# An involution of reals, discontinuous on rationals and whose derivative vanish almost everyhere

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**Abstract**

We study the involution of the real line, induced by Dyer's outer automorphism of PGL(2,Z). It is continuous at irrationals with jump discontinuities at rationals. We prove that its derivative exists almost everywhere and vanishes almost everywhere.

## 1 Introduction

A function discontinuous on a dense subset of $[0, 1]$ can not be differentiable everywhere on the residual set; it can be differentiable at most on a meager set (i.e. a countable union of nowhere dense sets), see [?]. However, meager does not mean negligible: there exists meager sets of full Lebesgue measure, and it was also exhibited in the location cited a function discontinuous at rationals and yet differentiable on a set of full measure.

Our aim in this paper is to show that the involution $\mathbf{J}$ (Jimm) of $\mathbf{R}$ recently introduced by us in [?], [?] is another function of this kind. Here, we shall work with the restriction of $\mathbf{J}$ to the unit interval $[0, 1]$. Our result is also valid for its extension to $\mathbf{R}$.

This involution is induced by the outer automorphism of $\mathrm{PGL}_2(\mathbf{Z})$ and satisfies a set of functional equations of "modular type". Moreover, it preserves the set of quadratic irrationals commuting with the Galois conjugation on them. It induces a duality of Beatty partitions of the set of positive integers. It conjugates the

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Gauss continued fraction map to the so-called Fibonacci map [**?**]. We refer the reader to [**?**] and to [**?**] for a wider perspective about $\mathbf{J}$ and for its connection to Dyer's outer automorphism.

# 2 Introducing the involution

As usual, denote the continued fraction $1/(n_1 + 1/\dots)$ by $[0, n_1, n_2, \dots]$. Let $x = [0, n_1, n_2, \dots]$ be a number with $2 \le n_1, n_2 \dots < \infty$. Then the value that $\mathbf{J}$ takes on $x$ is defined as

$$\mathbf{J}(x) = \mathbf{J}([0, n_1, n_2, \dots]) := [0, 1_{n_1-1}, 2, 1_{n_2-2}, 2, 1_{n_3-2}, \dots], \tag{1}$$

where $1_k$ denotes the sequence $1, 1, \dots, 1$ of length $k$. This formula extends to all irrational numbers, i.e. those with $x = [0, n_1, n_2, \dots]$ satisfying $1 \le n_1, n_2 \dots < 1$, if the emerging $1_{-1}$'s are eliminated in accordance with the rule $[\dots m, 1_{-1}, n, \dots] = [\dots m + n - 1, \dots]$ and $1_0$ with the rule $[\dots m, 1_0, n, \dots] = [\dots m, n, \dots]$.

See [**?**], [**?**] for a computation of some values of $\mathbf{J}$.

From its definition it is readily seen that $\mathbf{J}$ sends ultimately periodic continued fractions (i.e. quadratic irrationals) to itself. For example, if $x = [0, 1_\infty]$, then the definition gives

$$\mathbf{J}(x) = [0, 1_0, 2, 1_{-1}, 2, 1_{-1}, 2, \dots]$$

and applying the simplification rules we get

$$\mathbf{J}(x) = [0, 2, 1_{-2}, 2, \dots] = [0, 3, 1_{-1}, 2, \dots] = [0, 4, 1_{-1}, 2, \dots] = \dots = [0, \infty] = 0.$$

Similarly if $n_i$ is constantly 1 from some point on, i.e. $x = [0, n_1, n_2, \dots, n_k, 1_\infty]$ with $n_k > 1$, then $\mathbf{J}(x) = [0, \dots, 1_{n_k-2}, \infty] \in \mathbf{Q}$, i.e. noble numbers are sent to rationals under $\mathbf{J}$. Note that

$$\mathbf{J}([0, 3, 1_\infty]) = [0, 1_2, 2, 1_{-1}, 2, 1_{-1}, 2, \dots] = [0, 1, 1, \infty] = 1/2, \text{ and}$$
$$\mathbf{J}([0, 1, 2, 1_\infty]) = [0, 1_0, 2, 1_0, 2, 1_{-1}, 2, 1_{-1}, 2, \dots] = [0, 2, \infty] = 1/2.$$

In a similar manner, it is easy to see that $\mathbf{J}$ is two-to-one on the set of noble numbers in $[0, 1]$ (except that $\mathbf{J}^{-1}(0) = [0, 1_\infty]$ and $\mathbf{J}^{-1}(1) = [0, 2, 1_\infty]$). It is bijective and involutive on the set $[0, 1] \backslash \mathbf{Q} \cup \mathcal{N}$, where $\mathcal{N}$ denotes the set of noble numbers (see [**?**]).

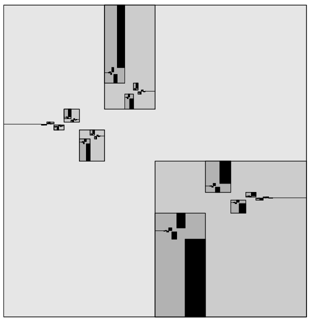

**Figure.** The graph of $\mathbf{J}$ lies inside the smaller (and darker) boxes.

If $x = [0, n_1, n_2, \dots]$ is an irrational and $x_k = [0, n_1^k, n_2^k, \dots]$ is a sequence tending to $x$, then for every $N$, there exists an $M$ such that $n_i^k = n_i$ for $k > N$ and $i < M$. This implies that longer and longer initial segments of $[0, \ell_1^k, \ell_2^k, \dots]$ coincides with that of $[0, \ell_1, \ell_2, \dots]$, where $\mathbf{J}(x) = [0, \ell_1, \ell_2, \dots]$ and $\mathbf{J}(x_k) = [0, \ell_1^k, \ell_2^k, \dots]$. Hence, $\mathbf{J}(x_k) \to \mathbf{J}(x)$, i.e. our involution $\mathbf{J}$ is continuous at every irrational $x$.

If $x = [0, n_1, n_2, \dots, n_m, \infty]$ is a rational with $m$ odd, let $x_k = [0, n_1^k, n_2^k, \dots]$ be a sequence tending to $x$ from below. Then there exists an $N$ such that $n_i^k = n_i$ for $k > N$, $i \le m$, and $n_{m+1}^k \to \infty$. This implies that longer and longer initial segments of $[0, \ell_1^k, \ell_2^k, \dots]$ coincides with that of $\mathbf{J}_-(x) := [0, \ell_1, \ell_2, \dots]$, where $\mathbf{J}(x_k) = [0, \ell_1^k, \ell_2^k, \dots]$ and

$$[0, \ell_1, \ell_2, \dots] = [0, 1_{n_1-1}, 2, 1_{n_2-2}, 2, 1_{n_3-2}, \dots, 2, 1_{n_m-2}, 2, 1_\infty]$$

Hence, $\mathbf{J}(x_k) \to \mathbf{J}_-(x)$, i.e. $\mathbf{J}$ is continuous from the left at $x$.

On the other hand, if $x_k \downarrow x$, then let $[0, p_1, p_2, \dots, p_r, \infty]$ be the other representation of $x$ as a continued fraction (which is $[0, n_1, n_2, \dots, n_m - 1, 1, \infty]$ if $n_m > 1$ and $[0, n_1, n_2, \dots, n_{m-1} + 1, \infty]$ if $n_m = 1$). Then there exists an $N$ such that $n_i^k = p_i$ for $k > N$, $i \le r$, and $n_{r+1}^k \to \infty$. This implies that longer and longer initial segments of $[0, \ell_1^k, \ell_2^k, \dots]$ coincides with that of $\mathbf{J}_+(x) := [0, \ell_1, \ell_2, \dots]$, where

$$[0, \ell_1, \ell_2, \dots] = [0, 1_{p_1-1}, 2, 1_{p_2-2}, 2, 1_{p_3-2}, \dots, 2, 1_{p_r-2}, 2, 1_\infty]$$

and $\mathbf{J}(x_k) = [0, \ell_1^k, \ell_2^k, \dots]$. Hence, $\mathbf{J}$ is continuous from the right at $x$.

Similar arguments show that $\mathbf{J}$ is continuous from left and right for $m$ even as well. The jump function $\delta(q) := \mathbf{J}(q)^+ - \mathbf{J}(q)^-$ can be taken as a measure of complexity of a rational number $q$.

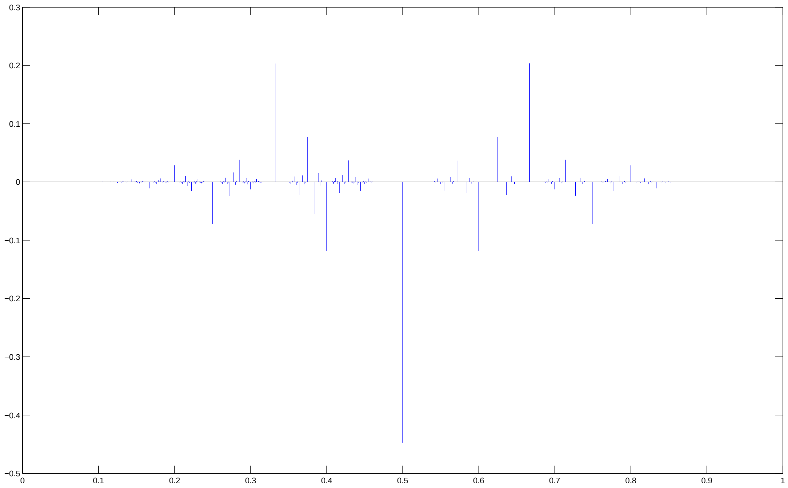


**Figure.** The plot of $\delta$, centered around the point $x = 1/2$.

# 3 The derivative of jimm.

It is known that for almost all $x$, the arithmetic mean of partial quotients of $x$ tends to infinity, i.e. if $x = [0, n_1, n_2, \dots]$ then

$$\lim_{k\to\infty} \frac{n_1 + \cdots + n_k}{k} = \infty \tag{2}$$

almost everywhere (see [?]). In other words, the set of numbers in the unit interval such that the above limit is infinite, is of full Lebesgue measure. Denote this set by $A$. Now since the first $k$ partial fractions of $x$ give rise to at most $n_1 + \cdots + n_k - k$ partial fractions of $\mathbf{J}(x)$ and at least $n_1 + \cdots + n_k - 2k$ of these are 1's, one has

$$\frac{n_1 + \cdots + n_k - k}{n_1 + \cdots + n_k - 2k} \to 1 \text{ as } k \to \infty.$$

This shows that the density of 1's in the continued fraction expansion of $\mathbf{J}(x)$ equals 1 a.e., and therefore the the continued fraction averages of $\mathbf{J}(x)$ tend to 1 a.e. We conclude that $\mathbf{J}(A)$ is a set of zero measure.

Suppose $x = [0, n_1, n_2, \dots]$ is an irrational satisfying (2). Then for every constant $M$, there is some $k$ with $n_1 + \cdots + n_k > kM$. But then the $\mathbf{J}$- transform of the initial length-$k$ segment of $x$ is of length at least $kM - k$. Hence if $y$ is any number whose continued fraction expansion coincide with that of $x$ up to the place $k$, then the continued fraction $\mathbf{J}(y)$ coincide with that of $\mathbf{J}(x)$ at least up to the place $kM - k$. Since $kM - k$ is arbitrarily big compared to $k$, and since longer continued fractions give exponentially better approximations, we see that $\mathbf{J}(y)$ a.e. is much closer to $\mathbf{J}(x)$ then $y$ is to $x$. Hence the idea of the following theorem.

**Theorem 1** *The derivative of* $\mathbf{J}(a)$ *exists almost everywhere and vanishes almost everywhere.*

To prove this, we need to show that for almost all $a$,

$$\lim_{x\to a} \frac{\mathbf{J}(a)-\mathbf{J}(x)}{a-x} = 0.$$

Assume that $x$ is irrational or equivalently its continued fraction expansion is non-terminating.

Let $x \in [0,1]$ with $0 < |x-a| < \delta$ for some $\delta$. Then there is a number $k = k_\delta$, such that the continued fractions of $a$ and $x$ coincide up to the $k$th element. Hence $x = [0, n_1, n_2, \ldots n_k, m_{k+1}, \ldots]$ with $m_{k+1} \neq n_{k+1}$. Note that this latter condition also guarantees that $0 < |x-a|$. Now let

$$M_k(z) := [n_1, n_2, \ldots, n_{k-1}, n_k + z] = \frac{\alpha_k z + \beta_k}{\gamma_k z + \theta_k}$$

and put $a_k := [0, n_{k+1}, n_{k+2}, \ldots]$, $\quad x_k := [0, m_{k+1}, m_{k+2}, \ldots]$. Then one has $0 < a_k < 1$ (with strict inequality since $a$ is irrational) and $0 \leq x_k < 1$ for every $k = 1, 2, \ldots$. One has

$$a = M_k(a_k), \quad x = M_k(x_k) \text{ and } \det(M_k) = (-1)^k.$$

**Lemma 2** *Let* $a := [0, n_0, n_1, \ldots]$ *and suppose that the continued fractions of* $a$ *and* $x$ *coincide up to the place* $k$ *(but not* $k+1$*), where* $x \in [0,1]$*. Put* $N_k := \sum_{i=1}^{k} n_i$*, and* $\mu_k := N/k$*. Then*

$$|a-x| > \frac{1}{24}(2\mu_{k+3})^{-2(k+3)}$$

*Proof.* One has

$$|a-x| = |M_k(a_k) - M_k(x_k)| = \left|\frac{\alpha_k a_k + \beta_k}{\gamma_k a_k + \theta_k} - \frac{\alpha_k x_k + \beta_k}{\gamma_k x_k + \theta_k}\right| = \frac{1}{(\gamma_k a_k + \theta_k)(\gamma_k x_k + \theta_k})$$

Since

$$M_{i+1}(z) = M_i\left(\frac{1}{n_{i+1}+z}\right) = \frac{\beta_i z + (\alpha_i + n_{i+1}\beta_i)}{\theta_i z + (\gamma + n_{i+1}\theta_i)},$$

one has $\gamma_{i+1} = \theta_i$ and $\theta_{i+1} = \gamma_i + n_{i+1}\theta_i$. Hence $\theta_{i+1} > \gamma_{i+1} \implies \theta_{i+1} > \theta_i(1+n_{i+1})$. This implies

$$\theta_i < (1+n_1)(1+n_2)\ldots(1+n_i),$$
$$\gamma_i < (1+n_1)(1+n_2)\ldots(1+n_{i-1}).$$

Since $0 \le a_k, x_k < 1$, this implies

$$\gamma_k a_k + \theta_k < \gamma_k + \theta_k < 2(1+n_1)(1+n_2)\dots(1+n_k),$$
$$\gamma_k x_k + \theta_k < \gamma_k + \theta_k < 2(1+n_1)(1+n_2)\dots(1+n_k).$$

Hence, we get

$$|a-x| > \frac{|a_k - x_k|}{4(1+n_1)^2(1+n_2)^2\dots(1+n_k)^2}$$

To estimate $|a_k - x_k|$, consider

$$a_k - x_k = \frac{1}{n_{k+1}+a_{k+1}} - \frac{1}{m_{k+1}+x_{k+1}} = \frac{m_{k+1}-n_{k+1}+x_{k+1}-a_{k+1}}{(n_{k+1}+a_{k+1})(m_{k+1}+x_{k+1})}$$
$$> \frac{m_{k+1}-n_{k+1}+x_{k+1}-a_{k+1}}{(1+n_{k+1})(1+m_{k+1})}.$$

Now, if $m_{k+1} < n_{k+1}$ then set $m_{k+1} = n_{k+1} - t$ with $t \ge 1$. Then one has

$$|a_k - x_k| > \frac{|-t+x_{k+1}-a_{k+1}|}{(1+n_{k+1})(1+n_{k+1}-t)} > \frac{a_{k+1}}{(1+n_{k+1})^2}$$

On the other hand, if $3n_{k+1} \ge m_{k+1} > n_{k+1}$ then

$$|a_k - x_k| > \frac{|1+x_{k+1}-a_{k+1}|}{(1+n_{k+1})(1+3n_{k+1})} > \frac{1-a_{k+1}}{3(1+n_{k+1})^2},$$

and if $m_{k+1} > 3n_{k+1}$ then

$$|a_k - x_k| = \frac{1 - \frac{n_{k+1}}{m_{k+1}} + \frac{x_{k+1}}{m_{k+1}} - \frac{a_{k+1}}{m_{k+1}}}{(1+n_{k+1})(1+\frac{1}{m_{k+1}})} > \frac{1}{6(1+n_{k+1})}$$

So one has

$$|a_k - x_k| > \frac{a_{k+1}(1-a_{k+1})}{6(1+n_{k+1})^2},$$

which gives the estimation from below

$$|a-x| > \frac{a_{k+1}(1-a_{k+1})}{24(1+n_1)^2(1+n_2)^2\dots(1+n_k)^2(1+n_{k+1})^2},$$

estimation obtained under the assumption that the continued fraction expansions of $x$ and $a$ coincide up until the $k$th term and differ for the $k+1$th term.

Now we have the crude estimate

$$\frac{1}{n_{k+2}+\cfrac{1}{n_{k+3}+1}} > a_{k+1} > \frac{1}{1+n_{k+2}} \implies a_{k+1}(1-a_{k+1}) > \frac{1}{(1+n_{k+2})^2}\frac{1}{(1+n_{k+3})^2}.$$

which gives

$$|a - x| > \frac{1}{24(1+n_1)^2(1+n_2)^2 \dots (1+n_{k+2})^2(1+n_{k+3})^2},$$

Now put $N_k := \sum_{i=1}^k n_i$, and $\mu_k := N/k$. Then

$$(1+n_1)^2(1+n_2)^2 \dots (1+n_k)^2 \leq (1+\mu_k)^{2k} \leq (2\mu_k)^{2k}$$

The last inequality follows from the fact that $\mu_k \geq 1$ for all $k$, since $n_i \geq 1$ for all $i$. We finally obtain the estimate

$$|a-x| > \frac{1}{24}(2\mu_{k+3})^{-2(k+3)} = \frac{1}{24}\exp\{-2(k+3)\log 2\mu_{k+3}\}. \qquad \square$$

On the other hand, if the c.f. expansions of $a$ and $x$ coincide up to the $k = k(x)$th place, then the c.f. expansions of $\mathbf{J}(a)$ and $\mathbf{J}(x_i)$ coincide up to the place $N_k$, and by Binet's formula we have

$$|\mathbf{J}(a) - \mathbf{J}(x)| < F_{N_k}^{-2} < \sqrt{5}\phi^{-2N_k} = \sqrt{5}\exp\{-2k\mu_k \log\phi\}$$

(this estimate should be close to optimal (a.e.), since the density of 1's in the c.f. expansion of $\mathbf{J}(a)$ equals one a.e.) This gives

$$\left|\frac{\mathbf{J}(a) - \mathbf{J}(x)}{a - x}\right| < 24\sqrt{5}\exp k\{2(1+3/k)\log 2\mu_{k+3} - 2\mu_k \log\phi\} \implies$$

$$\left|\frac{\mathbf{J}(a) - \mathbf{J}(x)}{a - x}\right| < A\exp\Big\{2k\log\phi\big(B\log 2\mu_{k+3} - \mu_k\big)\Big\}$$

where $A$ is some absolute constant and $B = (1+3/k)/\log\phi$ can be taken arbitrarily close to $1/\log\phi < 2.08$ by assuming $k$ is big enough.

We see immediately that, if $a = [0, n, n, n, n, \dots]$ then $\mu_k$ is constant $= n$, and if $n$ is taken big enough so that $2.08\log 2n - n < 0$, then the derivative exists and is zero. This is true for $n > 4$. We don't claim that our estimations are optimal in this respect, however.

On the other hand, since $\mu_k \to \infty$ almost surely, we see that $B\log 2\mu_{k+3} - \mu_k < 0$ for $k$ sufficiently big and the derivative exists and vanishes. This is because by choosing a sufficiently small neighborhood $\{|x - a| < \delta\}$, we can guarantee that $k = k(x)$ is always greater than a given number for any $x$ in this neighborhood. This concludes the proof of the theorem.

Note that, if $\mu_k \to \infty$ then the average partial quotient of $\mathbf{J}(a)$ tends to 1, and $\mathbf{J}$ is not differentiable at $\mathbf{J}(a)$. In other words, $\mathbf{J}$ is almost surely not differentiable at $\mathbf{J}(a)$. In the same vein, the derivative of $\mathbf{J}$ at $a = [0, n, n, n, n, \dots]$ vanish for $n > 4$, and we see that $\mathbf{J}$ is not differentiable at $\mathbf{J}(a) = [0, 1_{n-1}, \overline{2, n-2}]$ or at best it will be of infinite slope at this point.

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