**THÈSE**

pour obtenir le diplôme de

l' **Université De Galatasaray**
Spécialité : **Mathématiques**
Directeur : **Gülay KAYA**

# QU'EST-CE QU'UN YOUNG TABLEAU?

préparée par **Cansu TAN**

*Juin '11*

## Acknowledgements

I would like to express my sincere gratitude to my professors Gülay KAYA and Meral TOSUN for their supervision and guidance.

Also I would like to thank my professors Muhammed ULUDAĞ and Ayşegül ULUS in helping me to broaden my view and knowledge.

And I am deeply and forever indebted to my parents for their love, support and encouragement throughout my entire life.

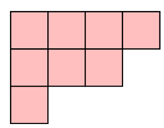
# Contents

# Chapter 1

# INTRODUCTION

Alfred Young who is a mathematician at Cambridge University introduced the subject of Young Tableau in 1900. In 1903, Georg Frobenius used Young tableaux to study the symmetric groups.

A *Young diagram* is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths weakly decreasing where each row has the same or shorter length than its predecessor.



A Young tableau is obtained by filling in the boxes of the Young diagram with symbols taken from some alphabet. It is a combinatorial object which is useful and important in representation theory, geometry, and algebra.

In this work, we deal with the Young tableaux in Representation Theory. First, we describe the construction of a Young tableau and some preliminaries about it. In Chapter 3, we present some algorithms and some examples of Young tableaux. In Chapter 4, we use the words to define a Young Tableau and Schur polynomials. In the last chapter, we try to apply Young Tableaux to understand the modules in Representation Theory.

# Chapter 2

# PRELIMINARIES

**Definition 2.1** A **total order** on a set $X$ is a binary relation which is transitive, antisymmetric and total. A set paired with a total order is called a **totally ordered set**.

In other words if $X$ is a totally ordered set under $\leq$ then the following statements hold for all $a$, $b$ and $c$ in $X$:

(1) If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)

(2) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)

(3) $a \leq b$ or $b \leq a$ (totality)

For example, we can say that the letters of alphabet ordered by the standard dictionary order.

$$A < B < C$$

**Definition 2.2** A **partition** of a positive integer $m$ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ satisfying $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n > 0$ and $m = \lambda_1 + \lambda_2 + ... + \lambda_n$. We write $\lambda \vdash m$ to denote that $\lambda$ is a partition of $m$.

The **Young diagram** associated to the partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is the one that has n rows, and $\lambda_i$ boxes on the i-th row. Listing the number of boxes in each row gives a partition of the integer $m$ that is the total number of boxes and it is symbolized $|\lambda|$ and named **shape** of the tableau.

A **Young tableau** is obtained by filling a Young diagram with some elements from any totaly ordered set under some rules. Elements should be:

(1) (Weakly) increasing across each row

(2) Strictly increasing down each column

Entries of tableaux can be taken from any totaly ordered set but we usually take positive integers.

**Example:** $\lambda = (5, 3, 1)$ is a partition of 8 and the following is a Young tableau associated to $\lambda$;

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&2&4&5\\ \hline 4&5&7\\ \cline{1-3} 8\\ \cline{1-1}\end{array} \tag{2.0.1}$$

The shape $|\lambda|$ of this tableau is 8 where $|\lambda| = 5 + 3 + 1$. Here 5 shows the number of boxes on the first row, 3 shows the number of boxes on the second row and 1 shows the number of the boxes on the last row.

**Example:** There are 5 possible partitions of 4: (4), (3,1), (2,2), (2,1,1), (1,1,1,1).

For example the Young tableau

$$\begin{array}{|c|c|}\hline 4&4\\ \hline 5\\ \cline{1-1} 6\\ \cline{1-1}\end{array} \tag{2.0.2}$$

corresponds to the partition of 4 into 2+1+1.

**Remark 2.3** There is a one-to-one correspondence between partitions and Young diagrams.

**Remark 2.4** The notation used here is known as the English notation. The French notation, which is the upside-down form of the English notation. Here is an example to see the difference between French and English notation (3,1) :

English notation: $\begin{array}{|c|c|c|}\hline \phantom{0}&\phantom{0}&\phantom{0}\\ \hline \phantom{0}\\ \cline{1-1}\end{array}$

French notation: $\begin{array}{|c|c|c|}\cline{1-1} \phantom{0}\\ \hline \phantom{0}&\phantom{0}&\phantom{0}\\ \hline\end{array}$

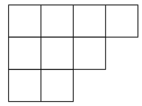
**Definition 2.5** A **standard Young tableau** is a tableau in which the entries are the numbers $1, \ldots, n$, each occurring once.

**Example:** For standard Young tableau:

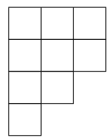
$$\begin{array}{|c|c|c|c|}\hline 1&3&4&6\\ \hline 2&7\\ \cline{1-2} 5\\ \cline{1-1}\end{array} \tag{2.0.3}$$

**Definition 2.6** Any numbering $T$ of a diagram determines a numbering of the conjugate called **the transpose** and denoted $T^t$. The transpose of a standard tableau is standard but the transpose of a tableau need not to be a tableau.

**Example:** We will take below the transpose of a Young tableau shaped $\lambda = (4, 3, 2)$.



After having taken transpose;



The other and more complicated type of Young tableau is **skew tableau**.

**Definition 2.7** A **skew shape** is obtained by removing a smaller diagram $\mu$ from a larger one $\lambda$ that contains it. It is denoted by $\lambda/\mu$.

**Example:**

| | | | | |
|---|---|---|---|---|
| | | 2 | 5 | 9 |
| | 3 | 4 | 6 | |
| 1 | 7 | | | |
| 8 | 9 | | | |

is a skew tableau of shape $\lambda/\mu$ where $\lambda = (5, 4, 2, 2)$ and $\mu = (2, 1)$

**Definition 2.8** Partial orderings in partitions has two types, besides that of inclusion $\mu \subset \lambda$. First is the **lexicographic ordering**, denoted $\mu \leq \lambda$, which means that the first $i$ for which $\mu_i \neq \lambda_i$, if any has $\mu_i \leq \lambda_i$. The other is the **dominance ordering**, denoted $\mu \trianglerighteq \lambda$, which means that :

$$\mu_1 + \ldots + \mu_i \leq \lambda_1 + \ldots + \lambda_i$$

for all $i$.

# Chapter 3

# OPERATIONS ON YOUNG TABLEAUX

Two fundamental operations on tableaux can be defined: **The Schensted bumping algorithm** and **The Schutzenberger sliding algorithm** . First one leads to the Robinson-Schensted-Knuth correspondence and the second one to the jeu du taquin.

## 3.1 The Schensted Bumping Algorithm

**Definition 3.1 (The Schensted algorithm)** Bumping algorithm constructs a new Young tableau by inserting new numbers to a given tableau.

To start to apply algorithm first suppose that we have a tableau named $T$ and a positive integer $x$ which will be inserted to the tableau $T$. We define the action $T \leftarrow x$ by the rules as follow:

(1) Look at the first row of $T$ and find the smallest number that is larger than $x$, replace this number with $x$. If the smallest number larger than $x$ occurs more than once in the row then choose the one furthest to the left. If no such number is larger than $x$ then simply place $x$ at the end of the first row.

(2) If an integer say $y$ was replaced by $x$ in the first row then bump $y$ into the second row using the same method as above. If there is no row to add $y$ to, then $y$ has been bumped out of the bottom, in which case it forms a new row with one entry.

(3) Keep going this process on each row of the tableau until either some number gets added to the end of a row or until it is bumped out of the bottom.

**Example:** We try to insert 3 into the Young tableau below:

$$T = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 6 & 7 & 8 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} \end{array} \tag{3.1.1}$$

**Step 1:** In the first row there is a number greater than 3 so we can not insert it to the end of the first row. 4 is the smallest number which is greater than 3. Therefore bump 4 from the first row to the second row. Process will continue until all the numbers inserted into the tableau.

$$\begin{array}{|c|c|c|c|}\hline 2&3&3&4\\ \hline 3&4&5\\ \cline{1-3} 6&7&8\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array} \quad \Longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&5\\ \cline{1-3} 6&7&8\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array} \tag{3.1.2}$$

**Step 2:** 4 is smaller than 5 so bump 5 from the second row to the third row.

$$\begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&5\\ \cline{1-3} 6&7&8\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array} \quad \Longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&4\\ \cline{1-3} 6&7&8\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array} \tag{3.1.3}$$

**Step 3:** 5 is smaller then 6 so bump 6 from the third row to the fourth row.

$$\begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&4\\ \cline{1-3} 6&7&8\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array} \quad \Longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&4\\ \cline{1-3} 5&7&8\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array} \tag{3.1.4}$$

**Step 4:** 6 is the smaller then 7 so bump 7 from the fourth row.

$$\begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&4\\ \cline{1-3} 5&7&8\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array} \quad \Longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&4\\ \cline{1-3} 5&7&8\\ \cline{1-3} 6&9\\ \cline{1-2}\end{array} \tag{3.1.5}$$

**Step 5:** Since there is no row to add 7, insert it as a new box at the end of the tableau.

$$\begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&4\\ \cline{1-3} 5&7&8\\ \cline{1-3} 6&9\\ \cline{1-2}\end{array} \quad \Longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 2&3&3&3\\ \hline 3&4&4\\ \cline{1-3} 5&7&8\\ \cline{1-3} 6&9\\ \cline{1-2} 7\\ \cline{1-1}\end{array} \tag{3.1.6}$$

There is an important sense in which this operation is invertible. If we are given the resulting tableau, together with the location of the box that has been added to the diagram, we can recover the original tableau $T$ and the element $x$. The bumping algorithm is simply run backwards. If $y$ is the entry added in the added box, it looks for its position in the row above the location of the box, finding the entry farthest to the right which is strictly less than $y$. It bumps this entry up to next row, and the process continues until an entry is bumped out of the top row.

**Definition 3.2** The route which includes the boxes we used in the algorithm is called the **bumping route** of the corresponding row-insertion.

**Example:** The *bumping route* for the previous example is:

| | | | |
|---|---|---|---|
| 2 | 3 | 3 | $\bullet$ |
| 3 | 4 | $\bullet$ | 5 |
| $\bullet$ | 7 | 8 | |
| $\bullet$ | 9 | | |

A bumping route has at most one box in each of several successive rows starting at the top. We say a route $R$ is **strictly left** (resp. **weakly left**) of a route $R^{'}$ if in each row which contains a box of $R^{'}$, $R$ has a box which is left of (resp. right of or equal to) the box in $R^{'}$.

**Lemma 3.3** *([2], p.9) Consider the actions $T \leftarrow x$ and $T \leftarrow x^{'}$. which give rise to two routes $R$ and $R^{'}$ and two next boxes $B$ and $B^{'}$ respectively.*

*(1) If $x \leq x^{'}$, then $R$ is strictly left of $R^{'}$ and $B$ is strictly left of and weakly below $B^{'}$.*

*(2) If $x > x^{'}$ then $R^{'}$ is weakly left of $R$ and $B^{'}$ i weakly left of and strictly below $B$.*

**Proof:** Suppose $x \leq x^{'}$, and $x$ bumps an element $y$ from the first row. The element $y^{'}$ bumped $x^{'}$ from the first row must lie strictly to the right of the box where $x$ bumped, since the elements in that box or to the left are no longer than $x$. In particular, $y \leq y^{'}$, and the same argument continues from row to row. Note that the route for $R$ cannot stop above that of $R^{'}$, and if $R^{'}$ stops first, the route for R never moves to the right, so the box B must be strictly left of and weakly below $B^{'}$.

On the other hand, if $x > x^{'}$, and $x$ and $x^{'}$ bumps elements $y$ and $y^{'}$, respectively, the box in the first row where $x$ bumped, and in either case, we must have $y > y^{'}$, so the argument can be repeated on successive rows. This time the route $R^{'}$ must continue at least one row below that of $R$.

**Proposition 3.4** *([2], p.11) Let $T$ be a tableau of shape $\lambda$, and let*

$$U = ((T \leftarrow x_1) \leftarrow x_2) \leftarrow ... \leftarrow x_p)$$

*for some $x_1, ..., x_p$. Let $\mu$ be the shape of $U$. If $x_1 \leq x_2 \leq ... \leq x_p$ (resp. $x_1 > x_2 > ... > x_p$), then no two of the boxes in $\mu/\lambda$ are in the same column (resp. row). Conversely, suppose $U$ is a tableau on a shape $\mu$, and $\lambda$ a Young diagram contained in $\mu$, with p boxes in $\mu/\lambda$. If no two boxes in $\mu/\lambda$ are in the same column (resp. row), then there is a unique tableau $T$ of shape $\lambda$, and unique $x_1, ..., x_p$ (resp. $x_1 > x_2 > ... > x_p$) such that $U = ((T \leftarrow x_1) \leftarrow x_2) \leftarrow ... \leftarrow x_p)$.*

**Proof:** The first assertion is a direct consequence of the lemma. For the converse, in the case where $\mu/\lambda$ has no two boxes in the same column, doing reverse row bumping on U, using the boxes in $\mu/\lambda$, starting from the right-most box and working to the left. The tableau $T$ is the tableau obtained after these operations are carried out, and $x_1, ..., x_p$ are the elements bumped out. The row bumping Lemma guarantees that the resulting sequence satisfies $x_1 \leq x_2 \leq ... \leq x_p$.

Similarly, if $\mu/\lambda$ has no two boxes in the same row, do p reverse bumping, starting from the lowest box in $\mu/\lambda$, and working up; again, The Row Bumping Lemma implies that the elements $x_1, ..., x_p$ bumped out satisfy $x_1 > x_2 > ... > x_p$

**Definition 3.5** The **product** of two tableaux $T$ and $U$ can be defined by using bumping algorithm as follows:

Start with the left-most entry in the bottom row of $U$ and make row insertion into $T$ one by one until there is no one left to bump. The product is denoted by $T.U$.

**Example:**

$$\begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 \\ 5 & 6 \end{ytableau} . \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} = \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 \\ 5 & 6 & 6 \end{ytableau} . \begin{ytableau} 1 & 3 \end{ytableau} = \begin{ytableau} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 4 & 4 & 5 \\ 5 & 5 & 6 \\ 6 \end{ytableau} . \begin{ytableau} 3 \end{ytableau} = \begin{ytableau} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 \\ 5 & 5 & 5 \\ 6 \end{ytableau} \tag{3.1.7}$$

**Claim 1** The multiplication operation makes the set of tableaux into an associative monoid. The empty tableau is a unit in this monoid: $\emptyset.T$=$T.\emptyset$=$T$.

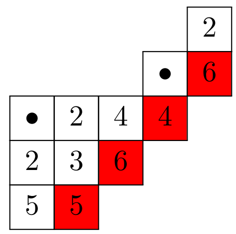
## 3.2 Sliding: Jeu de Taquin

There is another remarkable way to construct the product of two tableaux, using skew tableaux.

**Definition 3.6** An **inside corner** of a skew tableau is a box in the smaller (deleted) diagram $\mu$ such that the boxes below and to the right are not in $\mu$. An **outside corner** is a box in $\lambda$ such that neither box below or to the right is in $\lambda$.

A skew diagram $\lambda/\mu$ which is not a tableau has one or more inside corners.

**Example:**




where red boxes represent outside corners and boxes with black bullets represent inside corners.

To perform the operation we take a skew tableau $S$ and an inside corner which can be thought of as a hole or an empty box. The action is defined by the rules below:

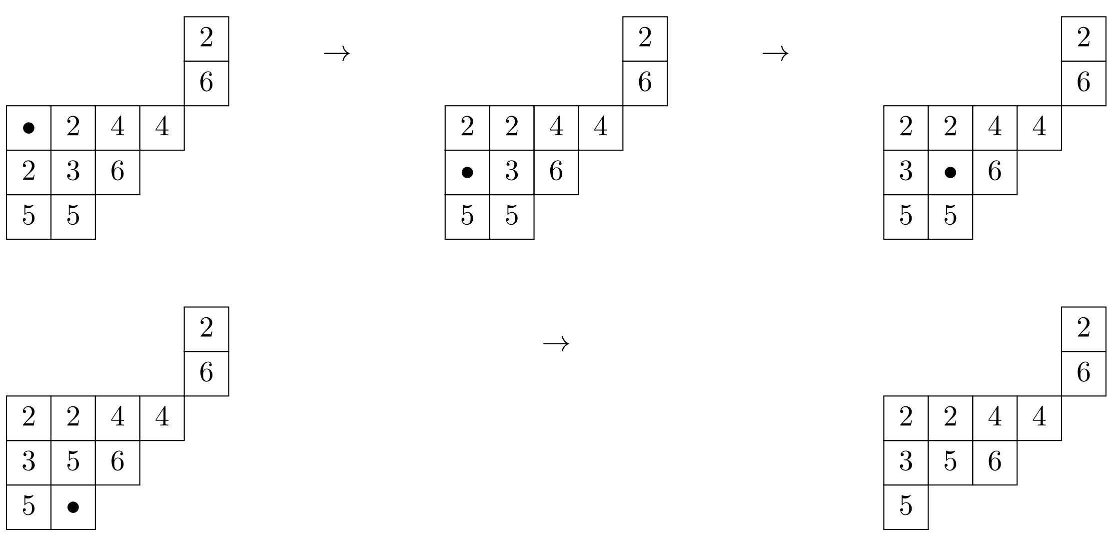
(1) Slide the smaller of its two neighbours, to the right or below, into the empty box.

(2) If only one of these two neighbours is in the skew diagram then we choose that one.

(3) In the situation that two neighbours have the same value then we choose the one below. This is very similar to bumping where we considered elements to the left as smaller than those elements to the right if two elements were the same.

(4) We repeat this process with the new inside corner (hole). If we get to a situation in which there are no boxes to the right or below, then we remove the box from diagram.

**Example:** We begin with the box with black bullet.



**Remark 3.7** The result of sliding operation is always a skew tableau.

As with the bumping algorithm, the sliding algorithm is reversible: if one is given the resulting skew tableau, together with the box that was removed, one can run the procedure backwards.

**Definition 3.8** Using sliding one can slide all inside corners of a skew tableau until there are no more inside corners. This is called **rectification**. The result is a tableau. The rectification of a skew tableau $S$ is denoted by $Rect(S)$.

The whole process is called **jeu de taquin**. It is a game where a player's move is to choose an inside corner. The name *jeu de taquin* refers to the French version of the *15 puzzle* in which one tries to rearrange the numbers by sliding neighboring squares into empty box.

**Claim 2** Starting with a given skew tableau, all choices of inner corners lead to the same rectified tableau.

In fact, jeu de taquin can also be used to give another construction of the product of two Young tableaux. Given two tableaux $T$ and $U$, form a skew tableau denoted $T*U$ as follows: Supposing $T$ is the smaller one, take a rectangle of empty squares with the same number of columns as $T$ and the same number of rows as $U$, and put $T$ below and $U$ to the right of this rectangle. Then the product of $T$ and $U$ is $T.U = Rect(T * U)$ uniquely. This product agrees with the first definition.

**Example:** Let

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 5 & 5 \\ \hline 4 & 4 & 6 \\ \cline{1-3} 5 & 6 \\ \cline{1-2} \end{array} \quad \text{and} \quad U = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Then

$$T * U = \begin{array}{cccccc} \cline{1-6} & & & & 1 & 3 \\ \cline{5-6} & & & & 2 & \\ \cline{1-5} 1 & 2 & 2 & 3 & & \\ \cline{1-4} 2 & 3 & 5 & 5 & & \\ \cline{1-4} 4 & 4 & 6 & & & \\ \cline{1-3} 5 & 6 & & & & \\ \cline{1-2} \end{array}$$

By rectifying this skew tableau one obtains

$$T.U = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3} 5 & 5 & 5 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}$$

# Chapter 4

# WORDS

## 4.1 Words and Elementary Transformations

We can study Young tableaux by using words which encodes the tableau by a sequence of integers.
Given a tableau $T$, we define the word of $T$, $w(T)$ or $w_{row}(T)$, by reading the entries of $T$ from left to right and bottom to top.

**Example:** If $t$ is:

| | | | |
|---|---|---|---|
| 1 | 2 | 2 | 3 |
| 2 | 3 | 5 | 5 |
| 4 | 4 | 6 | |
| 5 | 6 | | |

then the word of $T$ is $w(T) = 56|446|2355|1223$.

We will simply see what the bumping process does to the word of a tableau. This will eventually tell us how the word of a product of two tableaux is related to the words of its factors. Suppose an element $x$ is row inserted into a row. In *word* language, the Schensted algorithm says to factor the word of row into $u.x\prime.v$, where $u$ and $v$ are words, $x\prime$ is a letter, and each letter in $u$ is no larger than $x$, and $x\prime$ is strictly larger than $x$. The letter $x\prime$ is to be replaced by $x$, so the row with word $u.x\prime.v$ becomes $u.x.v$, and $x\prime$ is bumped to the next row. The resulting tableau has word $x\prime.u.x.v$. So in the word code, the basic algorithm is;

$$(u.x\prime.v.).x \rightsquigarrow x\prime.u.x.v \quad \text{if} \quad u \leq x < x\prime \leq v$$

Here $u$ and $v$ are weakly increasing and an inequality such as $u \leq v$ means that every letter in $u$ is smaller than or equal to every letter in $v$.

**Example:** The row insertion of 2 into the Young tableau with the word (56)(446)(2355)(1223) can be written;

$$\begin{aligned}(56)(446)(2355)(1223)2 &\mapsto (56)(446)(2355)3(1222)\\ &\mapsto (56)(466)(5(2335)(1222)\\ &\mapsto (56)6(445)(2335)(1222)\\ &\mapsto (566)(455)(2335)(1222)\end{aligned}$$

**Remark 4.1** The basic transformation for each step above is;

$(yz).x \mapsto (y)(xz)$ if $x < y \leq z \qquad (K)$

$(xz).y \mapsto (z)(xy)$ if $x \leq y < z \qquad (K^{'})$

Two elementary rules mentioned above can be illustrated by the simple products or row-bumping:

If $x < y \leq z$;

$$\boxed{y}\boxed{z}\,.\,\boxed{x} = \begin{array}{|c|c|}\hline x & z \\ \hline y & \\ \cline{1-1}\end{array}$$

If $x \leq y < z$;

$$\boxed{x}\boxed{z}\,.\,\boxed{y} = \begin{array}{|c|c|}\hline x & y \\ \hline z & \\ \cline{1-1}\end{array}$$

An **elementary Knuth transformation** on a word applies one of the transformations $K$ or $K^{'}$. We call two words **Knuth equivalent** if they can be changed into each other by a sequence of elementary Knuth transformations, and we write $w \equiv w^{'}$ to denote that the words $w$ and $w^{'}$ are Knuth equivalent.

**Proposition 4.2** *([2], p.19) For any tableau $T$ and positive integer $x$,*

$$w(T \leftarrow x) \equiv w(T).x$$

Since the first construction of the product $T.U$ of two tableaux was by successively row inserting the letters of the word of $U$ into $T$, we have:

**Corollary 4.3** *([2], p.20) If $T.U$ is the product of two tableaux $T$ and $U$, constructed by row inserting the word of $U$ into $T$ then*

$$w(T.U) \equiv w(T).w(U)$$

**Proposition 4.4** *([2], p.22) If one skew tableau can be obtained from another by a sequence of slides, then their words are Knuth equivalent.*

**Theorem 4.5** *([2], p.22) Every word is a Knuth equivalent to the word of a unique tableau.*

The assertion that every word is Knuth equivalent to the word of some tableau is an easy consequence of Proposition 4.2. Indeed, if $w = x_1 ... x_r$ is any word, Proposition 4.2 shows that it is Knuth equivalent to the word of the tableau:

$$((...((x_1 \leftarrow x_2) \leftarrow x_3) \leftarrow ...) \leftarrow x_{r-1}) \leftarrow x_r$$

We call this **canonical** procedure for constructing a tableau whose word is Knuth equivalent to a given word, and we denote the resulting tableau by $P(w)$.

## 4.2 Schur Polynomials

Any numbering T of a Young diagram we have a monomial, denoted $x^T$, which is the product of the variables $x$; corresponds to the $i$'s that occur in $T$. Formally we have;

$$x^T = \prod_{i=1}^{m} (x_i)$$

number of times $i$ occurs in $T$.

The **Schur polynomial** $s_\lambda(x_1, ..., x_m)$ is the sum;

$$s_\lambda(x_1, ..., x_m) = \Sigma x^T$$

of all monomials coming from tableaux $T$ of shape $\lambda$ using the numbers 1 to $m$.
**Example:** Suppose $\lambda = (2, 1)$ and $m = 3$, there are eight Young tableaux;

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$S_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2$$

# Chapter 5

# REPRESENTATION THEORY

## 5.1 Introduction

We have representations of $S_n$ and combinatorial objects called Young tableaux. In this chapter we will examine how are representations of $S_n$ related to Young tableaux.

**Definition 5.1** Each permutation in $S_n$ can be written in the form of product of cycles and the **cycle type** of a permutation is the unordered list of the sizes of the cycles in the cycle decomposition.

**Example:** $\pi = (123)(45)$ denotes the permutation that sends $1 \to 2 \to 3 \to 1$ and swaps 4 and 5. The cycle type of $\pi$ is the partition whose parts are the lengths of the cycles in the decomposition. So the cycle type of $(123)(45) \in S_5$ is $(3, 2)$.

The cycle type of $(123)(45) \in S_7$ is $(3, 2, 1, 1)$, since actually its cycle decomposition is of the form $(123)(45)(6)(7)$.

**Definition 5.2** Two permutations $a$ and $b$ in $S_n$ are said to be **conjugate** if there exists another permutation $\sigma \in S_n$ such that $\sigma a \sigma^{-1} = b$.

Conjugacy relation is an equivalence relation.

**Example:** In $S_3 = \{1, (12), (13), (23), (123), (132)\}$, there are three conjugacy classes which are: $\{1\}, \{(12), (13), (23)\}, \{(123), (132)\}$.

**Proposition 5.3** *Two elements of $S_n$ are conjugates if and only if they have the same cycle type. Indeed, if*

$$\pi = (a_1, a_2, ..., a_k)(b_1, b_2, ..., b_l)$$

*and $\sigma$ sends $x$ to $x^{'}$ then*

$$\sigma\pi\sigma^{-1} = (a_1^{'}, a_{2'}, ..., a_k^{'})(b_1^{'}, b_2^{'}, ..., b_l^{'})$$

**Proof:** $(\Longrightarrow)$ : Assume that $\sigma$ and $\sigma^{'} = \varsigma\sigma\varsigma$ are conjugate. If $\sigma$ has cycle type $(m_1, m_2, ..., m_t)$, then $\sigma = \sigma_1\sigma_2...\sigma_t$ where $\sigma_k$ is an $m_k$-cycle. The conjugate has cycle decomposition where $\sigma = \sigma_1^{'}\sigma_2^{'}...\sigma_t^{'}$ is also an $m_k$-cycle. Hence $\sigma_{'}$ also has cycle type $(m_1, m_2, ..., m_t)$.

($\Longleftarrow$) : Assume that $\sigma$ and $\sigma\prime$ have the same cycle type, namely $(m_1, m_2, ..., m_t)$. Write their cycle decompositions as $\sigma = \sigma_1\sigma_2...\sigma_t$ and $\sigma = \sigma_1^{'}\sigma_2^{'}...\sigma_t^{'}$ , where $\sigma_k$ and $\sigma_k^{'}$ are both $m_k$-cycles. Rather explicitly, say that

$$\sigma_k = (a_{k,1}, a_{k,2}, ..., a_{k,m_k})$$

and

$$\sigma_k^{'} = (b_{k,1}, b_{k,2}, ..., b_{k,m_k}).$$

The integers $..., a_{k,i}, ... = ..., b_{k,i}, ... = 1, 2, ..., n$, so we can find a permutation $\varsigma \in S_n$ that sends $a_{k,i} \longmapsto b_{k,i}$. This implies $\sigma_k^{'} = \varsigma\sigma_k\varsigma^{-1}$ for each $k$, so that $\sigma^{'} = \varsigma\sigma\varsigma^{-1}$. Hence $\sigma$ and $\sigma^{'}$ are conjugate.

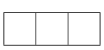
**Remark 5.4** We can say after having showed this proposition, the conjugacy classes of $S_n$ are characterized by the cycle types and thus they correspond bijectively to the partitions of $n$, which are equivalent to Young diagrams of size $n$.

The number of irreducible representations of a finite group is equal to the number of its conjugacy classes.

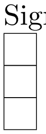
Our goal is to construct an irreducible representation of $\mathrm{S}_n$ corresponding to each Young diagram. We can describe a basis of each irreducible representation using standard Young tableaux.

**Example:** There are three irreducible representations of $S_3$. They can be described by using the set of Young diagrams with three boxes which are illustrated below:

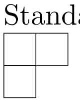
Trivial Representation;



Sign Representation;



Standard Representation;



**Example:** The standard representation of $S_3$ correspond to the following two standard Young tableaux.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \; , \; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

## 5.2 Tabloids and the Permutation Module $M^\lambda$

**Definition 5.5** Two tableaux $t_1$ and $t_2$ are **row equivalent**, denoted $t_1 \sim t_2$, if the corresponding rows of the two tableaux contain the same elements. A **tabloid** of shape $\lambda$ or $\lambda$-tabloid is such an equivalence class, denoted by $\{t\} = \{t_1 \sim t\}$ where $t$ is a $\lambda$-tabloid. The tabloid $\{t\}$ is drawn as the tableau $t$ without vertical bars separating the entries within each row.

**Example:** Suppose $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ then $\{t\}$ is the tabloid drawn as below;

$$\begin{array}{|cc|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}$$

which represents the equivalence class containing the following two tableaux;

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

**Remark 5.6** The order of the entries within each row is irrelevant so that each row may be shuffled arbitrarily. For example the Young tableau drawn below;

$$\begin{array}{|ccc|}\hline 1 & 4 & 7 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 5 \\ \cline{2-3}\end{array} = \begin{array}{|ccc|}\hline 4 & 1 & 7 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 5 \\ \cline{2-3}\end{array} \neq \begin{array}{|ccc|}\hline 4 & 1 & 7 \\ \hline \multicolumn{1}{c|}{} & 6 & 5 \\ \cline{2-3} \multicolumn{1}{c|}{} & 2 & 5 \\ \cline{2-3}\end{array}$$

We have to find a way for elements of $S_n$ to act on tabloids and we will do this by letting the permutations permutate the entries of the tabloid.

**Example:** The cycle $(123) \in S_3$ acts on a tabloid by changing at the same time its 1 by 2, 2 by 3 and 3 by 1 as shown below;

$$(123).\begin{array}{|cc|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array} = \begin{array}{|cc|}\hline 2 & 3 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1}\end{array}$$

**Remark 5.7** If $t_1$ and $t_2$ are row equivalent, the result of the permutation is the same that is $\pi t_1 = \pi t_2$.

**Definition 5.8** Suppose $\lambda \vdash n$. Let $M^\lambda$ denote the vector space whose basis is the set of $\lambda$-tabloids. Then $M^\lambda$ is a representation of $S_n$ known as the **permutation module** corresponding to $\lambda$.

**Example:** Considering $\lambda = (n)$ we see that $M^\lambda$ is the vector space generated by the single tabloid:

$$\begin{array}{|cccc|}\hline 1 & 2 & ... & \text{n} \\ \hline \end{array}$$

Since this tabloid is fixed by $S_n$, we see that $M^n$ is the one dimensional trivial representation.

**Example:** Suppose that $\lambda = (n-1, 1)$. Let $t_i$ be the $\lambda$-tabloid with i on the second row. Then $M^\lambda$ has basis $t_1, t_2, ..., t_n$. Also the action of $\pi \in S_n$ sends $t_i$ to $t_\pi(i)$ in the $n = 4$ case, the representation $M^{(3,1)}$ has the following basis:

$t_1=\begin{array}{|ccc|}\hline 2 & 3 & 4 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1} \end{array}$, $t_2=\begin{array}{|ccc|}\hline 1 & 3 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$, $t_3=\begin{array}{|ccc|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$, $t_4=\begin{array}{|ccc|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}$

**Proposition 5.9** *If* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ *then*

$$\dim M^\lambda = \frac{n!}{\lambda_1!...\lambda_n!}$$

**Proof:** Since the basis for $M^\lambda$ is the set of $\lambda$-tabloids, the dimension of $M^\lambda$ is equal to the number of distinct $\lambda$-tabloids.

Since there are $\lambda_i!$ ways to permute the $i$-th row, the number of tableaux in each row equivalence class is $\lambda_1!\lambda_2!...\lambda_l!$. Since there are $n!$ tableaux in total, the number of equivalence classes is given by $\dfrac{n!}{\lambda_1!\lambda_2!...\lambda_l}$

**Definition 5.10** For a tableau $t$ of size $n$, **the row group** of t, denoted $R_t$, is the subgroup of $S_n$ consisting of permutations which only permutes the elements within each row of $t$.

**Definition 5.11** For a tableau $t$ of size $n$, **the column group** of t, denoted $C_t$, is the subgroup of $S_n$ consisting of permutations which only permutes the elements within each column of $t$.

**Example:** Suppose that;

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \tag{5.2.1}$$

and then ;

$R_t = S_{\{1,2,4\}} \times S_{\{3,5\}}$ and $C_t = S_{\{3,4\}} \times S_{\{1,5\}} \times S_{\{2\}}$

where $S_{\{n_1,n_2,...,n_k\}}$ denotes the symmetric group of the set $\{n_1, n_2, ..., n_k\}$.

**Definition 5.12** If $t$ is a tableau, then the associated **polytabloid** is;

$$e_t = \sum_{\pi \in C_t} sgn(\pi)\pi\{t\}$$

We can find $e_t$ by summing all the tabloids that come from column-permutations of $t$, taking into account the sign of the column-permutation used.

**Example:** Suppose that

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

then the associated polytabloid is as below:

$$e_t = \begin{array}{|ccc|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|ccc|}\hline 3 & 1 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|ccc|}\hline 4 & 5 & 2 \\ \hline 3 & 1 \\ \cline{1-2}\end{array} + \begin{array}{|ccc|}\hline 3 & 5 & 2 \\ \hline 4 & 1 \\ \cline{1-2}\end{array}.$$

## 5.3 Specht Modules

We constructed representations $M^\lambda$ of $S^\lambda$ known as permutation modules. Now we will consider an irreducible subrepresentation of $M^\lambda$ that corresponds uniquely to $\lambda$.

**Lemma 5.13** *Let $t$ be a tableau and $\pi$ be a permutation. Then $e_{\pi t} = \pi e t$*

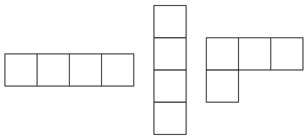
**Proof:** Suppose that $C_{\pi t} = \pi C_t \pi^{-1}$ . Then we can write;

$$\begin{aligned} e_{\pi t} &= \sum_{\sigma \in C_{\pi t}} sgn(\sigma)\sigma\{\pi t\} \\ &= \sum_{\sigma \in \pi C_t \pi^{-1}} sgn(\sigma)\sigma\{\pi t\} \\ &= \sum_{\sigma' \in C_t} sgn(\pi\sigma'\pi^{-1})\pi\sigma'\pi^{-1}\{\pi t\} \\ &= \pi \sum_{\sigma' \in C_t} sgn(\sigma')\sigma'\{t\} \\ &= \pi e_t. \end{aligned}$$

Now we can extract an irreducible representation from $M^\lambda$.

**Definition 5.14** For any partition $\lambda$, the corresponding **Specht module**, denoted $S^\lambda$, is the submodule of $M^\lambda$ spanned by the polytabloids $e_t$, here $t$ is taken over all tableaux of shape $\lambda$.

**Example:** We see that the Specht modules considered to the following Young diagrams are familiar irreducible representations.



**Example:** Consider $\lambda = (n)$. Then there is only one polytabloid, namely;

$$\boxed{1 \quad 2 \quad ... \quad n}$$

Since this tabloid is fixed by $S_n$, we see that $M^{(n)}$ is the one dimensional trivial representation.

**Theorem 5.15** *The Specht modules $S^\lambda$ for $\lambda \mapsto n$ form a complete list of irreducible representations of $S_n$ over $\mathbb{C}$.*

**Sketch of Proof:** We note that the number of irreducible representations of $S_n$ equals the number of Young diagrams with $n$ boxes. This theorem gives a *natural* bijection between the two sets.

**Theorem 5.16** *Let $\lambda$ be any partition. The set*

$$\{e_t : t \text{ is a standard } \lambda\text{−tableau}\}$$

*forms a basis for $S^\lambda$ as a vector space.*

**Sketch of Proof:** If some linear combinations $e_t$ is zero,summed over some standard tableaux $t$, then by looking at a maximal tabloid in the sum, one can deduce that its coefficient must be zero and conclude that $\{e_t : t \text{ is a standard } \lambda\text{−tableau}\}$ is independent.

Let $f^\lambda$ denote the number of standard $\lambda$-tableaux. Then the following result follows immediately from theorem above.

**Corollary 5.17** *Suppose $\lambda \vdash n$, then* $\dim S^\lambda = f^\lambda$

We have an easy way to calculate the number of dimension $S^\lambda$ which is the formula called **hook-length formula**.

**Definition 5.18** Let $\lambda$ be a Young diagram. For a square $u$ in the diagram (denoted by $u \in \lambda$), we define the **hook** of $u$ (or at $u$) to be the set of all squares directly to the right of $u$ or directly below $u$, including $u$ itself. The number of squares in the hook is called the **hook-length** of $u$ (or at $u$), and is denoted by $h_\lambda(u)$.

**Example:** Suppose $\lambda = (5, 5, 4, 2, 1)$ then the figure below shows a typical hook:

$$\begin{array}{|c|c|c|c|c|}\hline \phantom{u} & \phantom{u} & \phantom{u} & \phantom{u} & \phantom{u} \\ \hline & u & \bullet & \bullet & \bullet \\ \hline & \bullet & & & \multicolumn{1}{c}{} \\ \cline{1-4} & \bullet & \multicolumn{3}{c}{} \\ \cline{1-2} & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}$$

**Definition 5.19 (Hook-length formula)** The dimension of the irreducible representations can be easily computed from its Young diagram through the hook-length formula.

Let $\lambda \vdash n$ be a Young diagram . Then

$$\dim S^{\lambda} = f^{\lambda} = \frac{n!}{\prod_{u \in \lambda} h_{\lambda}(u)}.$$

**Example:**

$$\dim S^{(5,5,4,2,1)} = f^{(5,5,4,2,1)} = \frac{17!}{9.8.7.6^2.5.4^3.3^2.2.1^5}.$$

**Example:** Let $n = 4$. For each of the partitions of 4, we compute the dimension of the irreducible representation of $S_4$ corresponding to the associated Young Diagram. Here the numbers in each box is the hook-length of that box.

$$\lambda \;=\; \begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline\end{array} \quad \Rightarrow \quad f^{(4)} = 1$$

$$\lambda \;=\; \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \Rightarrow \quad f^{(3,1)} = 3$$

$$\lambda \;=\; \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array} \quad \Rightarrow \quad f^{(2,2)} = 2$$

$$\lambda \;=\; \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \Rightarrow \quad f^{(2,1,1)} = 3$$

$$\lambda \;=\; \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} \quad \Rightarrow \quad f^{(1,1,1,1)} = 1$$

## 5.4 Young's Rule

**Definition 5.20** The **content** of $T$ is the composition $\mu = (\mu_1, \mu_2, ..., \mu_m)$, where $\mu_i$ equals the number of $i$'s in $T$.

**Example:** The following is a tableau with shape (4,2,1) and content (2,2,1,0,1,1):

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 5 \\ \hline 2 & 3 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

**Definition 5.21** Suppose $\lambda, \mu$ is a partiton of $n$, the **Kostka number** $K_\lambda \mu$ is the number of Young tableax of shape $\lambda$ and content $\mu$.

**Example:** If $\lambda = (3,2)$ and $\mu = (2,2,1)$, then $K_\lambda \mu = 2$ since there are exactly two tableaux of shape $\lambda$ and content $\mu$:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$$

**Young's rule:** $M^\lambda \cong \bigoplus_{\lambda \trianglelefteq \mu} K_{\lambda\mu} S^\lambda$.

**Example:** Representation of $S_4$ is:

$$\mathbb{C}S_4 \cong S^{(4)} \bigoplus 3S^{(3,1)} \bigoplus 2S^{(2,2)} \bigoplus 3S^{(2,1,1)} \bigoplus 2S^{(2,2)} \bigoplus 2S^{(1,1,1,1)}$$

## 5.5 Conclusion

We have seen that the Young tableaux can be used to construct a unique representation of $S^n$ by using some rules. This gives an important application to representation theory. Actually, this is just an introduction which exposes the power of Young tableaux. There are more complicated applications to combinatorics and algebraic geometry. The study of Young tableaux is an active research topic in mathematics.

# Bibliography


[1] B. Sagan, *The ubiquitous Young tableau, Invariant theory and tableaux* . IMA Vol. Math. Appl.,19, Springer,New York, 1990.

[2] W. Fulton, *Young tableaux with applications to representation theory and geometry,* . London Mathematical Society Student Texts 35,Cambridge University Press, Cambridge, 1997.

[3] J-P. Serre, *Linear Representations of Finite Groups* New York,Springer 1977.

[4] W. Fulton and J. Harris, *Representation Theory: A First Course*, New York, Springer, 1991.