Hyperbolic Cauchy problem and Leray's residue formula.

Susumu Tanabé

Dedicated to the memeory of Bogdan Ziemian

Abstract

We give an algebraic descriptions of (wave) fronts that appear in strictly hyperbolic Cauchy problem. Concrete form of defining function of wave front issued from initial algebraic variety is obtained by the aid of Gauss-Manin systems satisfied by Leray's residues.

0. Introduction.

In one of his last works [20], Prof. Bogdan Ziemian pursued a possibility to express fundamental solutions to PDE by the aid of Leray residues. He used this technique to write down the Mellin transform of fundamental solutions to Fuchsian type PDE and proved that these solutions belong to so called a class of generalized analytic functions(GAF).

In this note, we show that the advantage to make use of Leray's residue formula in hyperbolic Cauchy problem is the fact that it facilitates the calculus of the representative integrals (i.e. a basis of certain cohomology group) whose summation gives fundamental solutions to the Cauchy problem.

Let us explain in short how the Leray's residue formula can be applied to the construtioin of fundamental solutions. Let us denote $V_x = \{\xi \in \mathbf{C}^n; F(\xi, x') + x_0 = 0\}$ a complex variety of dimension $n-1$ depending on $x = (x', x_0) \in \mathbf{R}^m$ defined by a polynomial $F(\xi, x') + x_0$. Choose a cycle $\gamma_x \in H_{n-1}(V_x)$. For x "in generic position" the variety V_x is smooth and γ_x is a family of cycles depending continuously on x. Suppose that $a(\xi)$ is a smooth function defined on $\mathbf{C}^n$. In this situation the following equality is called the Leray's residue formula:

$$I_a(x) = \int_{\gamma_x} a(\xi) \frac{d\xi}{dF} = \frac{1}{2\pi i} \int_{\partial\gamma_x} \frac{a(\xi)}{F(\xi, x') + x_0} d\xi, \tag{0.1}$$

where $\partial\gamma_x \in H_n(\mathbf{C}^n \setminus V_x)$ so called Leray coboundary of the cycle γ_x which is homotopically equivalent to a S^1- bundle over γ_x. See [14], [19].

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If $a(\xi)d\xi = d\psi \wedge dF$ for some $\psi \in \Omega^{n-2}_{\mathbf{C^n}}$, then the integral $I_a(x)$ defined above must be constantly zero:

$$I_a(x) = \int_{\gamma_x} \frac{d\psi \wedge dF}{dF} = \int_{\gamma_x} d\psi = 0, \tag{0.2}$$

that one can see by Stoke's theorem. Evidently

$$\int_{\gamma_x} F(\xi, x')a(\xi)\frac{d\xi}{dF} = -x_0 \int_{\gamma_x} a(\xi)\frac{d\xi}{dF}. \tag{0.3}$$

After (0.2) and (0.3), we conclude that the important forms $a(\xi)d\xi$ that will give non zero $I_a(x)$ not expressed by other Leray's residue must be of the space

$$''H = \frac{\Omega^n_{\mathbf{C}^n}}{dF \wedge d\Omega^{n-2}_{\mathbf{C}^n} + F\Omega^n_{\mathbf{C}^n}}. \tag{0.4}$$

Furthermore (0.1) yields the relation

$$\int_{\gamma_x} \frac{d\omega}{dF} = \frac{d}{dx_0}\int_{\gamma_x} \omega, \tag{0.5}$$

for $\omega \in \Omega^{n-1}_{\mathbf{C}^n}$. That is to say a differential equation satisfied by $I_a(x)$ does not depend on the choice of cycle along which one defines the integral.

In his famous work [3], E.Brieskorn has shown that for $F(\xi, x')$ whose singular fibre V_0 defines a hypersurface isolated singularity, the space $''H$ is a vector space of finite dimension μ that coincides with the Milnor number of the singularity $V_0 = \{\xi \in \mathbf{C}^n; F(\xi, 0) = 0\}$. Thus if a fundamental solution $I_a(x)$ is expressed by a sum of integrals of forms $b_1 d\xi, \cdots, b_\mu d\xi \in {}''H$, for certain $F(\xi, x')$ we can expect that analytic properties of the fundamental solution can be deduced from those of $I_{b_1}(x), \cdots, I_{b_\mu}(x)$ which in their turn can be described by informations of the singularity V_0.

To pursue further this study, we propose to make use of the Gauss-Manin system associated with fibre bundle structure that naturally arises in integration. Our main tool is concrete expressions of the overdetermined differential systems obtained from non trivial relations between base elements of $''H$ for complete intersection singularities (Proposition 5, Theorem 7). In our situation, Leray's residue formula can be written down like (2.10) below. The main Theorem 10 directly follows the Theorem 7 on reducing the situation to a specific mapping (2.15).

In our former work [18] we illustrated in concrete examples the possibility to interprete the fundamental solution to the Cauchy problem associated with the wave operator

$$P(D_t, D_x) = (\frac{\partial}{\partial t})^2 - \sum_{j=1}^{n}(\frac{\partial}{\partial x_j})^2,$$

as generalized hypergeometric functions. In this note we consider the Cauchy problem associated with strictly hyperbolic operators with constant coefficients in general. This procedure has been supported by the general theory of the Gauss-Manin systems for isolated complete intersection singularities [5]. More systematic explanation of this situation from singularity theoretical point of view is given in [1], [16], [17] and [18].

1. Preliminaries on Cauchy problem.

In this section we prepare fundamental notations and lemmata to develop our studies in further sections. Let $P(D_t, D_x)$ be a strictly hyperbolic operator with constant coefficients of degree m i.e. its total symbol

$$P(\tau,\xi) = \tau^m + \sum_{i=1}^{m} P_{m-i}(\xi)\tau^{m-i},$$

$$P_{m-i}(\xi) = \sum_{|\alpha|=i} P_{m-i,\alpha}\xi^{\alpha} \in \mathbf{R}[\xi],$$

satisfies the following decomposition

$$P(\tau,\xi) = \prod_{j=1}^{m}(\tau - \lambda_j(\xi)),$$

such that $\lambda_j(\xi) \in \mathbf{R}$ if $\xi \in \mathbf{R}^n$ and $\lambda_i(\xi) \neq \lambda_j(\xi)$ for $i \neq j$, $\xi \in \mathbf{R}^n \setminus \{0\}$.

Without loss of generality, we suppose that $P(\tau,\xi)$ is an irreducible polynomial of $\mathbf{R}[\tau,\xi]$. If $P(\tau,\xi) = P_1(\tau,\xi) \cdot P_2(\tau,\xi)$ fundamental solutions of $P(D_t, D_x)$ is a sum of those of $P_1(D_t, D_x)$ and $P_2(D_t, D_x)$ provided that their characteristic roots are mutually distinct out of the origin. Let us consider the following Cauchy problem (C.P.):

$$\text{(C.P.)} \begin{cases} P(D_t, D_x)u(t,x) &= 0 \\ D_t^{m-1}u(0,x) &= v(x) \\ D_t^{m-j}u(0,x) &= 0, 2 \leq j \leq m \end{cases}$$

where $t \in \mathbf{R}, x \in \mathbf{R}^n, D_t = \frac{\partial}{i\partial t}, D_x = (\frac{\partial}{i\partial x_1}, \cdots, \frac{\partial}{i\partial x_n}), i = \sqrt{-1}$. We will study the Cauchy problem (C.P.) under the following conditions (C.1), (C.2), (C.3) imposed on the initial data. In order to describe these conditions, we use notation $\chi_q^{\epsilon}(z)$ $(\epsilon = \pm 1)$ which stands for the following distributions defined as boundary values of an analytic function on $\mathbf{C}_z^1 \setminus \{0\}$ (cf. [4]) :

$$\chi_q^{\epsilon}(z) = \chi_q(z + i0) + \epsilon\chi_q(z - i0),$$

where

$$\chi_q(z) = z^q/\Gamma(q) \;\; q \notin \mathbf{Z} \;\text{ or } q \text{ negative integer}$$

$$\chi_q(z) = \frac{z^q}{q}(-\log z + C_q) \quad q \textit{ positive integer}$$

where $C_0 = 0,\ C_q = C_{q-1} + 1/q$. Remark that

$$\frac{d}{dz}\chi_q(z) = \chi_{q-1}(z).$$

(C.1) *The initial data are given by a distribution of finite order with singular support (see Definition 2.2.3 of [9]) located on cotangent bundle of a smooth algebraic surface* $S := \{x \in \mathbf{R}^n : F(x) - s = 0\}$ *defined by a real polynomial* $F(x)$,

$$v(x) = g(x)\chi_q^{\epsilon}(F(x) - s)$$

with a smooth function $g(x)$.

Further we shall denote the singular support of a distribution $v(x)$ by $S.S.v(x)$.

We impose several technical conditions also. These conditions will be used so that the reasoning on the isolated complete intersection singularities can be applied to our (C.P.)

(C.2)(Quasihomogeneity) *There exists a set of positive integers* $(w_1, \cdots, w_n)$, *that satisfies 1)* $w_i \neq w_j$ *for certain pair* $1 \leq i \neq j \leq n$, *2) G.C.D.*$(w_1, \cdots, w_n) = 1$ *and 3) for a positive integer* $w(F)$,

$$\Big(\sum_{1 \leq j \leq n} w_j x_j \frac{\partial}{\partial x_j}\Big)F(x) = w(F)F(x),$$

holds.

The above contdition 1) plays an essential rôle in establishing the Lemma 8 below.

(C.3) *The following is a vector space of finite dimension:*

$$\mathbf{R}[x]/\mathcal{I}$$

where $\mathcal{I} = \langle F(x), \frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x)\rangle$ *(ideal generated by the entries).* Let us introduce the following notations.

a) The phase function $\psi(x, t, z)$ is defined as follows:

$$\psi(x, t, z) = P(\langle x - z, grad_z F(z)\rangle, t\; grad_z F(z)).$$

b) The paired oscillatory integrals studied in [4] defined for the phase function $\psi(x, t, z)$ introduced in a):

$$I_p^{\epsilon}(x, t, s) = \int_{\{F(z)=s\}} H_p(z)\chi_p^{\epsilon}(\psi(x, t, z))\frac{dz}{dF},$$

with regular amplitude functions of pseudo-differential operator

$$H_p(z) \sim \sum_{r=p-m}^{-\infty} h_{p,r}(z) \in S^{p-m}(\mathbf{R}^n),$$

in which $h_{p,r}(z)$ is homogeneous of order r for large values of z. One shall understand $I_p^\epsilon(x,t,s)$ as the Gel'fand-Leray integral (see 1.5 [3]) defined on the real algebraic set $S = \{z \in \mathbf{R}^n; F(z) = s\}$.

$c)$ The function $\phi(x,t,s)$ denotes a defining function of the (wave) front Σ issued from S determined by $(C.P.)$.

Proposition 1 *The following assertions hold for solutions to the Cauchy problem (C.P.) with the notations introduced in a),b) and c) as above.*

d) The solution $u(x,t)$ to the Cauchy problem (C.P.) admits an asymptotic expansion

$$u(x,t) \sim \sum_{j=0}^{\infty} I^\epsilon_{-n/2+q+j}(x,t,s). \tag{1.1}$$

That is to say, for every $N >> 0$ there exists $C_N > 0$ such that

$$\mid u(x,t) - \sum_{j=0}^{N} I^\epsilon_{-n/2+q+j}(x,t,s) \mid \leq C_N \mid \phi(x,t,s) \mid^{q+N+1}, \tag{1.2}$$

in the neighbourhood of $S.S.u(x,t)$.

Proof We give only a sketch of proof while a detailed one shall appear in [17], [18]. First of all we show that the phase function of the integrals $I_p^\epsilon(x,t,s)$ coming into a) is given by c). In solving the Hamilton-Jacobi equation associated with the Hamiltonian $\tau - \lambda_\kappa(\xi)$, $1 \leq \kappa \leq m$ (in a symplectic coordinate with canonical symplectic form $dt \wedge d\tau + \sum_{j=1}^n dx_j \wedge d\xi_j$. For the symplectic geometry see Chapter XXI [9]),

$$\begin{cases} \dot{t} &= & 1 \\ \dot{\tau} &= & 0 \\ \dot{x}_i &= & \frac{\partial \lambda_\kappa(\xi)}{\partial \xi_i} \\ \dot{\xi}_i &= & 0 \\ x_i(0) &= & z_i, \quad \{z \in \mathbf{R}^n; F(z) = s\}, 1 \leq i \leq n, \end{cases}$$

we get

$$x_i = t\frac{\partial \lambda_\kappa(\xi)}{\partial \xi_i} + z_i \text{ with } z \in S = \{z \in \mathbf{R}^n; F(z) = s\}. \tag{1.3}$$

This means that the singularities of the solutions to (C.P.) lies on the rays (1.3). These lines are interpreted as rays issued from the initial front $S = \{z \in \mathbf{R}^n : F(z) = s\}$ in directions determined by the Hamiltonians

$$P(\tau,\xi) = \tau^m + \sum_{i=1}^{m} P_{m-i}(\xi)\tau^{m-i} = \prod_{j=1}^{m}(\tau - \lambda_j(\xi))$$

given in (1.2).

Consequently they are expressed by integrals with phase

$$\psi(x,t,z) = \prod_{j=1}^{m} (\langle x-z, grad_z F(z)\rangle - t\,\lambda_j(grad_z F(z))) \tag{1.4}$$

$$= P(\langle x-z, grad_z F(z)\rangle, t grad_z F(z)).$$

This is the "minimal" algebraic equation describing the wave front in view of the irreducibility of the polynomial $P(\tau,\xi)$. Here we remark that for every $p \in \mathbf{Q}$ and $H(z) \in \mathcal{D}'(\mathbf{R}^n_z)$, we have

$$P(D_t, D_x)\int_S H(z)(\psi(x,t,z))^p \frac{dz}{dF} = 0.$$

One can prove this equality with the aid of Gauss-Stokes' theorem. Thus the question is how to find a series of integrals

$$I^\epsilon_p(x,t,s) = \int_S H_p(z)\chi^\epsilon_p(\psi(x,t,z))\frac{dz}{dF}, \quad p \in \mathbf{Q}$$

whose suitably converging sum produces a distribution $u(x,t)$ satisfying (C.1). The possibility of an asymptotic expansion (1.2) consisting of terms like $b)$ can be proven by well known estimates on the stationary phase ([9], Theorem 7.7.12). More precisely, let us remind the following lemma.

Lemma 2 *Let $(G)^0(x,t,s)$ be a residue of a smooth function $G(x,t,z)$ after division by an Jacobi ideal generated by $\frac{\partial\psi(x,t,z)}{\partial z_j}$, $\frac{\partial F(z)}{\partial z_j}$, $1 \le j \le n$, and $F(z)-s$, i.e.*

$$G(x,t,z) = G^0(x,t,s) + \sum_{j=1}^n f_j(x,t,s,z)\frac{\partial\psi(x,t,z)}{\partial z_j} + \tag{1.5}$$

$$+\sum_{j=1}^{n} g_j(x,t,s,z)\frac{\partial F(z)}{\partial z_j} + h(x,t,s,z)(F(z)-s),$$

with some smooth functions $h(x,t,s,z)$, $f_j(x,t,s,z)$, $g_j(x,t,s,z)$, $1 \le j \le n$.

Then for every smooth function $a(z)$ the following asymptotic estimate with some $C_N > 0$ holds in the neighbourhood of the wave front $\Sigma = \{(x,t) \in \mathbf{R}^{n+1}; (\psi)^0(x,t,s) = 0\}$:

(1.6) $\mid \int_S a(z)\chi_q^\epsilon(\psi(x,t,z))dz - \sum_{j=0}^N (L_{\psi,j}a)^0(x,t,s)\chi_{q+n/2+j}^\epsilon((\psi)^0(x,t,s)) \mid$

$$< C_N \mid (\psi)^0(x,t,s) \mid^{n/2+N+1+q}$$

with differential operators $L_{\psi,j}$ of degree $2j$. Furthermore we have:

$$(L_{\psi,0}a)^0(x,t,s) = i^{\frac{n}{2}}(2\pi)^{\frac{n-1}{2}}(a)^o(x,t,s) \mid det(\frac{\psi_{zz}}{2\pi i})^0(x,t,s) \mid^{-1/2} .$$

In the literature concerning the singularity theory, one often calls the correspondence $G(x,t,z) \rightarrow G^0(x,t,s)$ Lyashko-Loojenga mapping.

Let us briefly sketch proof of the lemma. Malgrange's division theorem yields the decomposition (1.6) in connexion with the fact that the follwing $\mathcal{O}_{\mathbf{C}^n}-$ module is a finite dimensional vector space under the assumption (C.3):

$$\frac{\Omega^n_{\mathbf{C}^n}}{dF(z) \wedge \Omega^{n-1}_{\mathbf{C}^n} + d\psi(0,0,z) \wedge \Omega^{n-1}_{\mathbf{C}^n} + d_z\Omega^{n-1}_{\mathbf{C}^n} + F(z) \wedge \Omega^n_{\mathbf{C}^n}}.$$

Further it suffices to apply above mentioned stationary phase method.

After Lemma 2 the function $(\psi)^0(x,t,s)$ can be given by (1.5) for $G(x,t,z) = \psi(x,t,z)$ in (1.4). As a definig function $\phi(x,t,s)$ of the wave front issued from S, one shall take a polynomial such that $\{(x,t) \in \mathbf{R}^{n+1}; (\psi)^0(x,t,s) = 0\} \subset \{(x,t) \in \mathbf{R}^{n+1}; \phi(x,t,s) = 0\}$ and $S = \{x : \phi(x,0,s) = 0\}$. We adopt such $\phi(x,t,s)$ as needed one in c).

It remains to justify asymptotic estimates in b) and d). This can be achieved in view of (1.6) and well known construction of an elementary solution to strictly hyperbolic Cauchy problem (see for example [7], [8]). Especially due to the choice of $\phi(x,t,s)$, the inequality (1.2) satisfied with $\psi^0(x,t,s)$ in a local context holds with $\phi(x,t,s)$. Hence the assertion follows. **Q.E.D.**

We formulate a simple lemma before introducing necessary notations.

Lemma 3 *Under the assumptions (C.2), (C.3) imposed on $F(x)$ there exists a collection of polynomials degree $\leq m$, $W_1(x,t), \cdots, W_{\mu'}(x,t)$, with μ' an integer smaller than $m^n \prod_{i=1}^n (\frac{w(F)}{w_i})$, satisfying*

$$(1.7) \qquad \psi(x,t,z) = \langle z, grad_z F(z) \rangle^m + \sum_{i=1}^{\mu'} W_i(x,t) z^{\alpha^{(i)}}$$

for $\psi(x,t,z)$ of (1.4). Here $\alpha^{(i)} = (\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}) \in (\mathbf{Z}_{\geq 0})^n$ stands for multi-indiex under restriction $\sum_{j=1}^n w_j \alpha_j^{(i)} < m \cdot w(F)$.

The proof follows direct calculation of (1.4). Quasihomogeneous type of $F(z)$ yields the estimate on term number μ'. Let us denote by

$$w(z^{\alpha^{(i)}}) = \sum_{j=1}^{n} w_j \alpha_j^{(i)},$$

quasihomogeneous weight of monomial $z^{\alpha^{(i)}}$ for $\alpha^{(i)} \in \mathbf{N}^n$. In terms of the quasihomogeneous weight we distinguish two cases.

Case 1 If there is a term with $w(z^{\alpha^{(i)}}) = 0$, let us mark it as $\alpha^{(1)}$ and define a polynomial

$$f_1(y(x,t), z) = \langle z, grad_z F(z) \rangle^m + \sum_{i=1}^{\mu'} y_i(x,t) z^{\alpha^{(i)}}.$$

Here $y_i(x,t) = W_i(x,t), 1 \leq i \leq \mu'$, for polynomials introduced in Lemma 3, (1.7).

Case 2 If all terms of (1.7) has positive weight, we shall define

$$f_1(y(x,t), z) = \langle z, grad_z F(z) \rangle^m + \sum_{i=2}^{\mu'+1} y_i(x,t) z^{\alpha^{(i-1)}} + y_1.$$

with $y_{i+1}(x,t) = W_i(x,t), 1 \leq i \leq \mu'$.

For the sake of simplicity we adopt notation $\mu = \mu'$ for Case 1 and $\mu = \mu' + 1$ for Case 2.

Further we define integrals

$$I_p(y(x,t), s) = \int_S H_p(z) \chi_p(f_1(y(x,t), z)) \frac{dz}{dF}. \tag{1.8}$$

Hence if one denotes $y' = (y_2(x,t), \cdots, y_\mu(x,t))$,

$$I_p^\epsilon(x,t,s) = I_p(y_1 + i0, y'(x,t), s) + \epsilon I_p(y_1 - i0, y'(x,t), s)$$

on understanding that the boundary value shall be taken at $y_1 = 0$ in the above mentioned Case 2. Thus it is essential to study $I_p(y(x,t), s)$ of (1.8) to estimate asymptotic behaviour of $I_p^\epsilon(x,t,s)$. From now on we shall regard the integral (1.8) as a function in variables $y(x,t) = (y_1, y_2(x,t), \cdots, y_\mu(x,t))$. Therefore our main concern will be to investigate the differential equations that satisfy $I_p(y,s)$ corresponding to various amplitudes $H_p(z)$ w ith the aid of Gauss-Manin connexions associated to complete intersection singularities.

2. Gauss-Manin connexions for quasihomogeneous complete intersections.

Here we propose to study the integrals $I_p(y,s)$ defined in (1.8) by means of the Gauss-Manin system associated with complete intersection singularities. In effect, it is well known that the Gauss-Manin connexion can be defined on the relative de Rham cohomology groups. Instead of that here we propose to calculate it on spaces so called Brieskorn lattices (see [3], [5]).

The formulation of this section is a modification of [16], §1 adapted to our situation.

Let us observe a mapping between complex manifolds $X = (\mathbf{C}^{N+K}, 0), Y = (\mathbf{C}^K, 0)$,

$$f : X \to Y$$

that defines an isolated quasihomogeneous complete intersection singularity at the origin. That is to say, if we denote

$$X_y := \{u \in X; f_0(u) = y_0, \ldots, f_{K-1}(u) = y_{K-1}\}, \tag{2.1}$$

then $dim X_y = N \geq 0$ and the critical set of mapping $f : X_0 \to Y$ is isolated in X_0. Further we assume that polynomials $f_0(u), \cdots, f_{K-1}(u)$ are quasihomogeneous i.e. there exists a collection of positive integers $v_1, \cdots, v_{N+K}$ whose greatest common divisor equals 1 and

$$(v_1 u_1 \frac{\partial}{\partial u_1} + \cdots + v_{N+K} u_{N+K} \frac{\partial}{\partial u_{N+K}}) f_\ell(u) = p_\ell f_\ell(u), \quad \ell = 0, 1, \cdots, K-1,$$

for certain integers $p_0, \cdots, p_{K-1}$. We shall call the vector field

$$E = \sum_{i=1}^{N+K} v_i u_i \frac{\partial}{\partial u_i}, \tag{2.2}$$

Euler vector field and $v_1, \cdots, v_{N+K}$ (resp. $p_0, \cdots, p_{K-1}$) positive weights of variables $u_1, \cdots, u_{N+K}$ (resp. polynomials $f_0, \cdots, f_{K-1}$) i.e. $v_1 = w(u_1)$, $p_0 = w(f_0)$ etc.

In order to calculate the Gauss-Manin connexion for isolated complete intersection singularity X_0, we introduce two vector spaces V and F. After Greuel-Hamm [6], we look at a space whose dimension as a vector space over $\mathbf{C}$ is known to be the Minor number $\mu(X_0)$ of singularity X_0,

$$V := \frac{\Omega_X^N}{df_0 \wedge \Omega_X^{N-1} + \cdots + df_{K-1} \wedge \Omega_X^{N-1} + d\Omega_X^{N-1} + f_0 \Omega_X^N + \cdots + f_{K-1} \Omega_X^N}. \tag{2.3}$$

The second one will later turn out to be isomorphic to V (see Proposition 6),

$$F := \frac{\Omega_X^{N+1}}{df_0 \wedge \Omega_X^N + \cdots + df_{K-1} \wedge \Omega_X^N + i_E(\Omega_X^{N+2})}. \tag{2.4}$$

Here i_E means the inner contraction with Euler field E defined by (2.2). The third vector space associated with the singularity X_0 is defined as follows

$$(2.5)\qquad \Phi := \frac{\Omega_X^{N+K}}{df_0 \wedge \cdots \wedge df_{K-1} \wedge \Omega_X^N + f_0 \Omega_X^{N+K} + \cdots + f_{K-1}\Omega_X^{N+K}}.$$

Later we define period integrals as coupling of forms of V or of Φ with base element of homology groups $H_N(X_y)$. We remember also the definition of the Brieskorn lattice $''H$ from [5],

$$''H = \frac{\Omega_X^{N+K}}{df_0 \wedge \cdots \wedge df_{K-1} \wedge d\Omega_X^{N-1}},$$

whose rank as $\mathbf{O}_Y-$ module equals the Minor number $\mu(X_0)$ of the singularity X_0. It is easy to show

Lemma 4 *For $f_0, \cdots, f_{K-1}$ quasihomogeneous polynomials defining an isolated complete intersection singularity,*

$$\Phi \cong {}''H/(f_0, \cdots, f_{K-1}).$$

Thus $dim_{\mathbf{C}}\Phi = \mu(X_0)$.

Let us denote by $\tilde{\omega}_1, \cdots \tilde{\omega}_{\mu(X_0)}$ a basis of F such that each $\tilde{\omega}_i$ is a quasihomogeneous $N+1$ form. From definitions (2.4) and (2.5) we easily deduce the following.

Proposition 5 *For every form $\tilde{\omega}_i \in F$, one has the following decomposition,*

$$(2.6)\quad \tilde{\omega}_i \wedge df_0 \wedge \cdots^{\vee \ell} \wedge df_{K-1} = \sum_{j=1}^{\mu(X_0)} P_{ij}^{(\ell)} \phi_j(u) du \; mod(df_0 \wedge \cdots \wedge df_{K-1} \wedge d\Omega_X^{N-1})$$

with $P_{ij}^{(\ell)} \in \mathbf{C}[f_0, \cdots, f_{K-1}]$ and $\phi_j(u)du \in \Phi$, for $1 \leq i, j \leq \mu(X_0), 0 \leq \ell \leq K-1$ and $df_1 \wedge \cdots^{\vee \ell} \cdots \wedge df_{K-1} = \bigwedge_{i \neq \ell} df_i$.

From [16] we remember the following

Proposition 6 *Under the situation and definitions as above, the mapping,*

$$i_E : F \to V$$

induces an isomorphism. Consequently $dim_{\mathbf{C}}F = dim_{\mathbf{C}}V = \mu(X_0)$.

In view of Proposition 6, let us choose the base of F by $\tilde{\omega}_i$ (resp. V by ω_i) such that $\tilde{\omega}_i = \frac{1}{\ell_i} d\omega_i$ where ℓ_j denotes weight of the form ω_j. Remark that $i_E \tilde{\omega}_i \equiv \frac{1}{\ell_i}(di_E + i_E d)(\omega_i) \equiv \omega_i$, $1 \leq i \leq \mu(X_0)$ in F. To make a transition from $(N+K)-$ forms to period integrals, we introduce meromorphic $N-$forms ψ_i satisfying

$$df_0 \wedge \cdots \wedge df_{K-1} \wedge \psi_i = \phi_i(u)du, \quad 1 \leq i \leq \mu(X_0).$$

Then we derive the following relation from Proposition 5,

(2.7) $d\omega_j = \ell_j \tilde{\omega}_j \equiv \ell_j(\sum_{q=1}^{\mu(X_0)} P_{jq}^{(0)} df_0 \wedge \psi_q + \cdots$
$+(-1)^{K-1} \sum_{q=1}^{\mu(X_0)} P_{jq}^{(K-1)} df_{K-1} \wedge \psi_q) \; mod((df_0, \cdots, df_{K-1}) d\Omega_X^{N-1})$.

See (2.12) below to see that this relation calculates the "partial deriative" $\frac{d\omega}{df_i}$. Hence,

(2.8) $$\omega_j \equiv i_E(\tilde{\omega}_j) \equiv$$

$\sum_{i=0}^{K-1}(-1)^i [\sum_{q=1}^{\mu(X_0)} P_{jq}^{(i)} p_i f_i \psi_q - \sum_{q=1}^{\mu(X_0)} P_{jq}^{(i)} df_i \wedge i_E(\psi_q)]$
$mod((df_0, \cdots, df_{K-1}) i_E d\Omega_X^{N-1}, (f_0, \cdots, f_{K-1}) d\Omega_X^{N-1})$.

As a consequence

(2.9) $d\omega_j \equiv \sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i (d(p_i P_{jq}^{(i)} f_i) - w(\psi_q) P_{jq}^{(i)} df_i)] \wedge \psi_q +$
$\sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i p_i P_{jq}^{(i)} f_i] \wedge d\psi_q, \;\; mod((df_0, \cdots, df_{K-1}) d\Omega_X^{N-1})$,

where $w(\psi_q)$, quasihomogeneous weight of form ψ_q. The expression (2.9) can be simplified if one lets them couple with a vanishing $N-$cycle, say $\gamma(y)$ and attains non trivial relations between integrals $\int_{\gamma(y)} \psi_q$, instead of those between forms. One defines so called period integral $I_{\phi_q, \gamma(y)}(y)$ taken along a vanishing cycle $\gamma(y)$ whose ambiguity in homology class $H_N(X_y, \mathbf{Z})$ we do not care for the moment,

(2.10) $I_{\phi_q,\gamma(y)}(y) := \int_{\gamma(y)} \psi_q = (\frac{1}{2\pi i})^K \int_{\partial\gamma(y)} \frac{df_0 \wedge \cdots \wedge df_{K-1} \wedge \psi_q}{(f_0 - y_0) \cdots (f_{K-1} - y_{K-1})}$

$$= (\frac{1}{2\pi i})^K \int_{\partial\gamma(y)} \frac{\phi_q(u) du}{(f_0 - y_0) \cdots (f_{K-1} - y_{K-1})},$$

where $\partial\gamma(y) \in H_{N+K}(\mathbf{C}^{N+K} \setminus \cup_{i=0}^{K-1} \{f_i = y_i\}, \mathbf{Z})$ is a cycle obtained by the aid of Leray's coboundary operator ∂. That is to say, although ψ_q is in general a meromorphic form with poles along the critical set of the mapping f, $I_{\phi_q,\gamma(y)}(y)$ can be calculated as an integral of a holomorphic form on $\partial\gamma$.

One may consult a booklet by F.Pham [14], or a book by V.A.Vasiliev [19] on the coboundary operator. One understands (2.10) the Leray's residue formula in our situation (2.1).

From (2.8) we can deduce

(2.11) $$\int_{\gamma(y)} \omega_j = \sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i p_i y_i P_{jq}^{(i)}(y)] I_{\phi_q,\gamma(y)}(y).$$

It is easily seen from the following evident equality in view of the definition (2.10),

$$\int_{\partial\gamma(y)} \frac{df_0 \wedge \cdots \wedge df_{K-1}}{(f_0 - y_0) \cdots (f_{K-1} - y_{K-1})} \wedge df_i \wedge i_E(\psi_q) = 0, \; 0 \leq i \leq K-1.$$

$$\int_{\partial\gamma(y)} \frac{df_0 \wedge \cdots \wedge df_{K-1}}{(f_0 - y_0) \cdots (f_{K-1} - y_{K-1})} \wedge d\varphi = 0, \; \varphi \in \Omega_X^{N-1}.$$

Let us compare the relation

(2.12) $\quad d\int_{\gamma(y)}\omega_j = \ell_j \sum_{q=1}^{\mu(X_0)}[\sum_{i=0}^{K-1}(-1)^i P_{jq}^{(i)}(y)dy_i] I_{\phi_q,\gamma(y)}(y),$

obtained from (2.7) and (2.11). As a result we get equations between $I_{\phi_q}(y)$ and $\frac{\partial}{\partial y_\ell} I_{\phi_q}$, $0 \leq \ell \leq K-1$, (we omit to specify $\gamma(y)$ except necessary cases),

$$\frac{\partial}{\partial y_\ell}[\sum_{q=1}^{\mu(X_0)}\sum_{i=0}^{K-1}(-1)^i p_i y_i P_{jq}^{(i)} I_{\phi_q}]$$
$$= \ell_j \sum_{q=1}^{\mu(X_0)}[\sum_{i=0}^{K-1}(-1)^i P_{jq}^{(i)}(y)dy_i] I_{\phi_q}(y), 1 \leq j \leq \mu(X_0).$$

Thus we have obtained a system of differential equations to be understood as the Gauss-Manin connexion of the singularity X_0. To state the theorem in a simple form, we introduce the following notations: $\mathbf{I}_V = (\int_{\gamma(y)}\omega_1, \cdots, \int_{\gamma(y)}\omega_{\mu(X_0)})$, $\mathbf{I}_\Phi = (I_{\phi_1,\gamma(y)}(y), \cdots, I_{\phi_{\mu(X_0)},\gamma(y)}(y))$, i.e. vectors of integrals taken along a certain vanishing cycle $\gamma(y)$. We define several other $\mu(X_0) \times \mu(X_0)$ matrices as follows,

$$L_V = diag(\ell_1, \cdots, \ell_{\mu(X_0)})$$

with $\ell_i = w(\omega_i)$, $P^{(0)}(y) = (P_{jq}^{(0)}(y)), \cdots, P^{(K-1)}(y) = (P_{jq}^{(K-1)}(y)), 1 \leq j, q \leq \mu(X_0)$, matrices consisting of elements defined in (2.6).

In summing up the above arguments and the theory due to Greuel [5], we obtain the following.

Theorem 7 *1). For a quasihomogeneous mapping*

$$f : X \to Y$$

with isolated complete intersection singularities of dimension N like (2.1), the Gauss-Manin system satisfied by $\mathbf{I}_\Phi$ is described as follows:

(2.13) $\quad d[\sum_{i=0}^{K-1}(-1)^i p_i y_i P^{(i)}(y)\mathbf{I}_\Phi] = L_V[\sum_{i=0}^{K-1}(-1)^i P^{(i)}(y)dy_i]\mathbf{I}_\Phi.$

2). The critical value D (singular locus of the system (2.13)) of deformation X_y is given by $D = \{y \in Y : \Delta(y) = 0\}$ where

$$\Delta(y) = det(\sum_{i=0}^{K-1}(-1)^i p_i y_i P^{(i)}(y)). \tag{2.14}$$

3). The system of differential equations (2.13) is a holonomic system.

Let us return to the problem (C.P.) of §1. Our main concern is to understand the integral (1.8) as a sum of integrals like (2.10) for certain mapping f. To adapt our (C.P.) to the scheme explained before Theorem 7, we treat the following mapping $f : X \to Y$ for $X = (\mathbf{C}_u^{n+\mu}, 0), Y = (\mathbf{C}_y^{\mu+1}, 0)$. Concretely, it is defined as follows,

$$(2.15)\begin{cases} f_0(u) & = & F(z) & = & y_0 \\ f_1(u) & = & z_{n+\mu}^P + \langle z, grad_z F(z)\rangle^m + \sum_{i=1}^{\mu-1} z_{n+i} z^{\alpha^{(i)}} & = & y_1 \\ f_2(u) & = & z_{n+1} & = & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{i+1}(u) & = & z_{n+i} & = & y_{i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_\mu(u) & = & z_{n+\mu-1} & = & y_\mu \end{cases}$$

with notation $z = (z_1, \cdots, z_n), z' = (z_{n+1}, \cdots, z_{n+\mu-1}), u = (z, z', z_{n+\mu})$. Here the power P is an integer that corresponds to the denominator of $q \in \mathbf{Q}$.

Lemma 8 *For $F(z)$ under conditions (C.2), (C.3), the mapping (2.15) $(f_0, \cdots, f_\mu)$ defines an isolated quasihomogeneous complete intersection singularity,*

$$X_0 = \{u \in X : f_0(u) = \ldots = f_\mu(u) = 0\}.$$

Namely

$$V = \frac{\Omega_X^{n+\mu}}{f_0\Omega_X^{n+\mu} + f_1\Omega_X^{n+\mu} + \sum_{i=1}^{\mu} z_{i+n}\Omega_X^{n+\mu} + dF\wedge\Omega_X^{n+\mu-1} + df_1\wedge\Omega_X^{n+\mu-1} + \sum_{i=1}^{\mu} dz_{i+n}\wedge\Omega_X^{n+\mu-1}},$$

is a finite dimensional vector space.

Proof The complete intersection property follows from the fact that two polynomials $F(z)$ and $\langle z, grad_z F(z)\rangle$ are of the same quasihomogeneous weight but with different coefficients. This is a consequence of (C.2), 1) which supposes that $F(z)$ is not a homogeneous polynomial. The condition (C.3) entails immediately the finite dimensionality of V. **Q.E.D.**

To see that the components of $\mathbf{I}_\Phi$ defined for the mapping (2.15) give rise to integrals of type (1.8), we prepare the following.

Lemma 9 *1). Let us denote*

$$\Phi(z, z') = \frac{\Omega_X^{n+\mu+1}}{df_0 \wedge \ldots df_\mu \wedge \Omega_X^n + \sum_{i=0}^{\mu} f_i\Omega_X^{n+\mu+1}}|_{z_{n+\mu}=0}.$$

Then the following natural isomorphism holds

$$\Phi \cong \Phi(z, z') \otimes (\mathbf{C}[z_{n+\mu}]/ < z_{n+\mu}^P >).$$

2). For $\partial\gamma_{n-2} \in H_{n+\mu-1}(\mathbf{C}^{n+\mu-1} \setminus \cup_{i=0}^{\mu}\{f_i = y_i\} \mid_{z_{n+\mu}=0}, \mathbf{Z})$ a Leray coboundary of a vanishing cycle $\gamma_{n-2} \in H_{n-2}(X_y \mid_{z_{n+\mu}=0}, \mathbf{Z})$ one can choose a corresponding vanishing cycle $\tilde{\gamma}_{n-1} \in H_{n-1}(X_y, \mathbf{Z})$ such that an equality

(2.16) $\int_{\partial\gamma_{n-2}} \phi(z)(f_1(z, y_2, \cdots, y_\mu, 0) - y_1)^{\frac{r+1}{P}-1}\frac{dz}{dF} =$

$$= \epsilon(\frac{1}{2\pi i})^{\mu} \int_{\partial\tilde{\gamma}_{n-1}} \phi(z) z^r_{n+\mu} \frac{du}{(f_0 - u_0)\cdots(f_\mu - u_\mu)}$$

holds, where $\epsilon \in \mathbf{C}^\times$ *such that* $\epsilon^P = 1$. *Furthermore, the cycle* $\partial\tilde{\gamma}_{n-1} \in H_{n+\mu}(\mathbf{C}^{n+\mu} \setminus \cup_{i=0}^{\mu}\{f_i = u_i\}, \mathbf{Z})$ *is topologically equivalent to a product of a small circle on complex* $z_{n+\mu}-$ *plane and* $\partial\gamma_{n-2}$.

Proof The statement 1) is evident. The statement 2) is an integral version of statement 1), which can be shown by means of equality (2.10). **Q.E.D.**

Thus the singular locus of the integral (1.8) can be given by that of

$$\int_{\tilde{\gamma}_{n-1}} \phi(z) z^r_{n+\mu} \frac{du}{df_0 \wedge \cdots \wedge df_\mu}$$

with $\tilde{\gamma}_{n-1} \in H_{n-1}(X_y, \mathbf{Z})$ after substitution $y_1 = -W_1(x,t)$ (Case 1 after Lemma 3), or $y_1 = 0$ (Case 2 after Lemma 3), $y_i = W_i(x,t), 2 \leq i \leq \mu$. Let us remind that we denoted the quasihomogeneous weight of function f_i by p_i, $0 \leq i \leq \mu$. We define matrices $P^{(i)}(y), 2 \leq i \leq \mu$ for the mapping (2.15) after the master (2.6) and Theorem 7. Combining Theorem 7 with Lemma 9, we obtain the following.

Theorem 10 *The defining equation of wave front (Propositi on 1, c)) is given by the following polynomial:*

$$\phi(x,t,s) = det(\sum_{i=0}^{\mu}(-1)^i p_i y_i P^{(i)}(y)) \mid_{y_0=s, y_i=W_i(x,t), 2\leq i\leq\mu} . \tag{2.17}$$

Here the restriction shall be imposed in accordance with two cases treated just after Lemma 3 i.e. $y_1 = -W_1(x,t)$ *in Case 1 and* $y_1 = 0$ *in Case 2.*

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Moscow Independent University ,
Bol'shoj Vlasievskij pereulok 11,
121002, Moscow, Russia
E-mail: tanabe@mpim-bonn.mpg.de,
tanabe@mccme.ru