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ON FINITE BRANCHED UNIFORMIZATIONS OF THE PROJECTIVE PLANE

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We give a brief survey of the so-called Fenchel's problem for the projective plane, that is the problem of existence of finite Galois coverings of the complex projective plane branched along a given divisor and prove the following result: Let p, q be two integers greater than 1 and C be an irreducible plane curve. If there is a surjection of the fundamental group of the complement of C into a free product of cyclic groups of orders p and q, then there is a finite Galois covering of the projective plane branched along C with any given branching index.

Keywords: Fenchel problem; Uniformization; Branched covering; Generalized triangle group

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1. Introduction

Let M be a complex manifold, $C_1, C_2, \cdots, C_k \subset M$ be irreducible hypersurfaces, and $C := \cup_{i=1}^k C_i$. A morphism $X \to M$ is said to be a *Galois covering of* M *branched at the divisor* $D := \sum_{i=1}^k r_i C_i$ if it is a Galois covering of $M \setminus C$ in the usual sense, and is branched along C_i with branching index $r_i \geq 2$ for $1 \leq i \leq k$.

Given a divisor D on M, is there a finite Galois covering $X \to M$ branched at D? This problem was proposed by Fenchel in the case where M is a Riemann surface and is completely solved in this form: With two exceptions ("bad orbifolds" of Thurston) (I) $M = \mathbb{P}^1$, $D = rp$ and (II) $M = \mathbb{P}^1$, $D = r_1p_1 + r_2p_2$, $r_1 \neq r_2$, there always exists such a covering, see [2] and [5]. Here, we discuss the case $M = \mathbb{P}^2$.

Note that we are not concerned with the smoothness of the covering space X. Almost all pairs $(\mathbb{P}^2, D)$ that we consider in this paper does not admit finite *smooth* uniformizations; this is why we avoid the orbifold terminology.

By the Grauert-Remmert theorem [7], any unbranched finite covering $X' \to \mathbb{P}^2 \setminus C$ extends to a finite covering $X \to \mathbb{P}^2$ branched along C, which is unique up to isomorphism. Hence, there is a one-to-one correspondance between the normal subgroups of finite index in $\pi_1(\mathbb{P}^2 \setminus C)$ and the Galois coverings $X \to \mathbb{P}^2$ branched along C. The covering space X is a possibly singular algebraic surface.

Note that any finite branched cover is dominated by a finite branched Galois cover, but this point of view seems not to be very helpful in this problem, where we want to have a control on the branch curve and the ramification indices.

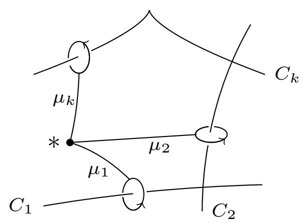

Fig. 1: Meridians of C_i

The map $X \to \mathbb{P}^2$ being branched at D leads one to study the orbifold fundamental group $\pi_1^{orb}(\mathbb{P}^2, D)$ defined as follows: First take a small analytic disc Δ intersecting C_i transversally at a smooth point of C, and define a *meridian* of C_i to be the homotopy class in $\pi_1(\mathbb{P}^2 \setminus C, *)$ of a loop obtained by joining $*$ to a point in $\partial\Delta$ along a path ω, turning once around $\partial\Delta$ in the positive sense, and going back to $*$ along ω (See Fig. 1). It is well known that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is generated by the meridians of C (see e.g. [22]). Define *orbifold fundamental group* of $(\mathbb{P}^2, D)$ as

$$\pi_1^{orb}(\mathbb{P}^2, D) := \pi_1(\mathbb{P}^2 \setminus C)/\langle\langle \mu_1^{r_1}, \mu_2^{r_2} \dots, \mu_k^{r_k} \rangle\rangle.$$

Since any two meridians of an irreducible component of C are conjugate elements in $\mathbb{P}^2 \setminus C$, the group $\pi_1^{orb}(\mathbb{P}^2, D)$ does not depend on the particular choice of the meridians μ_i, so $\pi_1^{orb}(\mathbb{P}^2, D)$ is a projective invariant of the curve C. Moreover, Fenchel's problem has a simple formulation in terms of this invariant: Is there a surjection $\phi : \pi_1^{orb}(\mathbb{P}^2, D) \twoheadrightarrow K$ onto a finite group K such that $|\phi(\mu_i)| = r_i$? In what follows, such a surjection will be called a *good image* of $\pi_1^{orb}(\mathbb{P}^2, D)$.

There is not much hope for a complete solution of the problem for a general complex manifold M, due to two main difficulties, first being topological, the other group theoretical: Firstly, it is not easy to determine the group $\pi_1(M \setminus C)$. The Zariski-Van Kampen method provides an algorithm to compute this group for $M = \mathbb{P}^2$, but does not give any further information on the gorup. Secondly, if M is a Riemann surface, then $\pi_1(M \setminus C)$ is a free group unless $C = \emptyset$, whereas even for $M = \mathbb{P}^2$, this group can be very complicated. The group $\pi_1^{orb}(\mathbb{P}^2, D)$ may even be trivial, consider for example $D = 2L_1 + 3L_2 + 5L_3$, where $L_i \subset \mathbb{P}^2$ intersect generically. The group $\pi_1(\mathbb{P}^2 \setminus C)$ in this case is the abelian group generated by μ_1 and μ_2, with $\mu_3 = \mu_1\mu_2$, the elements μ_i being meridians of L_i. Hence, the group

$\pi_1^{orb}(\mathbb{P}^2, D)$ is the trivial group with the presentation

$$\langle \mu_1, \mu_2, \mu_3 \,|\, \mu_1^2 = \mu_2^3 = \mu_3^5 = \mu_1\mu_2\mu_3 = [\mu_1, \mu_2] = [\mu_2, \mu_3] = [\mu_3, \mu_1] = 1 \rangle$$

For arbitrary M, the group $\pi_1(M \setminus C)$ can also be trivial, e.g. take M to be a simply connected surface and C to be a contractible curve. This is why we shall consider Fenchel's problem for the surface $M = \mathbb{P}^2$ only. Note that, in the algebraic case, the problem in dimension ≥ 3 can be reduced to the problem in dimension 2 by Zariski's hyperplane section theorem.

Many solutions to Fenchel's problem can be obtained by considering abelian coverings. For example, if C is a smooth curve of degree d and $D = nC$, then $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}/d\mathbb{Z}$ and $\pi_1^{orb}(\mathbb{P}^2, D) \simeq \mathbb{Z}/(d, n)\mathbb{Z}$, so that one can say that our problem is solved for smooth curves. All abelian finite smooth uniformizations of projective spaces of arbitray dimension have been effectively classified in [20]. At the other extreme, one can consider C to be an arrangement of d lines, one has then $H_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}^{d-1}$. However, if one takes $D = 2L_1 + 3L_2 + 5L_3$, where this time both three of the lines L_i pass through a common point, then as above there is no abelian solution, whereas it is readily seen that

$$\pi_1^{orb}(\mathbb{P}^2, D) \simeq \langle \mu_1, \mu_2, \mu_3 \,|\, \mu_1^2 = \mu_2^3 = \mu_3^5 = \mu_1\mu_2\mu_3 = 1 \rangle \simeq T_{2,3,5},$$

the latter group being the triangle group, which is finite of order 60. This suggests to look for the non-abelian solutions to the problem. Note however that the corresponding affine problem in $\mathbb{C}^2$ has always a positive solution, given by an abelian covering.

Non-abelian solutions to Fenchel's problem have been studied mainly by Kato [9] and Namba [14]. The following result of Kato on line arrangements is well known:

Theorem 1.1 (Kato [9]**).** *Let $\mathcal{A} = \{L_1, L_2, \dots, L_k\}$ be a line arrangement. If on each L_i lies at least one triple or higher point of $\cup_{i=1}^k L_i$, then there is a finite Galois covering of $\mathbb{P}^2$ branched at $D = \sum_{i=1}^k r_i L_i$ for any $r_i \geq 2$, $1 \leq i \leq k$.*

Note that applying some birational transformations to $\mathcal{A}$ we get a divisor D whose support consists of curves of higher degree and Fenchel's problem is soluable under the same conditions. There is a version of Kato's theorem for conics, proved by Namba.

Theorem 1.2 (Namba [14]**).** *Let $C_1, C_2, \dots, C_k$ be distinct irreducible conics in $\mathbb{P}^2$. Suppose that for each C_i there is another C_j such that they are tangent at two distinct points (See Fig. 2). Then, for any integers $r_i \geq 2$, $1 \leq i \leq k$, there is a finite Galois covering $X \to \mathbb{P}^2$ branching at $D = \sum_{i=1}^k r_i C_i$.*

There is very special conic-line arrangement which has been studied in depth, namely the *Apollonius configuration*, which is an arrangement of a smooth conic Q together with its k-distinct tangent lines T_i. This configuration is special since the fundamental group of this space is the second braid group of a k-times punctured

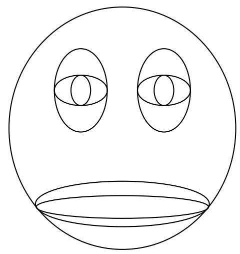

Fig. 2: Conics each has tacnodes

sphere. First result of uniformization problem branching along Apollonius configuration, stated in Theorem 1.3, is obtained by Namba [12]. Later, it has been studied in detail by Ueno [17] and Uludag [19], and more general result, stated in Theorem 1.4, was obtained.

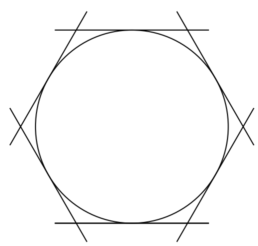

Fig. 3: Apollonious configuration

Theorem 1.3 (Namba [12]**).** *Let T_1, T_2 and T_3 be 3 distinct lines on $\mathbb{P}^2$ circumscribing an irreducible conic Q. Then for any integers $r, s \geq 2$, there is a finite Galois covering $X \to \mathbb{P}^2$ branching at $D = r(\sum_{i=1}^{3} T_i) + sQ$*

Theorem 1.4 (Ueno [17]**, Uludag** [19]**).** *Let $T_1, T_2, \ldots, T_k$ be k-distinct tangent lines of a smooth conic Q in $\mathbb{P}^2$, and $r_i, s \geq 2$ be integers ($1 \leq i \leq k$). Then, there is a finite Galois covering $X \to \mathbb{P}^2$ branching at $D = \sum_{i=1}^{k} r_i T_i + 2sQ$ if $(r_1, \ldots, r_k, s)$ is one of the followings:*

(i) $k = 2$ *and* $r_1 = r_2 \leq \infty$
(ii) $k = 3$ *and* $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \leq 1$
(iii) $k = 3$, $s = 1$ *and* $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} > 1$
(iv) $k \geq 4$

To the authors' knowledge, the rest of the literature available on Fenchel's problem are [11], [12] and [13].

Finally, note that the real version of this problem is also interesting and well-studied, i.e. in the theory of knots.

2. A result on coverings branched along an irreducible curve.

The first fact to notice about Fenchel's problem is the following trivial proposition.

Proposition 2.1. *Let D_1, D_2 be two divisors in $\mathbb{P}^2$ without any common component. If there are finite Galois coverings $X_i \to \mathbb{P}^2$ branched at D_i for $i = 1, 2$, then there is a finite Galois covering $X \to \mathbb{P}^2$ branched at $D_1 + D_2$.*

The covering $X \to \mathbb{P}^2$ can be constructed as the fibered product $X_1 \times_{\mathbb{P}^2} X_2$. Observe that Kato's theorem can not be derived from Theorem 2.1, since there are no coverings of $\mathbb{P}^2$ branched along a unique line L; obviously, $\mathbb{P}^2 \setminus L$ is simply connected.

In view of Proposition 2.1, it is natural to study Fenchel's problem for divisors $D = rC$ with C being irreducible. Unfortunately, for such divisors we are still at the point where Zariski gave the complete solution for the three-cuspidal quartic curve [22]. The group $\pi_1(\mathbb{P}^2 \setminus C)$ for this curve is a non-abelian group of order 12, so that all the Galois coverings branched along it can be characterized. For curves with an infinite non-abelian group, we have the result below.

Theorem 2.1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve. If there is a surjection $\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ for some $p \geq 2$, $q \geq 2$, then there is a finite Galois covering of $\mathbb{P}^2$ branched at rC for any $r \in \mathbb{N}$.*

Observe that there are irreducible curves C with $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ by a result of Oka [15]. Examples of curves with non-trivial surjections as in the hypothesis of the theorem are given in [18].

Proof of Theorem 2.1 makes use of the following result of Namba. Let $D = \sum_{i=1}^{k} r_i C_i$ be a divisor, with meridians μ_i of C_i, and let $\rho : \pi_1^{orb}(\mathbb{P}^2, D) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ be a representation of $\pi_1^{orb}(\mathbb{P}^2, D)$. We say that ρ is *essential* if $|\rho(\mu_i)| = r_i$.

Lemma 2.1 (Namba [11]). *If $\pi_1^{orb}(\mathbb{P}^2, D)$ has an essential representation $\rho : \pi_1^{orb}(\mathbb{P}^2, D) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$, then $\pi_1^{orb}(\mathbb{P}^2, D)$ has a good image $\pi_1^{orb}(\mathbb{P}^2, D) \twoheadrightarrow K$. In other words, there is a finite Galois covering of $\mathbb{P}^2$ branched at D.*

This lemma is a direct consequence of the following result:

Theorem 2.2 (Selberg [16]). *Let R be a non-trivial, finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then there exists a torsion-free normal subgroup N of R of finite index.*

Indeed, putting $R := \rho(\pi_1^{orb}(\mathbb{P}^2, D))$ and $K := R/N$ yields Namba's lemma.

Definition 2.1. A *generalized triangle group* is a group given by the presentation

$$G_{p,q,r} := \langle a, b |\, a^p = b^q = w^r = 1 \rangle,$$

where $2 \leq p, q, r \leq \infty$ and w is a cyclically reduced word involving both of a, b.

Remark 2.1. In the definition of qeneralized triangle group, we have omitted the case $r = 1$. But, the group $G_{p,q,1}$ is still non-trivial and intresting. One has $G_{2,2,1} \simeq$

$\mathbb{Z}/2\mathbb{Z}$ regardless of w. The group $G_{2,3,1} = \langle a, b \,|\, a^2 = b^3 = w(a,b) = 1\rangle$ is a one relator quotient of the modular group $\Gamma := \langle a, b \,|\, a^2 = b^3 = 1\rangle$. Setting $ab = x$ and $ab^{-1} = y$ one can obtain an alternative representation $\langle x, y \,|\, (yx^{-1}y)^2 = (x^{-1}y)^3 = 1\rangle$ of Γ. Using this alternate relation Conder, Havas and Newman [3] checked the relator w of length up to 36, determined character of the group $G_{2,3,1}$ and got some very interesting results.

Theorem 2.3 (Baumslag, Morgan, Shalen [1]**).** *The generalized triangle group $G_{p,q,r}$ has a representation $\rho : G \to \mathrm{PSL}(2,\mathbb{C})$, such that the orders of $\rho(a)$, $\rho(b)$ and $\rho(w)$ are p, q, and r, respectively. Moreover, the group $G_{p,q,r}$*

(i) has a non-abelian free subgroup if $\kappa := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.
(ii) is infinite if $\kappa = 1$.
(iii) is infinite if it has a special cyclic representation, or it has a special dihedral representation with at most one of p, q and r equal to 2. Infact the kernel of the given special representation (is of finite index) and maps to $\mathbb{Z} \times \mathbb{Z}$.

Proof. [Proof of Theorem 2.1] Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d, with a surjection $\phi : \pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, with $p, q \geq 2$. Observe that ϕ induce a surjection of the abelianized groups

$$\mathbb{Z}/d\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}.$$

As the latter group should be cyclic, one has $(p,q) = 1$. Now let μ be a meridian of C, and put $w := \phi(\mu)$. Then there is a surjection

$$\pi_1^{orb}(\mathbb{P}^2, rC) \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}) \,/\, \langle\langle w \rangle\rangle \simeq \langle a, b |\, a^p = b^q = w^r = 1\rangle.$$

In the letter presentation, w cannot be conjugate to a nor to b. Indeed, if w were conjugate to, say, a, then setting $a = 1$ we would obtain a surjection

$$\pi_1(\mathbb{P}^2 \setminus C)/\langle\langle \mu \rangle\rangle \;\twoheadrightarrow\; \mathbb{Z}/q\mathbb{Z},$$

which contradict the fact that the conjugacy class of the meridian μ generate $\pi_1(\mathbb{P}^2 \setminus C)$. This implies that the word w, which can be assumed to be cyclically reduced, involves both of the letters a and b. This matches with the Definition 2.1. Hence, Theorem 2.1 follows then by an application of the Theorem 2.3 to the generalized triangle group $G_{p,q,r}$, and by Namba's lemma. □

Remark 2.2. The following direct consequence of the Theorem 2.3 is noteworthy. If $C_{p,q}$ is an Oka curve (see [15]), with $\pi_1(\mathbb{P}^2 \setminus C_{p,q}) \simeq \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, then the group $\pi_1^{orb}(\mathbb{P}^2, rC_{p,q})$ contains a non-abelian free subgroup for any $r \geq 2$ provided that $p, q \geq 5$. On the other hand, for an irreducible curve C, the group $\pi_1^{orb}(\mathbb{P}^2, rC)$ may be trivial for infinitely many $r \in \mathbb{N}$, even if the group $\pi_1(\mathbb{P}^2 \setminus C)$ contains a non-abelian free subgroup. Such examples are discussed in [18], where the following question is raised:

Question 2.1. Let $C \subset \mathbb{P}^2$ be an irreducible curve, such that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is infinite. Is it true that there are infinitely many $r \in \mathbb{N}$ such that there exists a finite Galois covering of $\mathbb{P}$ branched at rC?

In contrast with the Remark 2.2, it can be proved that the group $\pi_1^{orb}(\mathbb{P}^2, 2C)$ is finite under some rather restrictive hypothesis:

Proposition 2.2. *If C is an irreducible curve such that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is generated by only two meridians of C, then $\pi_1^{orb}(\mathbb{P}^2, 2C)$ is a finite group (it can be trivial).*

Proof. Suppose that the meridians μ and ν generate $\pi_1(\mathbb{P}^2 \setminus C)$. Then, since any two meridians are conjugate elements of $\pi_1(\mathbb{P}^2 \setminus C)$, one has $\mu = x\nu x^{-1}$, where x is a word in μ and ν. This implies that $\pi_1^{orb}(\mathbb{P}^2, 2C)$ is a quotient of the group

$$K := \langle \mu, \nu \mid \mu^2 = \nu^2 = 1, \quad \mu = x\nu x^{-1} \rangle.$$

Since $\mu^2 = \nu^2 = 1$ in this latter group, the relation $\mu = x\nu x^{-1}$ can be written in the form $(\mu\nu)^n \mu (\mu\nu)^{-n} = \nu$ for some n. Hence,

$$K = \langle \mu, \nu \mid (\mu\nu)^{2n+1} = \mu^2 = \nu^2 = 1 \rangle,$$

that is, K is the dihedral group of order $4n + 2$. □

A direct application of the Zariski-Van Kampen theorem [22] shows that if an irreducible curve C of degree d has a flex F or a singular point p of order $(d - 2)$, then the group $G = \pi_1(\mathbb{P}^2 \setminus C)$ is generated by two meridians. Indeed, in the former case, considering projection with center $O \in F \setminus C$, one sees that $d - 2$ of the generators of $\pi_1(\mathbb{P}^2 \setminus C)$ are equal, so that there remains 3 generators. One of these generators can be eliminated by the projective relation. In the latter case, putting the center of projection at the singular point p yields the result.

3. Fenchel's problem under equisingular deformations

Another basic fact concerning Fenchel's problem will be obtained as a corollary to the following theorem.

Theorem 3.1 (Zariski [22]). *If the family of curves $\{C_t\}_{0<|t|\leq 1}$ is equisingular, and the limit curve C_0 is reduced, then there is a surjection*

$$\phi : \pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1).$$

The surjection ϕ is *natural* in the sense that ϕ sends meridians to meridians. Hence, under the hypothesis of Zariski's theorem, one has the induced surjections

$$\pi_1^{orb}(\mathbb{P}^2, rC_0) \twoheadrightarrow \pi_1^{orb}(\mathbb{P}^2, rC_1)$$

for any $r \in \mathbb{N}$. Assume that C_0, C_1 are irreducible. If we suppose that $\pi_1^{orb}(\mathbb{P}^2, rC_1)$ has a good image $\pi_1^{orb}(\mathbb{P}^2, rC_1) \twoheadrightarrow K$, we obtain the following corollary:

Corollary 3.1. *Suppose that C_0 is an irreducible curve. Under the hypothesis of Zariski's theorem, if there is a finite Galois covering of $\mathbb{P}^2$ branched at rC_1, then there is a finite Galois covering of $\mathbb{P}^2$ branched at rC_0.*

Remark 3.1. To conclude, let us give an example illustrating the utility of the group $\pi_1^{orb}(\mathbb{P}^2, D)$ as a projective invariant. In [4], Dimca gives an equisingular deformation of the Oka curve $C_{2,3}$ of degree $d = 6$ to a sextic with a unique singular point of multiplicity $d-2 = 4$. Let $p, q \in \mathbb{N}$ be two coprime numbers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{2} \leq 1$. *Then the Oka curve $C_{p,q}$ (of degree $d = pq$) cannot be equisingularly deformed to a reduced irreducible curve C' with a singular point of multiplicity $d - 2$.* Indeed, by Corollary 3.1, such a deformation would induce a surjection $\pi_1^{orb}(\mathbb{P}^2, 2C') \twoheadrightarrow \pi_1^{orb}(\mathbb{P}^2, 2C_{p,q})$. By Proposition 2.2, $\pi_1^{orb}(\mathbb{P}^2, 2C')$ is finite, whereas by the Theorem 2.3, $\pi_1^{orb}(\mathbb{P}^2, 2C_{p,q})$ is infinite, contradiction.

Acknowledgments

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