/R, \overline{d_X}\} \approx \{50.054, 50.0435, 49.52\}$. The circles in this figure are centered at the origin and have radii given by $\{q^{-1/2}, R^{1/2}, p^{-1/2} \approx \{0.130189, 0.142794, 0.166667\}\}$. The poles of the zeta function satisfy the weak Riemann Hypothesis but not the Riemann Hypothesis. However, the Riemann Hypothesis seems to be approximately true.

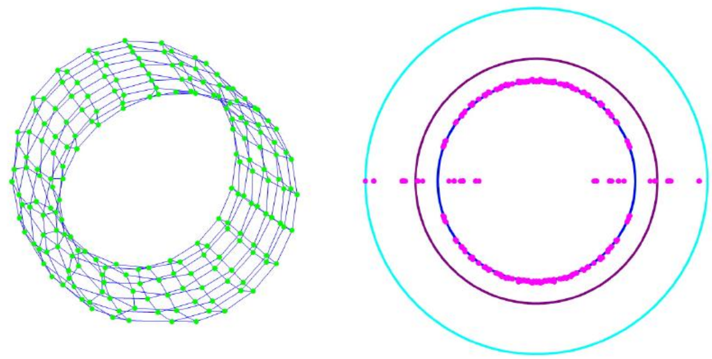

Figure 1.10: *The graph N on the left results from deleting 6 edges from the product of a 10-cycle and a 20-cycle. In the picture on the right, magenta points indicate the poles* ($\neq -1$) *of the Ihara zeta function of N. The circles are centered at the origina with radii* $\{q^{-1/2}, R^{1/2}, p^{-1/2}\}$ *The Ihara zeta function satisfies neither the Riemann Hypothesis nor the weak Riemann Hypothesis.*

Example 9

Torus minus some edges. From the torus graph T which is the product of a 10-cycle and a 20-cycle, we delete 6 edges to obtain the graph N in Figure 1.10. The spectrum of the adjacency matrix of N satisfies neither the Hoory inequality nor the naive Ramanujan inequality. We find that $\{\rho, 1 + 1/R, \overline{d}\} \approx \{3.98749, 3.98568, 3.98\}$ and $\rho' \approx 3.90275$. The right hand side of the figure shows the poles of the Ihara zeta for N as magenta points. The circles are centered at the origin and have radii $\{q^{-1/2}, R^{1/2}, p^{-1/2} \approx \{0.57735, 0.57873, 0.70711\}\}$. The zeta poles satisfy neither the graph theory weak Riemann Hypothesis nor the Riemann Hypothesis.

Example 10

Figures 1.11 and 1.12 show the results of some Mathematica experiments on the distribution of the poles of zeta for 2 graphs. The top row shows the graph. The middle row shows the histogram of degrees. In the bottom row, the magenta points are poles of the Ihara zeta function of the graph.
Figure 1.11: The middle green circle is the Riemann hypothesis circle with radius $\sqrt{R}$ where R is the closest pole to 0. The inner circle has radius $1/\sqrt{q}$, where $q+1$ is the maximum degree of the graph. The outer circle has radius 1. For this graph $p = 1$ and thus the circle of radius $1/\sqrt{p}$ coincides with the circle of radius 1. Many poles are inside the green middle circle and thus violate the Riemann hypothesis. For this graph, the Riemann hypothesis and the weak Riemann hypothesis are false as is the naive Ramanujan inequality. The probability of an edge is 0.119177.
Figure 1.12: The inner circle has radius $1/\sqrt{q}$ where $q+1$ is the maximum degree of the graph. The next circle out (the green circle) is the Riemann hypothesis circle with radius $\sqrt{R}$; where R is the closest pole to 0. The outer circle has radius 1. The circle just inside this one has radius $1/\sqrt{p}$; where $p+1$ is the minimum degree of the graph. For this graph, the Riemann hypothesis is false, but the weak Riemann hypothesis is true as well as the naive Ramanujan inequality. The probability of an edge is 0.339901 for this graph.

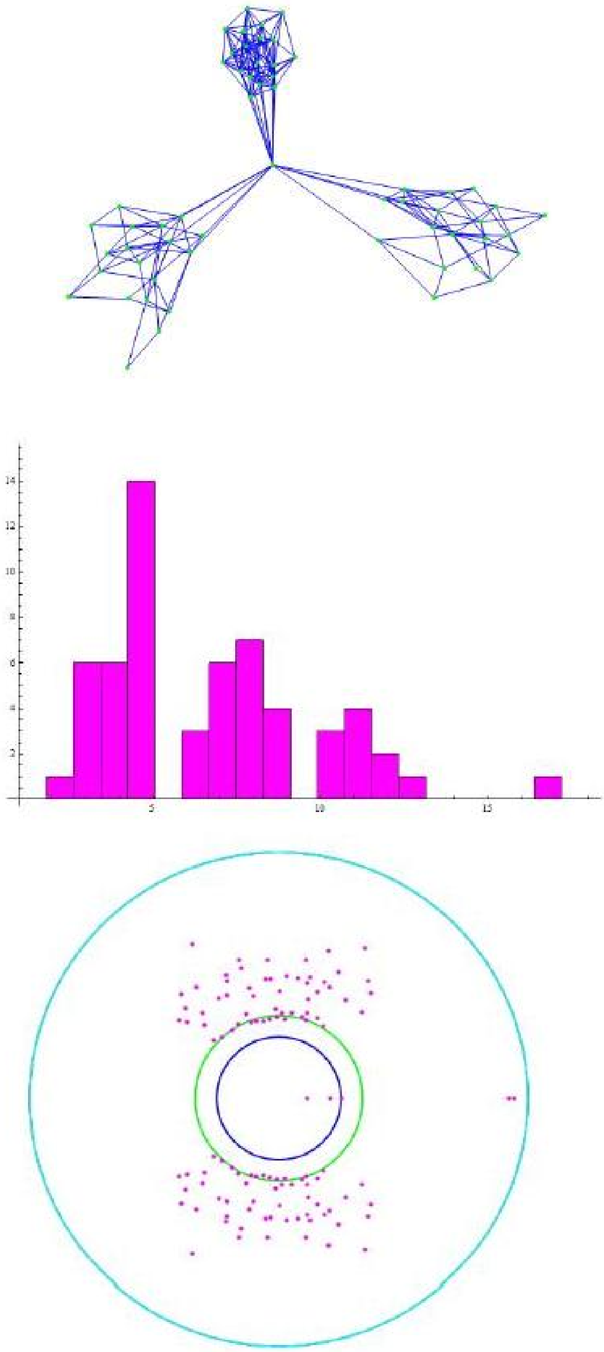

Figure 1.11: *A Mathematica Experiment.*

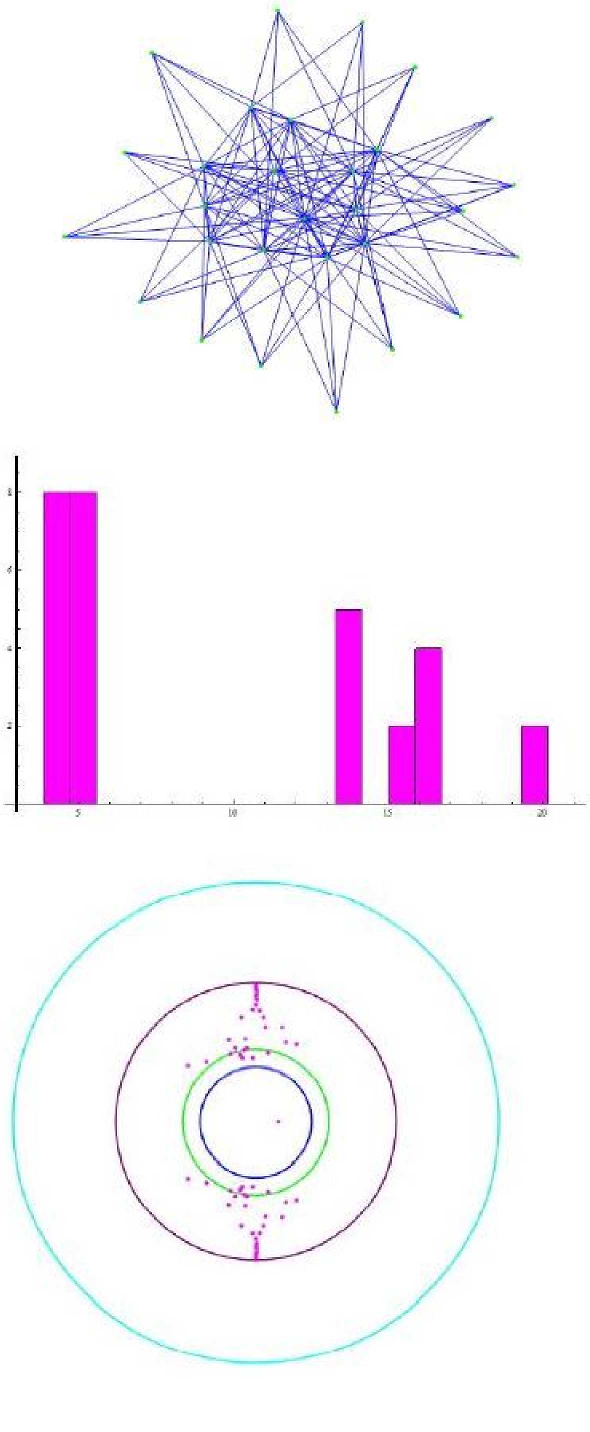

Figure 1.12: *A Mathematica Experiment.*

Chapter 2

A Family of the Covering Graphs and a Conjecture

In this section, we will examine the graph zeta functions of a family of the covering graphs. We will search for the poles and ramification points of these functions and visualize the computations that we made.

2.1 Description of the Covering Graphs and Their Graph Zeta Functions

First we construct the first graph of the family. We have three steps to follow. In the first step, we see the figure in which there are two triangles glued from their one side. Next, we put a vertex to the center of each triangle and we draw an edge to middle of each side of the triangle from the vertex. After we put a vertex to each point of intersection of the sides of triangles and

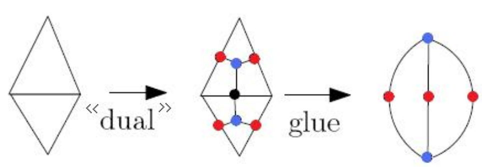

Figure 2.1: *The construction of the first graph* Γ_1.

the edges that we draw. In the last step, we glue the remaining two sides of the triangle respecting orientation (Figure 2.1).
By using Theorem 1.2.7 (Ihara's determinant formula) one can easily see that

$$\zeta_1(u)^{-1} = -(u-1)^2(u+1)^2(1+u^2)^2(2u^2-1)(2u^2+1)$$

The following three graphs are shown below.

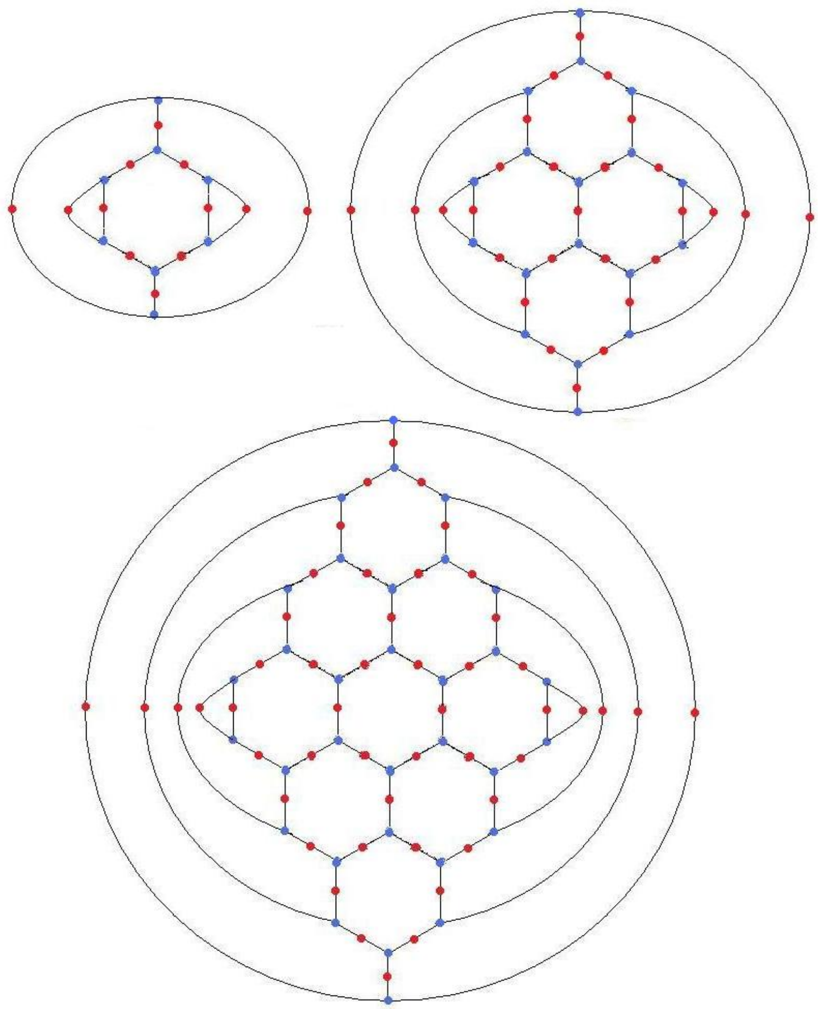

Figure 2.2: *The graphs* Γ_2, Γ_3, Γ_4.

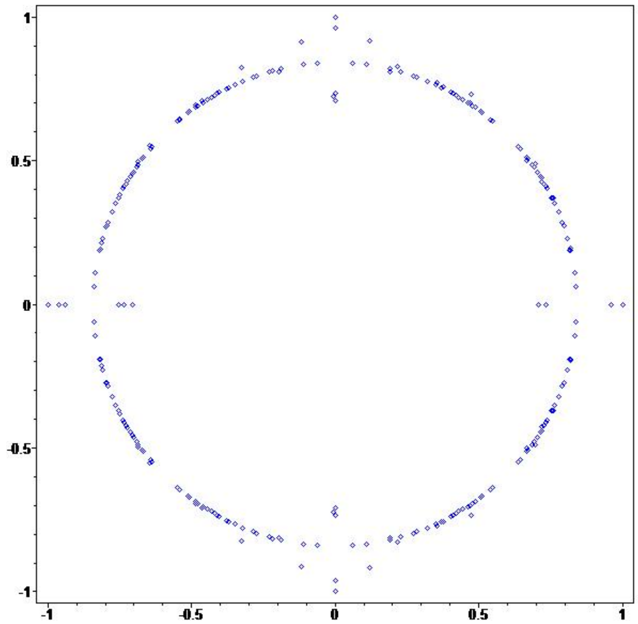

Figure 2.3: *The zeroes of the first four graph zeta functions in the same figure.*

The graph zeta functions of these three graphs can be calculated again by the Ihara's determinant formula.

$\zeta_2(u)^{-1} = (u-1)^5(u+1)^5(2u^2-1)(2u^4+u^2+1)(2u^4-2u^2+1)^2(2u^2+1)(1+u^2)^5(2u^4-u^2+1)(2u^2+1+2u^4)^2$

$\zeta_3(u)^{-1} = -(u-1)^{10}(u+1)^9(-1-5u^4-u+458752u^{70}+720896u^{68}-2u^3-8u^6-14u^8-2u^2-20u^{10}-5u^5-8u^7-14u^9-14u^{11}+1040384u^{62}+851968u^{67}-2u^{12}+10u^{13}+22u^{14}+248u^{18}+177u^{17}+125u^{16}+54u^{15}+851968u^{66}+1015808u^{64}+563456u^{53}+921600u^{59}+818176u^{57}+681216u^{55}+262144u^{72}+1064960u^{63}+1040384u^{61}+432512u^{51}+10487u^{30}+364u^{19}+673u^{20}+1153u^{22}+2323u^{24}+3556u^{26}+6911u^{28}+64232u^{38}+26848u^{34}+18419u^{32}+44919u^{36}+7983u^{29}+96948u^{40}+853u^{21}+1435u^{23}+2707u^{25}+4284u^{27}+72292u^{39}+870400u^{58}+764928u^{56}+631552u^{54}+517376u^{52}+346688u^{49}+262320u^{47}+203440u^{45}+143532u^{43}+106948u^{41}+991232u^{60}+31158u^{35}+20919u^{33}+12273u^{31}+397184u^{50}+322112u^{48}+241456u^{46}+186288u^{44}+129956u^{42}+51039u^{37}+720896u^{69}+458752u^{71}+1015808u^{65}+262144u^{73})(1+u^2)^8$

$\zeta_4(u)^{-1} = (u-1)^{17}(u+1)^{17}(2u^2-1)(2u^4-2u^2+1)^2(2u^4+u^2+1)^3(4u^8-u^4+1)^2(4u^8+4u^6+2u^4+2u^2+1)^2(4u^8+2u^4+1)^4(2u^2+1)(4u^8-4u^6+2u^4-2u^2+1)^2(1+u^2)^{17}(2u^4-u^2+1)^3(2u^2+1+2u^4)^2$

As we see, the graph zeta functions $\zeta_1(u)$, $\zeta_2(u)$, $\zeta_4(u)$ are growing regularly and related to each other. But in the third graph zeta function there is an irregularity.

2.2 Ramification Points of the Graph Zeta Functions

Definition 2.2.1 *Let $F : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ be an analytic function. The multiplicity of F at p, denoted $mult_p(F)$, is the unique integer m such that locally near p and $F(p)$ F has the form $z \to z^m$.*

Definition 2.2.2 *Let $F : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ be a nonconstant analytic map. A point p is a ramification point for F if $mult_p(F) \geq 2$.*

First we think the multiplicative inverse i.e. $\zeta_s(u)^{-1}$ of these graph zeta functions and write their derivatives till they vanish. At this time, we calculate their zeroes in each step (Figure 2.4).

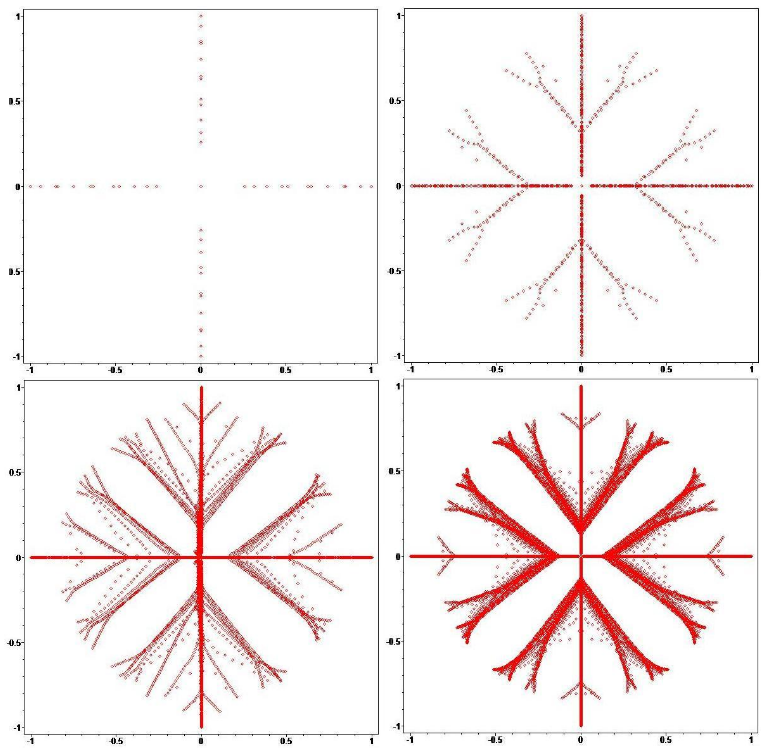

Figure 2.4: *Maple Experience - The red points are the zeroes of the derivatives* $\zeta_n(u)^{-1}$

Now we apply the same method to the graph zeta functions themselves and we obtain this exquisite figure.

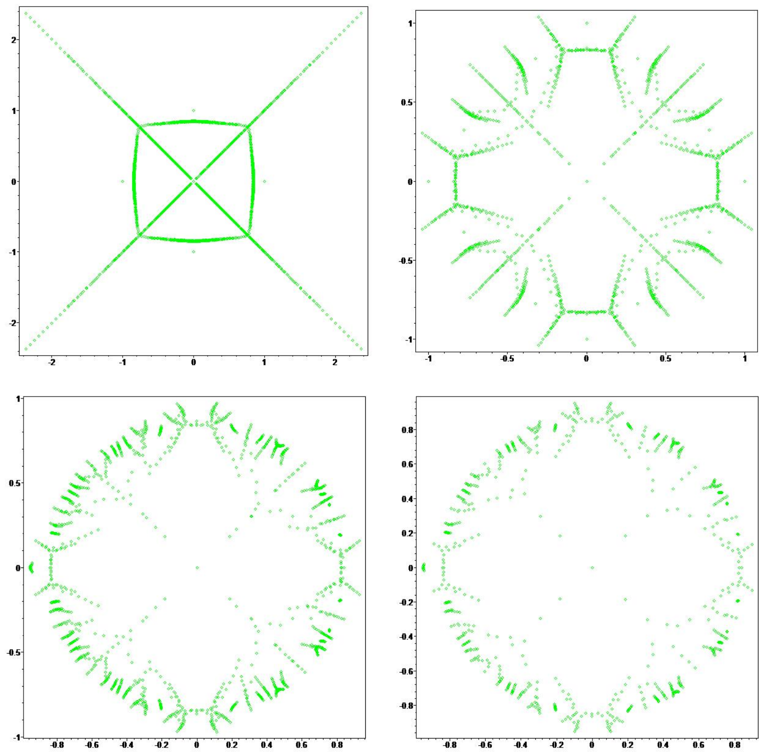

Figure 2.5: *Maple Experience - The green points are the zeroes of the derivatives* $\zeta_n(u)$

2.3 Conjecture

The experiment that we made with Maple shows that there is a division relation in first eight graphs.

$$\zeta_1(u)^{-1} \mid \zeta_2(u)^{-1},\ \zeta_2(u)^{-1} \mid \zeta_4(u)^{-1},\ \zeta_4(u)^{-1} \mid \zeta_8(u)^{-1}$$

Furthermore, one can see a regular extension between the functions if we decompose these four functions with real coefficients.

These results leads us to a possible relation of the graph zeta functions of the family members. We can write it as follows:

Conjecture 2.3.1 $\forall k \in \mathbb{N},\ \zeta_{2^{k+1}}(u) \mid \zeta_{2^k}(u)$

This is a conjecture which we work for its proof.

Bibliography

[1] Audrey Terras, *A Stroll Through The Garden Of Graph Zeta Functions.* Cambridge U. Press, Cambridge, U.K. 1st Edition, 2011.

[2] Audrey Terras, *What are primes in graphs and how many of them have a given length?.* Talk, U.C.S.D Math Club 2008.

[3] Bernhard Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, 671–680, 1859.

[4] Klaus Kirsten, Floyd L. Williams, *A Window Into Zeta and Modular Physics.* Cambridge University Press, 2010.

[5] Jean-Pierre Serre, *Trees.* Springer, 2003.