INVARIANTS OF HYPERGEOMETRIC GROUPS FOR CALABI-YAU COMPLETE INTERSECTIONS IN WEIGHTED PROJECTIVE SPACES

SUSUMU TANABÉ AND KAZUSHI UEDA

Abstract. Let Y be a Calabi-Yau complete intersection in a weighted projective space. We show that the space of quadratic invariants of the hypergeometric group associated with the twisted I-function is one-dimensional, and spanned by the Gram matrix of a split-generator of the derived category of coherent sheaves on Y with respect to the Euler form.

1. Introduction

Let $\boldsymbol{q} = (q_0, \dots, q_N)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$ be sequences of positive integers such that

$$Q := q_0 + \cdots + q_N = d_1 + \cdots + d_r,$$

and consider a complete intersection Y of degree $(d_1, \dots, d_r)$ in the weighted projective space $\mathbb{P} = \mathbb{P}(q_0, \dots, q_N)$. If Y is smooth, then it is a Calabi-Yau manifold of dimension $n = N - r$. The derived category $D^b \operatorname{coh} \mathbb{P}$ of coherent sheaves on $\mathbb{P}$ has a full strong exceptional collection

$$(\mathcal{E}_i)_{i=1}^Q = (\mathcal{O}_\mathbb{P}, \mathcal{O}_\mathbb{P}(1), \dots, \mathcal{O}_\mathbb{P}(Q-1))$$

of line bundles [Beĭ78, AKO08]. Let $(\mathcal{F}_i)_{i=1}^Q$ be the full exceptional collection dual to $(\mathcal{E}_i)_{i=1}^Q$, so that

$$\chi(\mathcal{E}_{Q-i+1}, \mathcal{F}_j) = \delta_{ij}$$

where

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_k (-1)^k \dim \operatorname{Ext}^k(\mathcal{E}, \mathcal{F}) \tag{1.1}$$

is the Euler form. The derived restrictions $\{\overline{\mathcal{F}}_i\}_{i=1}^Q$ of $\{\mathcal{F}_i\}_{i=1}^Q$ to Y split-generate the derived category $D^b \operatorname{coh} Y$ of coherent sheaves on Y [Sei11, Lemma 5.4].

Following [CLCT09, Equation (4)], let us introduce the *twisted I-function*

$$I_{\mathbb{P},Y}(t) = \sum_\alpha e^{P_\alpha \log t} t^{\rho_\alpha} \sum_{n=0}^\infty t^n \frac{\prod_{k=1}^r \prod_{\substack{b:\, \langle b \rangle = \langle \rho_\alpha d_k \rangle \\ 0 < b \le (n+\rho_\alpha) d_k}} (d_k P_\alpha + b)}{\prod_{\nu=0}^N \prod_{\substack{b:\, \langle b \rangle = \langle \rho_\alpha q_\nu \rangle \\ 0 < b \le (n+\rho_\alpha) q_\nu}} (q_\nu P_\alpha + b)}, \tag{1.2}$$

where $\langle x \rangle = x - \lfloor x \rfloor$ is the fractional part of x. This is an element of the ring $\bigoplus_\alpha \mathbb{C}[P_\alpha]_\alpha / (P_\alpha^{\mu_\alpha})_\alpha$, which is isomorphic to the orbifold cohomology of $\mathbb{P}$ as a vector space. See Section 2 for the definition of $\rho_\alpha \in \mathbb{Q}$ and $\mu_\alpha \in \mathbb{N}$. The twisted I-function $I_{\mathbb{P},Y}$ contains the information of twisted Gromov-Witten invariant of the bundle $\mathcal{O}_\mathbb{P}(d_1) \oplus \cdots \oplus \mathcal{O}_\mathbb{P}(d_r) \to \mathbb{P}$ through the twisted J-function [CCIT09].

2010 *Mathematics Subject Classification.* 32S40, 33C20, 53D37.
Key words and phrases. hypergeometric function; monodromy; mirror symmetry.

The components of the twisted I-function span the space of solutions of the hypergeometric differential equation

$$\left[\prod_{\nu=0}^{N}\prod_{a=0}^{q_\nu-1}(q_\nu\theta_t-a)-t\prod_{k=1}^{r}\prod_{b=1}^{d_k}(d_k\theta_t+b)\right]I=0 \tag{1.3}$$

where $\theta_t=t\dfrac{\partial}{\partial t}$. Let $\mathcal{H}$ be the differential operator on the left hand side and $\mathcal{H}^{\mathrm{red}}$ be the operator obtained from $\mathcal{H}$ by removing common factors from the two summands of $\mathcal{H}$. Both $\mathcal{H}$ and $\mathcal{H}^{\mathrm{red}}$ have regular singularities at 0, ∞ and $\lambda=\prod_{\nu=0}^{N}q_\nu^{q_\nu}\Big/\prod_{k=1}^{r}d_k^{d_k}$. The local system $\mathcal{L}^{\mathrm{red}}$ defined by $\mathcal{H}^{\mathrm{red}}$ is irreducible, and its rank Q^{red} is smaller than the rank Q of the local system $\mathcal{L}$ defined by $\mathcal{H}$. The irreducible system $\mathcal{L}^{\mathrm{red}}$ supports a pure and polarized variation of Hodge structures, whose Hodge numbers are computed by Corti and Golyshev [CG11, Theorem 1.3].

The mirror of Y is identified by Batyrev and Borisov [BB96] as the family of toric complete intersections whose affine part is given by

$$X_t=\{(x_0,\cdots,x_N)\in(\mathbb{C}^\times)^{N+1}\mid f_i(x)=0,\ i=0,\dots,r\} \tag{1.4}$$

where

$$\begin{aligned} f_0(x)&=x_0^{q_0}x_1^{q_1}\dots x_N^{q_N}-t,\\ f_i(x)&=\sum_{k\in S_i}x_k-1, \qquad\qquad i=1,\dots,r,\end{aligned}$$

the variable t is the parameter of the family, and $S_1\sqcup\cdots\sqcup S_r=\{0,1,\dots,N\}$ is a partition of $\{0,1,\dots,N\}$ into r disjoint subsets such that $d_k=\sum_{i\in S_k}q_i$. The period integral

$$I(t)=\int_\gamma\frac{x_0^{q_0}\dots x_N^{q_N}}{df_0\wedge\cdots\wedge df_r}\frac{dx_0}{x_0}\wedge\cdots\wedge\frac{dx_n}{x_n} \tag{1.5}$$

for a middle-dimensional cycle $\gamma\in H_n(X_t)$ satisfies the irreducible hypergeometric differential equation $\mathcal{H}^{\mathrm{red}}I=0$.

Define the *hypergeometric group* $H_{\boldsymbol{q},\boldsymbol{d}}$ as the subgroup of $GL(Q,\mathbb{Z})$ generated by

$$h_\infty=\begin{pmatrix}0&0&\dots&0&-A_Q\\1&0&\dots&0&-A_{Q-1}\\0&1&\dots&0&-A_{Q-2}\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&1&-A_1\end{pmatrix} \tag{1.6}$$

and

$$h_0^{-1}=\begin{pmatrix}0&0&\dots&0&-B_Q\\1&0&\dots&0&-B_{Q-1}\\0&1&\dots&0&-B_{Q-2}\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&1&-B_1\end{pmatrix}, \tag{1.7}$$

where

$$\prod_{k=1}^{r}(T^{d_k}-1)=T^Q+A_1T^{Q-1}+A_2T^{Q-2}+\cdots+A_Q \tag{1.8}$$

and

$$\prod_{\nu=0}^{N}(T^{q_\nu}-1) = T^Q + B_1 T^{Q-1} + B_2 T^{Q-2} + \cdots + B_Q \tag{1.9}$$

are the characteristic polynomials of the monodromy of (1.3) at infinity and zero respectively. If the system is irreducible, then a result of Levelt [Lev61] states that the monodromy group is conjugate to the hypergeometric group $H_{\boldsymbol{q},\boldsymbol{d}}$. Although the system $\mathcal{L}$ is reducible and one can not apply the result of Levelt directly, we can show the following:

Theorem 1.1. *For any sequences $\boldsymbol{q} = (q_0, \dots, q_N)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$ of positive integers satisfying*

$$Q := q_0 + \cdots + q_N = d_1 + \cdots + d_r,$$

the monodromy group of (1.3) *is conjugate to the hypergeometric group $H_{\boldsymbol{q},\boldsymbol{d}}$*

An element $h \in H_{\boldsymbol{q},\boldsymbol{d}}$ acts naturally on the space of $Q \times Q$-matrices by

$$H_{\boldsymbol{q},\boldsymbol{d}} \ni h : X \mapsto h \cdot X \cdot h^T,$$

where h^T is the transpose of h. The following is a corollary of Theorem 1.1:

Theorem 1.2. *The space of $Q \times Q$-matrices invariant under the action of the monodromy group $H_{\boldsymbol{q},\boldsymbol{d}}$ of* (1.3) *is one-dimensional and spanned by the Gram matrix*

$$\left(\chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_j)\right)_{i,j=1}^{Q}$$

of the split-generator $\{\overline{\mathcal{F}}_i\}_{i=1}^{Q}$ with respect to the Euler form.

This theorem is closely related to the works of Horja [Hor, Theorem 4.9] and Golyshev [Gol01, §3.5], which goes back to Kontsevich [Kon98]. The main difference from their works is that we work with the reducible system $\mathcal{L}$ which contains solutions not coming from period integrals on the mirror manifold. Although the geometric meaning of these extra solutions is unclear, Theorem 1.1 shows that the monodromy of the reducible system is controlled by the derived category of coherent sheaves on Y just as in the case of the irreducible system.

The organization of this paper is as follows: The proof of Theorem 1.1 is given in Section 2. The essential step is to show the existence of a cyclic vector for the monodromy around the origin, which satisfies additional condition with respect to the monodromy at infinity. The uniqueness of the invariant of the hypergeometric group is shown in Section 3, and the invariance of the Gram matrix of the split-generator with respect to the Euler form is shown in Section 4. In Section 5, we discuss the relationship between the Gram matrix in Theorem 1.2 and the Stokes matrix for the quantum cohomology of the weighted projective space.

Acknowledgment: We thank Hiroshi Iritani for valuable discussions. This work is supported by Grant-in-Aid for Scientific Research (No. 20540086 and No. 20740037).

2. Monodromy of hypergeometric equation

We prove Theorem 1.1 in this section. Let h_0, h_1 and h_∞ be the global monodromy matrix of the hypergeometric differential equation (1.3) around the origin, one and infinity with respect to some basis of solutions satisfying $h_0 \cdot h_1 \cdot h_\infty = 1$. Recall that a vector $v \in \mathbb{C}^Q$ is said to be *cyclic* with respect to $h \in GL(Q, \mathbb{C})$ if the set $\{h^i \cdot v\}_{i=0}^{Q-1}$ spans $\mathbb{C}^Q$. The following lemma is used by Levelt [Lev61] to compute the monodromy of hypergeometric functions (see also Beukers and Heckman [BH89, Theorem 3.5]).

Lemma 2.1. *Assume that there exists a vector satisfying*

$$h_0^i v = h_\infty^{-i} v, \qquad i = 1, \dots, Q-1, \tag{2.1}$$

which is cyclic with respect to h_0*. Then the monodromy group of* (1.3) *is isomorphic to* $H_{\boldsymbol{q},\boldsymbol{d}}$*.*

Proof. The condition (2.1) shows that the action of h_0 and h_∞^{-1} with respect to the basis $\{h_\infty^{-i} v\}_{i=0}^{Q-1}$ of $\mathbb{C}^Q$ is given by

$$\begin{pmatrix} 0 & 0 & \dots & 0 & * \\ 1 & 0 & \dots & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \end{pmatrix}.$$

The last line is determined by the characteristic equations

$$\det(T - h_0) = T^Q + A_1 T^{Q-1} + A_2 T^{Q-2} + \dots + A_Q$$

and

$$\det(T - h_\infty^{-1}) = T^Q + B_1 T^{Q-1} + B_2 T^{Q-2} + \dots + B_Q.$$

□

Remark 2.2. Even if there is no vector satisfying (2.1) which is cyclic with respect to h_0, one can consider the subspace generated from any vector satisfying (2.1) by the action of h_0, and the resulting matrix presentation the monodromy action with respect to $\{h_\infty^{-i} v\}_{i=0}^{Q-1}$ will be given by (1.7) and (1.6). Since $\{h_\infty^{-i} v\}_{i=0}^{Q-1}$ is not a basis but only a generator in such a case, this matrix presentation is not unique.

Hence the proof of Theorem 1.1 is reduced to the following:

Proposition 2.3. *There exists a vector* v *in the space of solutions of* (1.3) *which is cyclic with respect to* h_0 *and satisfies* (2.1)*.*

The rest of this section is devoted to the proof of Proposition 2.3. The hypergeometric differential equation (1.3) has regular singularities at $t = 0, \infty$ and λ where $\lambda = \prod_{\nu=0}^N q_\nu^{q_\nu} \Big/ \prod_{k=1}^r d_k^{d_k}$. To simplify notations, we introduce another variable z by $t = \lambda z$. Then the local exponents are given by

$$\begin{array}{llr} \dfrac{b}{d_k}, \quad k = 1, \dots, r, \quad b = 1, \dots, d_k & & \text{at } z = \infty, \\ \dfrac{a}{q_\nu}, \quad \nu = 1, \dots, N, \quad a = 0, \dots, q_\nu - 1 & & \text{at } z = 0, \text{ and} \\ 0,\ 1,\ 2, \dots, Q-2,\ \dfrac{n-1}{2} & & \text{at } z = 1. \end{array}$$

Let

$$1 > \rho_1 > \rho_2 > \dots > \rho_p = 0$$

be the characteristic exponents of (1.3) at $z = 0$ so that

$$\{\rho_1, \cdots, \rho_p\} = \bigcup_{0 \le \nu \le N} \left\{ 0, \frac{1}{q_\nu}, \dots, \frac{q_\nu - 1}{q_\nu} \right\}.$$

Let further

$$\mu_\alpha = \#\left\{ (q_\nu, a) \,\middle|\, \rho_\alpha = \frac{a}{q_\nu}, \quad 0 \le a \le q_\nu - 1, \quad 0 \le \nu \le N \right\}$$

be the multiplicity of the exponent ρ_α and put

$$e_\alpha = \exp(2\pi\sqrt{-1}\rho_\alpha), \qquad 1 \le \alpha \le p.$$

We remark that $\mu_p = N+1$. For the quantity defined by

$$\nu_\alpha = \#\left\{ (d_k, b) \,\middle|\, \rho_\alpha = 1 - \frac{b}{d_k}, \quad 1 \le b \le d_k - 1, \quad 1 \le k \le r \right\}, \qquad 1 \le \alpha \le p$$

the following relation holds

$$Q^{\mathrm{red}} = \sum_{\alpha=1}^{p} (\mu_\alpha - \nu_\alpha).$$

Let us introduce the matrices

$$M_0 = \begin{pmatrix} \rho_1 \operatorname{id}_{\mu_1} + J_{\mu_1,-} & 0 & \cdots & 0 \\ 0 & \rho_2 \operatorname{id}_{\mu_2} + J_{\mu_2,-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_p \operatorname{id}_{\mu_p} + J_{\mu_p,-} \end{pmatrix}$$

$$E_0 = \begin{pmatrix} e_1 \operatorname{id}_{\mu_1} + J_{\mu_1,-} & 0 & \cdots & 0 \\ 0 & e_2 \operatorname{id}_{\mu_2} + J_{\mu_2,-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_p \operatorname{id}_{\mu_p} + J_{\mu_p,-} \end{pmatrix}$$

where $J_{i,\pm}$ are $i \times i$ matrices defined by

$$J_{i,+} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$J_{i,-} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

A series solution to (1.3) at the origin can be obtained by the Frobenius method:

Lemma 2.4. *A basis of solutions to* (1.3) *can be obtained as the coefficient of* P_α^i *for* $\alpha = 1, \ldots, p$ *and* $i = 0, \ldots, \mu_\alpha - 1$ *in the* Γ*-series in* (1.2). *Solutions to the irreducible equation* $H^{red}u = 0$ *correspond to the cofficient of* P_α^i *for* $i = 0, \ldots, \mu_\alpha - \nu_\alpha - 1$ *in* (1.2).

Proof. Let

$$H = e^{P_\alpha \log t} t^{\rho_\alpha} \sum_{n=0}^{\infty} t^n \frac{\prod_{k=1}^{r} \prod_{\substack{b:\ \langle b \rangle = \langle \rho_\alpha d_k \rangle \\ 0 < b \le (n+\rho_\alpha) d_k}} (d_k P_\alpha + b)}{\prod_{\nu=0}^{N} \prod_{\substack{b:\ \langle b \rangle = \langle \rho_\alpha q_\nu \rangle \\ 0 < b \le (n+\rho_\alpha) q_\nu}} (q_\nu P_\alpha + b)}$$

be the Γ-series in (1.2) considered as a formal power series in P_α and $\log t$. A direct calculation shows

$$\left[\prod_{\nu=0}^{N}\prod_{a=0}^{q_\nu-1}(q_\nu\theta-a)-t\prod_{k=1}^{r}\prod_{b=1}^{d_k}(d_k\theta+b)\right]I=\left(\prod_{\nu=0}^{N}\prod_{a=0}^{q_\nu-1}(q_\nu P_\alpha+q_\nu\rho_\nu-a)\right)e^{P\log t}t^{\rho_\alpha},$$

where the right hand side is proportional to $P_\alpha^{\mu_\alpha}$, so that the coefficients of P^i for $i=0,1,\ldots,\mu_\alpha-1$ give solutions to the hypergeometric equation (1.3).

Alternatively, one can also argue as follows: For the set of poles $\Pi_\alpha=\{w\in\mathbb{C}:|w+\rho_\alpha+n|=\epsilon,n\in\mathbb{N},0<\epsilon\ll 1\}$ the Mellin-Barnes integral

$$\frac{1}{2\pi i}\int_{\Pi_\alpha}\prod_{\nu=0}^{N}\Gamma(q_\nu w)\prod_{k=1}^{r}\Gamma(1-d_kw)s^{-w}dw,$$

with $s=(-1)^Qt$ gives us a solution that is the $P_\alpha^{\mu_\alpha-1}$ part of (1.2). This can be seen from the following calculation

$$\sum_{n\geq 0}Res_{w=-\rho_\alpha-n}\prod_{\nu=0}^{N}\Gamma(q_\nu w)\prod_{k=1}^{r}\Gamma(1-d_kw)s^{-w}$$

$$=\sum_{n\geq 0}\frac{1}{(\mu_\alpha-1)!}(\frac{d}{dw})^{\mu_\alpha-1}((w+\rho_\alpha+n)^{\mu_\alpha}\prod_{\nu=0}^{N}\Gamma(q_\nu w)\prod_{k=1}^{r}\Gamma(1-d_kw)s^{-w})|_{w=-\rho_\alpha-n}$$

$$=\sum_{n\geq 0}\frac{1}{(\mu_\alpha-1)!}(-\frac{d}{dP})^{\mu_\alpha-1}((-P)^{\mu_\alpha}\prod_{\nu=0}^{N}\Gamma(-q_\nu(\rho_\alpha+n+P))\prod_{k=1}^{r}\Gamma(1+d_k(\rho_\alpha+n+P))s^{\rho_\alpha+n+P}|_{P=0}$$

$$=-\sum_{n\geq 0}\frac{(-1)^{nQ}}{(\mu_\alpha-1)!}\sum_{\kappa=0}^{\mu_\alpha-1}\binom{\mu_\alpha-1}{\kappa}\left((\frac{d}{dP})^{\mu_\alpha-1-\kappa}\frac{\prod_{k=1}^{r}\prod_{j=1}^{nd_k}(d_k(\rho_\alpha+P)+j)}{\prod_{\nu=0}^{N}\prod_{i=1}^{nq_\nu}(q_\nu(\rho_\alpha+P)+i)}((-1)^Qt)^{\rho_\alpha+n+P}\right)|_{P=0}$$

$$\times\left((\frac{d}{dP})^{\kappa}P^{\mu_\alpha}\prod_{\nu=0}^{N}\Gamma(-q_\nu(\rho_\alpha+P))\prod_{k=1}^{r}\Gamma(1+d_k(\rho_\alpha+n+P))\right)|_{P=0}$$

To get a solution with $P_\alpha^{\mu_\alpha-2}$ part of (1.2) we choose $\nu_1\in[0,N]$ such that $\rho_\alpha=\frac{a}{q_{\nu_1}}$ for some $a\in[0,q_{\nu_1}-1]$ and calculate

$$\frac{1}{2\pi i}\int_{\Pi_\alpha}(-1)^{q_{\nu_1}w}\frac{\prod_{\nu=0,\nu\neq\nu_1}^{N}\Gamma(q_\nu w)\prod_{k=1}^{r}\Gamma(1-d_kw)s^{-w}}{\Gamma(1-q_{\nu_1}w)}dw$$

In this way we increase the number of Γ-factors in the denominator. The factor $\Gamma(q_\nu w)$ multiplied by a function $\frac{sin\ (\pi q_\nu w)}{\pi}(-1)^{q_\nu w}$ with period $2\pi i$ gives $\frac{(-1)^{q_\nu w}}{\Gamma(1-q_\nu w)}$. Thus we obtain a μ_α tuple of Mellin-Barnes integral solutions to (1.3) that are linear combinations of (1.2). To get (1.2) solutions from the Mellin-Barnes integral solutions we need only to solve a system of linear equations determined by a $\mu_\alpha\times\mu_\alpha$ upper triangle matrix with non-zero diagonal entries.

As for the statement on the solutions to the irreducible operator H^{red} we shall consider the Mellin-Barnes integrals

$$\frac{1}{2\pi i}\int_{\Pi_\alpha}\frac{\prod_{\nu=0}^{N}\Gamma(q_\nu w)}{\prod_{k=1}^{r}\Gamma(d_kw)}s^{-w}dw,$$

whose poles $w \in \Pi_\alpha$ are atmost of order $\mu_\alpha - \nu_\alpha$. On calculating its residues, we obtain a subspace of solutions to (1.3) of dimension $Q^{red} = \sum_{\alpha=1}^{p}(\mu_\alpha - \nu_\alpha)$. □

This immediately gives the following:

Corollary 2.5. *There is a basis*

$$\boldsymbol{X}(z) = (X_1(z), \ldots X_Q(z))$$

of solutions to (1.3) *such that the monodromy around $s = 0$ is given by*

$$\boldsymbol{X}(z) \to \boldsymbol{X}(z) \cdot E_0.$$

Set $\sigma_0 = 0$ and $\sigma_i = \sum_{\alpha=1}^{i} \mu_\alpha$ for $i = 1, \ldots, p$.

Lemma 2.6. *$X_{\sigma_i}(z)$ is singular at $z = 1$ for any $1 \le i \le p$.*

Proof. Assume that $X_{\sigma_i}(z)$ is holomorphic at $z = 1$. Since $X_{\sigma_i}(z)$ is a solution to (1.3), its only possible singular points on $\mathbb{C}$ are $z = 0$ and 1, so that $z^{-\rho_i} X_{\sigma_i}(z)$ is in fact an entire function. Since (1.3) has a regular singularity at infinity, $X_{\sigma_i}(z)$ has at most polynomial growth at infinity. This implies that $z^{-\rho_i} X_{\sigma_i}(z)$ is a polynomial, which cannot be the case since the series (1.2) defining $X_{\sigma_i}(z)$ around the origin is infinite. □

Lemma 2.7. *There is a fundamental solution $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_Q(z))$ of* (1.3) *around $z = 1$ such that $Y_i(z)$ is holomorphic for $i = 1, \ldots, Q-1$.*

Proof. We prove the following stronger result; Y_i has a series expansion

$$Y_i = (z-1)^{i-1} \sum_{m \ge 0} G_m (z-1)^m$$

for $i = 1, \ldots, Q-1$, and $Y_Q(z)$ has the series expansion

$$Y_Q(z) = (z-1)^{\frac{n-1}{2}} \sum_{m \ge 0} G'_m (z-1)^m + \sum_{m \ge 0} G''_m (z-1)^m$$

when n is even, and

$$Y_Q(z) = (z-1)^{\frac{n-1}{2}} \log(z-1) \left(\sum_{m \ge 0} G'_m (z-1)^m \right) + \sum_{m \ge 0} G''_m (z-1)^m$$

when n is odd. Since $Q-2$ is the largest exponent, one can find a series solution

$$Y_{Q-1} = (z-1)^{Q-2} \sum_{m \ge 0} G_m (z-1)^m$$

to (1.3). Then one can remove a common factor in (1.3) from the left to obtain a differential equation, whose set of local exponents at $z = 1$ is given by

$$\left\{ 0, 1, \ldots, Q-3, \frac{n-1}{2} \right\}.$$

Now $Q-3$ is the largest exponent, and one can find a series solution

$$Y_{Q-2} = (z-1)^{Q-3} \sum_{m \ge 0} G_m (z-1)^m$$

to this equation. One can continue this process until the differential equation becomes irreducible one with the rank Q^{red} whose set of exponents is given by $\left\{ 0, 1, \ldots, Q^{red} - 2, \frac{n-1}{2} \right\}$.

This irreducible differential equation describes the period of the Calabi-Yau manifold obtained by compactifying Y_t (cf. [CG11, Theorem 1.1]). This Calabi-Yau manifold has an ordinary double point at $z = 1$, and the period integral along the vanishing cycle gives the singular solution $Y_Q(z)$, while integrals against classes orthogonal to the vanishing cycle give holomorphic solutions $Y_1(z), \dots, Y_{Q^{red}-1}(z)$. □

Lemma 2.6 and Lemma 2.7 implies the following:

Lemma 2.8. *One can choose a fundamental solution* $\boldsymbol{Y}(z) = (Y_1(z), \dots, Y_Q(z))$ *around* $z = 1$ *so that the connection matrix*

$$\boldsymbol{X}(z) = \boldsymbol{Y}(z) \cdot L_1 \tag{2.2}$$

is given by

$$L_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ c_1 & c_2 & \cdots & c_{Q-1} & 1 \end{pmatrix} \tag{2.3}$$

where $c_{\sigma_i} \neq 0$ *for any* $i = 1, \dots, p$.

When n is odd, the monodromy of Y_Q around $s = 1$ is given by

$$Y_Q(z) \to Y_Q(z) + 2\pi\sqrt{-1}(z-1)^{(n-1)/2} \sum_{m=0}^{\infty} G'_m (z-1)^m.$$

The second term is holomorphic at $z = 1$ and can be expressed as a linear combination of $Y_1(z), \dots, Y_{Q-1}(z)$. Hence the monodromy around $z = 1$ is given by

$$\boldsymbol{Y}(z) \to \boldsymbol{Y}(z) \cdot E_1$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & c'_1 \\ 0 & 1 & \cdots & 0 & c'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c'_{Q-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

When n is even,

$$Y_Q(z) \to -Y_Q(z) + 2 \sum_{m=0}^{\infty} G'_m (z-1)^m,$$

so that the monodromy around $z = 1$ is given by

$$\boldsymbol{Y}(z) \to \boldsymbol{Y}(z) \cdot E_1.$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & c'_1 \\ 0 & 1 & \cdots & 0 & c'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c'_{Q-1} \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Note that the monodromy of $\boldsymbol{Y}(z)$ around $z=0$ is given by

$$\begin{aligned}\boldsymbol{Y}(z) &= \boldsymbol{X}(z)\cdot L_1^{-1}\\ &\to \boldsymbol{X}(z)\cdot E_0\cdot L_1^{-1} = \boldsymbol{Y}(z)\cdot L_1\cdot E_0\cdot L_1^{-1}.\end{aligned}$$

By a straightforward calculation, we have the following:

Proposition 2.9. *The monodromy matrices h_0, h_1 and h_∞ around $z=0$, 1 and ∞ with respect to the basis $\boldsymbol{Y}(z)$ of solutions of (1.3) are given by*

$$h_0 = E_0 + \begin{pmatrix} 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\\ \gamma_1 & \gamma_2 & \cdots & \gamma_Q\end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & g_1\\ 0 & 1 & \cdots & 0 & g_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & g_{Q-1}\\ 0 & 0 & \cdots & 0 & (-1)^{n-1}\end{pmatrix},$$

$$h_\infty^{-1} = h_0 + \begin{pmatrix} 0 & 0 & \cdots & 0 & \delta_1\\ 0 & 0 & \cdots & 0 & \delta_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \delta_Q\end{pmatrix}.$$

Lemma 2.10. *Let $v=(v_1,\dots,v_Q)^T$ be a column vector and define a $Q\times Q$ matrix by*

$$T = (v, h_0\cdot v, \dots, h_0^{Q-1}\cdot v).$$

Then one has

$$\det T = \pm \prod_{1\le\alpha<\beta\le p}(e_\alpha - e_\beta)^{\mu_\alpha\cdot\mu_\beta}\cdot\prod_{\alpha=1}^{p}(v_{\sigma_{\alpha-1}+1})^{\mu_\alpha}.$$

Proof. Let $T(\alpha,j)\in \mathrm{SL}_Q(\mathbb{C})$ be the block diagonal matrix defined by

$$T(\alpha,j) = \begin{pmatrix}\mathrm{id}_{Q-j-1} & 0\\ 0 & \mathrm{id}_{j+1} - e_\alpha\cdot J_{j+1,+}\end{pmatrix}.$$

Then

$$\begin{aligned} T\cdot T(1,Q-1)\cdot T(1,Q-2)\cdot\dots\cdot T(1,Q-\mu_1)\\ \cdot T(2,Q-\mu_1-1)\cdot\dots\cdot T(2,Q-\mu_1-\mu_2)\\ \cdot T(p,Q-\sigma_{p-1}-1)\cdot\dots\cdot T(p,1)\end{aligned}$$

is a lower-triangular matrix whose i-th diagonal component for $\sigma_{\alpha-1}<i\le\sigma_\alpha$ is given by

$$\prod_{\beta<\alpha}(e_\alpha-e_\beta)^{\mu_\beta}\cdot v_{\sigma_{\alpha-1}+1}.$$

□

Corollary 2.11. *$v = (v_1, \dots, v_Q)^T$ is a cyclic vector with respect to h_0 if and only if the condition*

$$\prod_{\alpha=1}^{p} v_{\sigma_{\alpha-1}+1} \neq 0 \tag{2.4}$$

is satisfied.

Lemma 2.12. *If $v \in \mathbb{C}^Q$ satisfies*

$$h_\infty^{-i} \cdot v = h_0^i v, \qquad i = 1, 2, \dots, Q-1, \tag{2.5}$$

then (2.4) holds.

Proof. Since the kernel of $h_\infty^{-1} - h_0$ is the orthogonal complement of the last coordinate vector $\boldsymbol{e}_Q = (0, \dots, 0, 1) \in \mathbb{C}^Q$, the equations (2.5) for $v = (\boldsymbol{v}, 0)$ where $\boldsymbol{v} = (v_1, \cdots, v_{Q-1})$ can be rewritten as

$$\Sigma \cdot \boldsymbol{v} = 0$$

where Σ is a $(Q-1) \times (Q-1)$ matrix whose j-th row vector is the first $Q-1$ components of the last row vector of h_0^j. Define a block diagonal $(Q-1) \times (Q-1)$ matrix by

$$S(\alpha, j) = \begin{pmatrix} \mathrm{id}_{Q-j-2} & 0 \\ 0 & S' \end{pmatrix}$$

where $S' \in \mathrm{SL}_{j+1}(\mathbb{C})$ is given by

$$S' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -e_\alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -e_\alpha & 1 \end{pmatrix}.$$

Then the components of the matrix

$$\begin{aligned} \widetilde{\Sigma} = S(1,1) \cdots S(1, \mu_1 - 1) \cdot S(2, \mu_1) \cdots S(2, \sigma_2 - 1) \cdot S(3, \sigma_2) \cdots S(3, \sigma_3 - 1) \\ \cdots S(p, \sigma_{p-1}) \cdots S(p, \sigma_p - 2) \cdot \Sigma \end{aligned}$$

are zero below the anti-diagonal (i.e., $\widetilde{\Sigma}_{ij} = 0$ if $i + j > Q$) and the i-th anti-diagonal component $\widetilde{\Sigma}_{i,Q-i-1}$ for $\sigma_{\alpha-1} < i \leq \sigma_\alpha$ is given by

$$\prod_{\beta > \alpha} (e_\alpha - e_\beta)^{\mu_\beta} c_{\sigma_\alpha}.$$

The $(Q-1)$-st equation

$$(\mathrm{const}) \cdot v_1 + \prod_{\beta > 1} (e_1 - e_\beta)^{\mu_\beta} c_{\mu_1} v_2 = 0$$

together with Lemma 2.8 implies that $v_2 = 0$ if $v_1 = 0$. By repeating this type of argument, one shows that $v_1 = 0$ implies $\boldsymbol{v} = 0$. Moreover, one can run the same argument by interchanging the role of (v_1, e_1, c_{μ_1}) with $(v_{\sigma_{\alpha-1}+1}, e_\alpha, c_{\sigma_\alpha})$ to show that $v_{\sigma_{\alpha-1}+1} = 0$ implies $\boldsymbol{v} = 0$. Hence a non-trivial solution to (2.5) must satisfy (2.4). □

This concludes the proof of Theorem 1.1.

3. Invariants of the hypergeometric group

We prove the following in this section:

Proposition 3.1. *Let $\boldsymbol{q} = (q_0, \dots, q_N)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$ be sequences of positive integers such that $Q := \sum_{i=0}^N q_i = \sum_{k=1}^r d_r$. Then the space of $Q \times Q$ matrices invariant under the action*

$$H_{\boldsymbol{q},\boldsymbol{d}} \ni h : X \mapsto h \cdot X \cdot h^T$$

is at most one-dimensional.

Proof. Let X be a $Q \times Q$ matrix invariant under the hypergeometric group $H_{\boldsymbol{q},\boldsymbol{d}}$, so that

$$h \cdot X \cdot h^T = X \tag{3.1}$$

for any $h \in H_{\boldsymbol{q},\boldsymbol{d}}$. Let $e_1 = (1, 0, \dots, 0)^T$ be the first coordinate vector. Since $\{(h_\infty^T)^i e_1\}_{i=0}^Q$ spans $\mathbb{C}^Q$, X_{ij} is determined by the $H_{\boldsymbol{q},\boldsymbol{d}}$-invariance once we know X_{i1} for $i = 1, \dots, Q$. Put

$$h_1 = h_0^{-1} \cdot h_\infty^{-1} = \begin{pmatrix} (-1)^r B_Q & 0 & \cdots & 0 & 0 \\ (-1)^r (B_{Q-1} - A_{Q-1}) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^r (B_2 - A_2) & 0 & \cdots & 1 & 0 \\ (-1)^r (B_1 - A_1) & 0 & \cdots & 0 & 1 \end{pmatrix} \in H_{\boldsymbol{q},\boldsymbol{d}} \tag{3.2}$$

and consider (3.1) for $h = h_1$. Since

$$\begin{aligned} (h_1 \cdot X \cdot h_1^T)_{i1} &= \sum_{k,l=1}^Q (h_1)_{ik} X_{kl} (h_1)_{1l} \\ &= \sum_{k,l=1}^Q (h_1)_{ik} X_{kl} (-1)^{r+N+1} \delta_{1l} \\ &= \sum_{k=1}^Q (-1)^{N+r+1} (h_1)_{ik} X_{k1}, \end{aligned}$$

the first column of (3.1) reduces to

$$(-1)^{N+r+1} ((h_1)_{i1} X_{11} + X_{i1}) = X_{i1} \tag{3.3}$$

for $2 \le i \le Q$. If $n = N - r$ is even, then (3.3) implies

$$X_{i1} = -\frac{1}{2} (h_1)_{i1} X_{11},$$

so that the space of $H_{\boldsymbol{q},\boldsymbol{d}}$-invariants is at most one-dimensional. If $N+r$ is odd, then (3.3) gives $X_{11}=0$. Fix $j\neq 1$ such that $(h_1)_{j1}=(-1)^r(B_{Q-j+1}-A_{Q-j+1})\neq 0$. Since

$$\begin{aligned}(h_1\cdot X\cdot h_1^T)_{ij} &= \sum_{k,l=1}^{Q}(h_1)_{ik}X_{kl}(h_1)_{jl} \\ &= \sum_{k=1}^{Q}(h_1)_{ik}(X_{k1}(h_1)_{j1}+X_{kj}(h_1)_{jj}) \\ &= \sum_{k=1}^{Q}(h_1)_{ik}(X_{k1}(h_1)_{j1}+X_{kj}) \\ &= (h_1)_{i1}(X_{11}(h_1)_{j1}+X_{1j})+(X_{i1}(h_1)_{j1}+X_{ij}) \\ &= (h_1)_{i1}X_{1j}+X_{i1}(h_1)_{j1}+X_{ij},\end{aligned}$$

the j-th column of (3.1) gives

$$(h_1)_{i1}X_{1j}+(h_1)_{j1}X_{i1}=0$$

for $2\leq i\leq Q$. Since $(h_1)_{j1}\neq 0$, one obtains

$$X_{i1}=-\frac{(h_1)_{1i}}{(h_1)_{j1}}X_{1j}$$

for $2\leq i\leq Q$, so that the space of H-invariants is at most one-dimensional also in this case. $\square$

4. Coherent sheaves on Calabi-Yau complete intersections in weighted projective spaces

We prove the $H_{\boldsymbol{q},\boldsymbol{d}}$-invariance of the Gram matrix in Theorem 1.2 in this section. The proof is closely related to the discussion of Golyshev [Gol01, §1], although the use of the right dual collection $(\mathcal{F}_i)_{i=1}^{Q}$ seems to be new.

Let Y be a smooth complete intersection of degree $(d_1,\dots,d_r)$ in the weighted projective space $\mathbb{P}=\mathbb{P}(q_0,\dots,q_N)$. We use the Koszul resolution

$$\begin{aligned}0\longrightarrow \mathcal{O}_{\mathbb{P}}(-d_1-\cdots-d_r)\longrightarrow \bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}}(-d_1-\cdots-\widehat{d_i}-\cdots-d_r) \\ \longrightarrow\cdots\longrightarrow \bigoplus_{1\leq i<j\leq r}\mathcal{O}_{\mathbb{P}}(-d_i-d_j)\longrightarrow\bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}}(-d_i)\longrightarrow\mathcal{O}_{\mathbb{P}}\longrightarrow\mathcal{O}_Y\longrightarrow 0\end{aligned}$$

of the structure sheaf $\mathcal{O}_Y$ of Y to compute the derived restriction $\bullet\otimes_{\mathcal{O}_{\mathbb{P}}}\mathcal{O}_Y$.

Let $(\mathcal{E}_i)_{i=1}^{Q}$ be the full strong exceptional collection on $D^b\operatorname{coh}\mathbb{P}$ given as

$$(\mathcal{E}_1,\dots,\mathcal{E}_Q)=(\mathcal{O},\dots,\mathcal{O}(Q-1)),$$

and $(\mathcal{F}_1,\dots,\mathcal{F}_Q)$ be its right dual exceptional collection characterized by the condition

$$\operatorname{Ext}^k(\mathcal{E}_{Q-i+1},\mathcal{F}_j)=\begin{cases}\mathbb{C} & i=j,\text{ and } k=0,\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\mathcal{F}_1=\mathcal{O}_{\mathbb{P}}(-1)[N]$ and $\mathcal{F}_Q=\mathcal{E}_1=\mathcal{O}_{\mathbb{P}}$. The Euler form on the Grothendieck group $K(\mathbb{P})$ defined by (1.1) is neither symmetric nor anti-symmetric, whereas that on

$K(Y)$ is either symmetric or anti-symmetric depending on the dimension of Y. The bases $\{[\mathcal{E}_i]\}_{i=1}^Q$ and $\{[\mathcal{F}_i]\}_{i=1}^Q$ of $K(\mathbb{P})$ are dual to each other in the sense that

$$\chi(\mathcal{E}_{Q-i+1}, \mathcal{F}_j) = \delta_{ij}.$$

We will write the restrictions of $\mathcal{E}_i$ and $\mathcal{F}_i$ to Y as $\overline{\mathcal{E}}_i$ and $\overline{\mathcal{F}}_i$ respectively. Unlike $\{[\mathcal{E}_i]\}_{i=1}^Q$ and $\{[\mathcal{F}_i]\}_{i=1}^Q$, $\{[\overline{\mathcal{E}}_i]\}_{i=1}^Q$ and $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ are not bases of $K(Y)$, and their images in the numerical Grothendieck group are linearly dependent. Put

$$\overline{X}_{ij} = \chi([\overline{\mathcal{F}}_i], [\overline{\mathcal{F}}_j])$$

and let $(a_{ij})_{i,j=1}^Q$ be the transformation matrix between two bases $\{[\mathcal{E}_i]\}_{i=1}^Q$ and $\{[\mathcal{F}_i]\}_{i=1}^Q$ so that

$$[\mathcal{F}_i] = \sum_{j=1}^Q [\mathcal{E}_j] a_{ji}.$$

We prove the following in this section:

Proposition 4.1. *$\overline{X}$ is an invariant of the hypergeometric group $H_{\boldsymbol{q},\boldsymbol{d}}$.*

We divide the proof into three steps.

Lemma 4.2. *Let Φ be an autoequivalence of $D^b \operatorname{coh} Y$ such that its action on $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ is given by*

$$[\overline{\mathcal{F}}_i] \mapsto \sum_{j=1}^Q h_{ij} [\overline{\mathcal{F}}_j].$$

Then $\overline{X}$ is invariant under the action of $h = (h_{ij})_{i,j=1}^Q$;

$$\overline{X} = h \cdot \overline{X} \cdot h^T.$$

Proof. Since an autoequivalence Φ induces an isometry of $K(Y)$, one has

$$\begin{aligned}
\overline{X}_{ij} &= \chi([\overline{\mathcal{F}}_i], [\overline{\mathcal{F}}_j]) \\
&= \chi([\Phi(\overline{\mathcal{F}}_i)], [\Phi(\overline{\mathcal{F}}_j)]) \\
&= \sum_{k,l=1}^Q h_{ik} \chi([\overline{\mathcal{F}}_k], [\overline{\mathcal{F}}_l]) h_{jl} \\
&= \sum_{k,l=1}^Q h_{ik} \overline{X}_{kl} h_{jl}
\end{aligned}$$

for any $1 \le i, j \le Q$. □

Remark 4.3. Since $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ are not linearly independent, the choice of h in Lemma 4.2 is not unique.

Lemma 4.4. *The action of the autoequivalence of $D^b \operatorname{coh} Y$ defined by the tensor product with $\mathcal{O}_Y(-1)$ on $\{\overline{\mathcal{F}}_i\}_{i=1}^Q$ is given by h_0;*

$$[\overline{\mathcal{F}}_i \otimes \mathcal{O}_Y(-1)] = \sum_{j=1}^Q (h_0)_{ij} [\overline{\mathcal{F}}_j].$$

Proof. Since tensor product with $\mathcal{O}(-1)$ commutes with restriction, it suffices to show

$$[\mathcal{F}_i \otimes \mathcal{O}_{\mathbb{P}}(-1)] = \sum_{j=1}^{Q} (h_0)_{ij} [\mathcal{F}_j].$$

Since $\{[\mathcal{E}_{Q-i+1}]\}_{i=1}^{Q}$ and $\{[\mathcal{F}_i]\}_{i=1}^{Q}$ are dual bases, this is equivalent to

$$[\mathcal{E}_{Q-i+1} \otimes \mathcal{O}(-1)] = \sum_{j=1}^{Q} [\mathcal{E}_{Q-j+1}](h_0^{-1})_{ji}. \tag{4.1}$$

Recall from (1.7) that

$$(h_0^{-1})_{ji} = \delta_{j,i+1} - \delta_{i,Q} B_{Q-j+1}.$$

Since $\mathcal{E}_i = \mathcal{O}(i-1)$, Equation (4.1) for $i \neq Q$ gives

$$\mathcal{O}_{\mathbb{P}}(Q-i) \otimes \mathcal{O}_{\mathbb{P}}(-1) = \mathcal{O}_{\mathbb{P}}(Q-i-1),$$

which is obvious. Equation (4.1) for $i = Q$ gives

$$[\mathcal{O}_{\mathbb{P}}(-1)] + \sum_{j=1}^{Q} B_{Q-j+1}[\mathcal{O}_{\mathbb{P}}(Q-j)] = 0,$$

which is $\mathcal{O}_{\mathbb{P}}(-1)$ times the relation

$$[\mathcal{O}_{\mathbb{P}}] + B_1[\mathcal{O}_{\mathbb{P}}(1)] + \cdots + B_{Q-1}[\mathcal{O}_{\mathbb{P}}(Q-1)] + B_Q[\mathcal{O}_{\mathbb{P}}(Q)] = 0$$

coming from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \bigoplus_{i=0}^{N} \mathcal{O}_{\mathbb{P}}(q_i) \to \bigoplus_{0 \le i < j \le N} \mathcal{O}_{\mathbb{P}}(q_i + q_j) \to \cdots \to \mathcal{O}_{\mathbb{P}}\left(\sum_{i=0}^{N} q_i\right) \to 0$$

obtained by sheafifying the Koszul resolution

$$\begin{aligned} 0 \to \Lambda^N V \otimes \operatorname{Sym}^* V^* \to \cdots \to \Lambda^2 V \otimes \operatorname{Sym}^* V^* \\ \to V \otimes \operatorname{Sym}^* V^* \to \operatorname{Sym}^* V^* \to \mathbb{C} \to 0, \end{aligned}$$

where V is a graded vector space such that $\mathbb{P} = \operatorname{Proj}(\operatorname{Sym}^* V^*)$. □

Lemma 4.5. *The action of the autoequivalence of $D^b \operatorname{coh} Y$ given by the dual spherical twist $T^{\vee}_{\overline{\mathcal{F}}_1}$ along $\overline{\mathcal{F}}_1$ is given on $\{\overline{\mathcal{F}}_i\}_{i=1}^{Q}$ by h_1;*

$$[T^{\vee}_{\overline{\mathcal{F}}_1}(\overline{\mathcal{F}}_i)] = \sum_{i=1}^{Q} (h_1)_{ij} [\overline{\mathcal{F}}_j].$$

Proof. Recall that for a spherical object $\mathcal{E}$ and an object $\mathcal{F}$, the dual spherical twist $T^{\vee}_{\mathcal{E}}\mathcal{F}$ of $\mathcal{F}$ along $\mathcal{E}$ is defined as the mapping cone

$$T^{\vee}_{\mathcal{E}}\mathcal{F} = \{\mathcal{F} \to \hom(\mathcal{F}, \mathcal{E})^{\vee} \otimes \mathcal{F}\}$$

of the dual evaluation map. Since the induced action on the Grothendieck group is given by the reflection

$$[T^{\vee}_{\mathcal{E}}(\mathcal{F})] = [\mathcal{F}] - \chi(\mathcal{F}, \mathcal{E})[\mathcal{E}],$$

it suffices to show that

$$(h_1)_{ij} = \delta_{ij} - \overline{X}_{i1}\delta_{j1}. \tag{4.2}$$

Recall from (3.2) that

$$(h_1)_{ij} = \begin{cases} (-1)^r B_Q & i = j = 1, \\ \delta_{ij} & j \neq 1, \\ (-1)^r (B_{Q-i+1} - A_{Q-i+1}) & i \neq 1 \text{ and } j = 1. \end{cases}$$

Equation (4.2) for $j \neq 1$ is obvious, and that for $i = j = 1$ follows from

$$(-1)^r B_Q = (-1)^{r+N+1} = (-1)^{N-r+1} = (-1)^{n+1}$$

and

$$\overline{X}_{11} = \begin{cases} 0 & n \text{ is odd,} \\ 2 & n \text{ is even.} \end{cases}$$

To prove (4.2) for $i \neq 1$ and $j = 1$, one can use

$$\chi(\mathcal{F}_i(1)) = \sum_{j=1}^{Q} \chi((h_0^{-1})_{ij} \mathcal{F}_j) = \sum_{j=1}^{Q} (h_0^{-1})_{ij} \chi(\mathcal{E}_1, \mathcal{F}_j) = (h_\infty^{-1})_{iQ} = -B_{Q-i+1}.$$

and

$$\chi(\mathcal{F}_i(j)) = \chi(\mathcal{O}(-j), \mathcal{F}_i) = \chi(\mathcal{E}_{-j+1}, \mathcal{F}_i) = \delta_{Q+j,i}$$

for $-Q+1 \leq j \leq 0$ to show

$$\begin{aligned} (-1)^r \overline{X}_{i1} &= (-1)^r \chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_1) \\ &= (-1)^N \chi(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_i) \\ &= (-1)^N \chi(\mathcal{O}_Y(-1)[N], \overline{\mathcal{F}}_i) \\ &= \chi(\mathcal{O}_Y(-1), \overline{\mathcal{F}}_i) \\ &= \chi(\overline{\mathcal{F}}_i(1)) \\ &= \chi(\mathcal{F}_i(1)) - \sum_{k=1}^{r} \chi(\mathcal{F}_i(1-d_k)) + \sum_{1 \leq k < l \leq r} \chi(\mathcal{F}_i(1-d_k-d_l)) \\ &\qquad - \cdots + (-1)^r \chi(\mathcal{F}_i(1-d_1-\cdots-d_r)) \\ &= -B_{Q-i+1} - \sum_{k=1}^{r} \delta_{Q-d_k+1,i} + \sum_{1 \leq k < l \leq r} \delta_{Q-d_k-d_l+1,i} - \cdots + (-1)^r \delta_{Q-d_1-\cdots-d_r+1,i} \\ &= -B_{Q-i+1} - \sum_{k=1}^{r} \delta_{Q-i+1,d_k} + \sum_{1 \leq k < l \leq r} \delta_{Q-i+1,d_k+d_l} - \cdots + (-1)^r \delta_{Q-i+1,d_1+\cdots+d_r} \\ &= -B_{Q-i+1} + A_{Q-i+1}, \end{aligned}$$

where we have used (1.8) in the last equality. □

5. Mirror manifolds and Stokes matrices

In this section, we discuss the relation between the Gram matrix in Theorem 1.2 in the case when Y is a hypersurface and the Stokes matrix for the quantum cohomology of the weighted projective space. By [CLCT09, Corollary 1.8], the quantum differential equation for the small J-function of $\mathbb{P}$ is given by

$$\prod_{i=0}^{n} \prod_{k=0}^{q_i-1} \left(q_i z \frac{\partial}{\partial t_1} - kz \right) J_{\mathbb{P}} = e^{t_1} J_{\mathbb{P}},$$

where t_1 is the flat coordinate associated with the positive generator of $H^2(\mathbb{P};\mathbb{Z}) \subset H^*_{\text{orb}}(\mathbb{P};\mathbb{C})$ and z is the quantization parameter. It follows that the stationary-phase integrals

$$J_i(t_1;z) = \int_{\Gamma_i} e^{f/z}\Omega \tag{5.1}$$

span the identity component of the space of flat sections of the first structure connection, where f is the function $f(x) = \sum_{i=0}^N q_i x_i$ on

$$\mathbb{T} = \{(x_0,\dots,x_N) \in (\mathbb{C}^\times)^{N+1} \mid x_0^{q_0}\cdots x_N^{q_N} = e^{t_1}\},$$

Ω is the holomorphic volume form $\Omega = dx_0 \wedge \cdots \wedge dx_N / d(x_0^{q_0}\cdots x_N^{q_N})$ on $\mathbb{T}$, and $\{\Gamma_i\}_{i=1}^Q$ is a basis of flat sections of the local system whose fiber is the relative homology group $H_N(\mathbb{T},\ \mathfrak{Re}(f/z) \ll 0;\mathbb{Z})$.

The function f has Q critical points

$$p_i = e^{(t_1 - 2i\pi\sqrt{-1})/Q}\cdot(1,\dots,1), \qquad i = 1,\dots,Q$$

with critical values

$$f(p_i) = Qe^{(t_1-2i\pi\sqrt{-1})/Q},$$

where the minus sign comes from the clockwise order on the distinguished set $(c_i)_{i=1}^Q$ of vanishing paths, which we choose as straight line segments from the origin to the critical values as in Figure 5.1. See e.g. [AGZV88] for vanishing cycles and the Picard-Lefschetz formula. Let $(\gamma_i)_{i=1}^Q$ be the corresponding distinguished basis of vanishing cycles in $H_{N-1}(f^{-1}(0);\mathbb{Z})$. We choose Lefschetz thimbles $(\Gamma_i)_{i=1}^Q$ as in Figure 5.2, which gives a basis of the relative homology group $H_N(\mathbb{T},\ \mathfrak{Re}(f/z) \ll 0;\mathbb{Z})$ for $\arg(z) > 0$. They corresponds to the full exceptional collection $(\mathcal{F}_i)_{i=1}^Q$ in the derived category of coherent sheaves on $\mathbb{P}$. The thimbles $(\Gamma'_i)_{i=1}^Q$ shown in dotted lines are the dual Lefschetz thimbles, which is a basis of $H_N(\mathbb{T},\ \mathfrak{Re}(f/z) \ll 0;\mathbb{Z})$ for $\arg(z) < 0$ and correspond to the dual exceptional collection $(\mathcal{E}_i)_{i=1}^Q$. The stationary-phase integral (5.1) is the Laplace transform

$$J_i(t_1;z) = \int_{\ell_i} e^{s/z}\widetilde{I}_i(t_1;s)ds$$

of the period integral

$$\widetilde{I}_i(t_1;s) = \int_{\gamma_i \subset f^{-1}(s)} \Omega/df,$$

where ℓ_i is a path on the s-plane starting from a critical value underlying the Lefschetz thimble Γ_i and Ω/df is the Gelfand-Leray form on $f^{-1}(s)$.

The Stokes matrix $(S_{ij})_{i,j=1}^Q$ is a part of the monodromy data for the stationary-phase integrals in (5.1), which is related to intersection numbers of vanishing cycles as follows (cf. e.g. [Dub98, Section 4.1] or [Ued05, Section 5]): Let $(\Gamma_j^+)_{j=1}^Q$ be a basis of $H_N(\mathbb{T},\ \mathfrak{Re}(f/z) \ll 0;\mathbb{Z})$ for $\arg(z) > 0$, which is obtained from the basis $(\Gamma'_{Q+1-i})_{i=1}^Q$ of $H_N(\mathbb{T},\ \mathfrak{Re}(f/z) \ll 0;\mathbb{Z})$ for $\arg(z) < 0$ by parallel transport along a path in the upper half plane $\{z \in \mathbb{C}^\times \mid \mathfrak{Im}\, z \geq 0\}$ with respect to the Gauss-Manin connection on the relative homology bundle. Then the Stokes matrix is given by

$$\Gamma_j^+ = \sum_{i=1}^Q S_{ij}\Gamma_i.$$

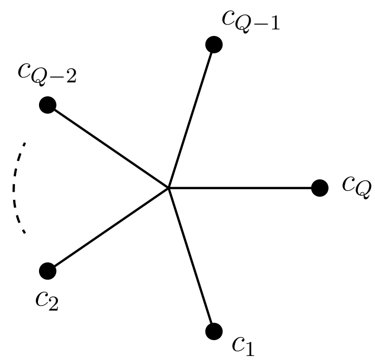

FIGURE 5.1. Vanishing paths

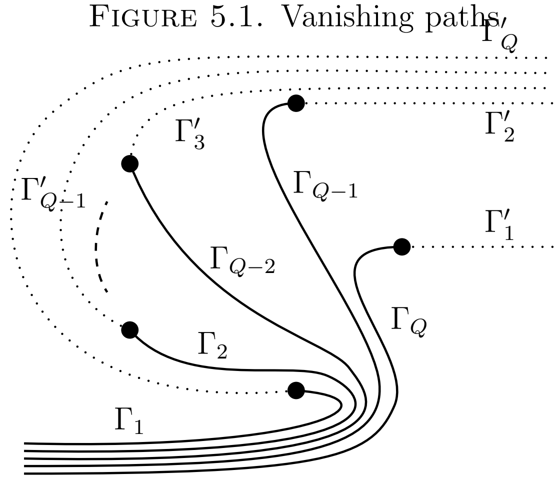

FIGURE 5.2. Lefschetz thimbles

On the other hand, Picard-Lefschetz formula (see e.g. [Pha85, Ebe87, AGZV88]) gives

$$\Gamma_j^+ - \Gamma_j = \sum_{i<j} (\gamma_i, \gamma_j) \Gamma_i,$$

where $(\gamma_i, \gamma_j) = (-1)^{N(N+1)/2} (\gamma_i \circ \gamma_j)$ is $(-1)^{N(N+1)/2}$ times the intersection number of vanishing cycles γ_i and γ_j. This shows that

$$S_{ij} = \begin{cases} (\gamma_i, \gamma_j) & i < j, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

Simultaneous multiplication $x_i \mapsto \alpha x_i$ by a constant $\alpha \in \mathbb{C}^\times$ induces an isomorphism from $f^{-1}(s)$ at $e^{t_1} = a$ to $f^{-1}(\alpha s)$ at $e^{t_1} = \alpha^Q a$, so that the period integral $\widetilde{I}_i(s; t_1)$ depends only on the ratio of s^Q and e^{t_1};

$$\widetilde{I}_i(s; t_1) = I_i(t), \qquad t = \lambda\, Q^Q\, e^{t_1} / s^Q, \qquad \lambda = \prod_{\nu=0}^{N} q_\nu^{q_\nu} \Big/ Q^Q.$$

Here, the factor Q^Q is chosen so that the critical values $s = f(p_i)$ go to $t = \lambda$. For any fixed value of t_1, the function $\widetilde{I}_i(s; t_1)$ is holomorphic at $s = 0$ and has singularities at $s = \infty$ and the critical values $s = f(p_i)$. On the other hand, the function $I_i(t)$ satisfies the irreducible hypergeometric differential equation $\mathcal{H}^{\mathrm{red}} I_i(t) = 0$ and has singularites at $t = 0$, λ and ∞. The singularities of $I_i(t)$ at $t = 0$ and λ come from those of $\widetilde{I}_i(s; t_1)$, whereas the singularity at $t = \infty$ comes from the Q-fold Kummer covering $t \sim 1/s^Q$.

The irreducible local system $\mathcal{L}^{\mathrm{red}}$ on $\mathbb{P}^1\backslash\{0,1,\infty\}$ associated with $\mathcal{H}^{\mathrm{red}}I=0$ is described by Golyshev [Gol01] as follows: Let $\iota : Y \to \mathbb{P}$ be the anticanonical hypersurface and K be the subgroup of the Grothendieck group $K(\mathbb{P})$ generated by

$$[\iota_* \mathcal{O}_Y(i)] = \sum_{k=0}^{r} (-1)^k \sum_{1\le j_1<\cdots<j_k\le r} \left[\mathcal{O}_{\mathbb{P}}\left(i - \sum_{\ell=1}^{k} d_{j_\ell}\right)\right].$$

Since $K(\mathbb{P})$ is generated by $\{\mathcal{O}_{\mathbb{P}}(i)\}_{i\in\mathbb{Z}}$ with relations

$$\sum_{k=0}^{N+1} (-1)^k \sum_{0\le j_1<\cdots<j_k\le N} \left[\mathcal{O}_{\mathbb{P}}\left(i - \sum_{\ell=1}^{k} q_{j_\ell}\right)\right] = 0,$$

one has an isomorphism

$$K(\mathbb{P}) \xrightarrow{\sim} \mathbb{Z}[x, x^{-1}] \Big/ \prod_{i=0}^{N} (1 - x^{q_i})$$

sending x^i to $[\mathcal{O}_{\mathbb{P}}(i)]$, and K is isomorphic to the subgroup of this group generated by $x^i \prod_{j=1}^{r}(1-x^{d_j})$ for $i \in \mathbb{Z}$. It follows that the rank of $K(Y)$ is given by

$$Q^{\mathrm{red}} = Q - \deg\left[\gcd\left(\prod_{i=0}^{N}(1-x^{q_i}), \prod_{j=1}^{r}(1-x^{d_j})\right)\right].$$

Let $\mathcal{K}$ be the local system associated with $K \otimes_{\mathbb{Z}} \mathbb{C}$ such that the monodromy at the origin acts by $[\iota_* \mathcal{O}_Y(i)] \mapsto [\iota_* \mathcal{O}_Y(i-1)]$ and the monodormy at λ acts by $[\iota_* \mathcal{O}_Y(i)] \mapsto [\iota_*(T^{\vee}_{\overline{\mathcal{F}}_1} \mathcal{O}_Y(i))]$. Since the irreducible local system $\mathcal{L}^{\mathrm{red}}$ is characterized by the eigenvalues of the monodromy at zero and infinity together with the fact that the monodromy h_1 at λ is a pseudo-reflection [BH89, Theorem 3.5], one has an isomorphism of the local systems $\mathcal{L}^{\mathrm{red}}$ and $\mathcal{K}$ such that the local section of $\mathcal{L}^{\mathrm{red}}$ coming from the integration along the vanishing cycle γ_1 corresponds to $[\overline{\mathcal{F}}_1] \in K \otimes_{\mathbb{Z}} \mathbb{C}$. By virtue of Lemmas 4.4 and 4.5, the monodromy at infinity acts by cyclic permutation

$$[\overline{\mathcal{F}}_i] \mapsto \sum_{j=1}^{Q} (h_\infty)_{ij} [\overline{\mathcal{F}}_j] = [\overline{\mathcal{F}}_{i-1}],$$

so that the basis $(\gamma_i)_{i=1}^{Q}$ corresponds to $([\overline{\mathcal{F}}_i])_{i=1}^{Q}$ under this isomorphism. Since the intersection form on vanishing cycles is monodromy-invariant, the Gram matrix $(\gamma_i, \gamma_j)_{ij}$ is an invariant of $H_{\boldsymbol{q},\boldsymbol{d}}$ so that it must be proportional to $(\chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_j))_{i,j}$ by Theorem 1.2. The multiplicative constant can be fixed from the fact that the monodromy of $\mathcal{L}^{\mathrm{red}}$ at λ is the pseudo-reflection by γ_1 and the monodromy of $\mathcal{K}$ at λ is the pseudo-reflection by $[\overline{\mathcal{F}}_1]$. If n is even, then the multiplicative constant can also be fixed by noting that $(\gamma_1, \gamma_1) = 2 = \chi(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_1)$. It follows that the Stokes matrix is given by

$$S_{ij} = (\gamma_i, \gamma_j) = \chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_j) = \chi(\mathcal{F}_i, \mathcal{F}_j) + (-1)^n \chi(\mathcal{F}_j, \mathcal{F}_i) = \chi(\mathcal{F}_i, \mathcal{F}_j)$$

for $i < j$, so that we have the following:

Theorem 5.1. *The Stokes matrix $(S_{ij})_{i,j=1}^{Q}$ for the quantum cohomology of the weighted projective space is given by the Gram matrix of the full exceptional collection $(\mathcal{F}_i)_{i=1}^{Q}$ with respect to the Euler form;*

$$S_{ij} = \chi(\mathcal{F}_i, \mathcal{F}_j). \tag{5.2}$$

This generalizes the case of the projective space proved in [Guz99, Tan04] and surely known to experts (see e.g. [Iri09, Remark 4.13]). The relation between Stokes matrices and exceptional collections originates from [CV93] and was developed by Kontsevich [Kon98], Zaslow [Zas96] and Dubrovin [Dub98].

References

[AGZV88] V. I. Arnol′d, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. II*, Monographs in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1988, Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous, Translation revised by the authors and James Montaldi. MR MR966191 (89g:58024)

[AKO08] Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. of Math. (2) **167** (2008), no. 3, 867–943. MR MR2415388 (2009f:53142)

[BB96] Victor V. Batyrev and Lev A. Borisov, *On Calabi-Yau complete intersections in toric varieties*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 39–65. MR 1463173 (98j:14052)

[Beĭ78] A. A. Beĭlinson, *Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69. MR MR509388 (80c:14010b)

[BH89] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function ${}_nF_{n-1}$*, Invent. Math. **95** (1989), no. 2, 325–354. MR MR974906 (90f:11034)

[CCIT09] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *Computing genus-zero twisted Gromov-Witten invariants*, Duke Math. J. **147** (2009), no. 3, 377–438. MR 2510741 (2010a:14090)

[CG11] Alessio Corti and Vasily Golyshev, *Hypergeometric equations and weighted projective spaces*, Sci. China Math. **54** (2011), no. 8, 1577–1590. MR 2824960

[CLCT09] Tom Coates, Yuan-Pin Lee, Alessio Corti, and Hsian-Hua Tseng, *The quantum orbifold cohomology of weighted projective spaces*, Acta Math. **202** (2009), no. 2, 139–193. MR 2506749 (2010f:53155)

[CV93] Sergio Cecotti and Cumrun Vafa, *On classification of $N = 2$ supersymmetric theories*, Comm. Math. Phys. **158** (1993), no. 3, 569–644. MR MR1255428 (95g:81198)

[Dub98] Boris Dubrovin, *Geometry and analytic theory of Frobenius manifolds*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 315–326 (electronic). MR MR1648082 (99j:32025)

[Ebe87] Wolfgang Ebeling, *The monodromy groups of isolated singularities of complete intersections*, Lecture Notes in Mathematics, vol. 1293, Springer-Verlag, Berlin, 1987. MR 923114 (89d:32051)

[Gol01] V. V. Golyshev, *Riemann-Roch variations*, Izv. Ross. Akad. Nauk Ser. Mat. **65** (2001), no. 5, 3–32. MR 1874352 (2002m:14032)

[Guz99] Davide Guzzetti, *Stokes matrices and monodromy of the quantum cohomology of projective spaces*, Comm. Math. Phys. **207** (1999), no. 2, 341–383. MR MR1724842 (2001e:32017)

[Hor] Richard Paul Horja, *Hypergeometric functions and mirror symmetry in toric varieties*, arXiv:math/9912109.

[Iri09] Hiroshi Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), no. 3, 1016–1079. MR 2553377 (2010j:53182)

[Kon98] Maxim Kontsevich, *Lectures at ENS Paris, spring 1998*, set of notes taken by J. Bellaiche, J.-F. Dat, I. Martin, G. Rachinet and H. Randriambololona, 1998.

[Lev61] A. H. M. Levelt, *Hypergeometric functions*, Indag. Math. **23** (1961), 361–403. MR 0137856 (25 #1304)

[Pha85] Frédéric Pham, *La descente des cols par les onglets de Lefschetz, avec vues sur Gauss-Manin*, Astérisque (1985), no. 130, 11–47, Differential systems and singularities (Luminy, 1983). MR 804048 (87h:32017)

[Sei11] Paul Seidel, *Homological mirror symmetry for the quartic surface*, math.AG/0310414, 2011.

[Tan04] Susumu Tanabé, *Invariant of the hypergeometric group associated to the quantum cohomology of the projective space*, Bull. Sci. Math. **128** (2004), no. 10, 811–827. MR MR2100846

[Ued05] Kazushi Ueda, *Stokes matrix for the quantum cohomology of cubic surfaces*, math.AG/0505350, 2005.

[Zas96] Eric Zaslow, *Solitons and helices: the search for a math-physics bridge*, Comm. Math. Phys. **175** (1996), no. 2, 337–375. MR MR1370099 (97c:81178)

Susumu Tanabé
Department of Mathematics,
Galatasaray University
Çırağan cad. 36,
Beşiktaş, Istanbul, 34357, Turkey
e-mail address : tanabesusumu@hotmail.com, tanabe@gsu.edu.tr

Kazushi Ueda
Department of Mathematics,
Graduate School of Science,
Osaka University,
Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.
e-mail address : kazushi@math.sci.osaka-u.ac.jp