# Testing the transcendence conjectures of a modular involution of the real line and its continued fraction statistics

Hakan Ayral, A. Muhammed Uludağ

**Abstract**

We study the values of the recently introduced involution **J** (jimm) of the real line, which is equivariant with the action of the group PGL(2,Z). We test our conjecture that this involution sends algebraic numbers of degree at least three to transcendental values. We also deduce some theoretical results concerning the continued fraction statistics of the generic values of this involution and compare them with the experimental results.

## 1 Introduction

Every irrational real number admits a unique simple continued fraction representation

$$x = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\cdots}}} \quad (n_0 \in \mathbf{Z}, n_1, n_2 \dots \in \mathbf{Z}_{>0})$$

denoted shortly as $x = [n_0, n_1, n_2, \dots]$. The numbers $n_0, n_1, \dots$ are called the partial quotients of $x$. If $x$ is a real quadratic irrational, i.e. if $x = a+\sqrt{b}$ with $a, b \in \mathbf{Q}$ and $b > 0$ is a non-square, then $x$ is known to have an eventually periodic continued fraction representation. In contrast with this, not much is known about the continued fraction representations of other real algebraic numbers.

Denote by $\overline{\mathbf{Q}}$ the field of algebraic numbers, i.e. the set of roots of polynomials with coefficients in $\mathbf{Q}$. If $x \in \overline{\mathbf{Q}} \cap \mathbf{R}$ is not quadratic, numerical evidence suggest that this representation should behave like the expansion of a "normal" number (i.e. its partial quotients must obey the Gauss-Kuzmin statistics). In particular, the partial quotients averages are expected to tend to infinity. However, to our knowledge the answer to the much weaker question "are partial quotients of $x \in \overline{\mathbf{Q}} \cap \mathbf{R}$ unbounded if $n$ is not quadratic?" is currently not known [1].

We have recently introduced and studied ([16], [17], [15]) a certain continued fraction transformation

$$\mathbf{J} : \mathbf{R} - \mathbf{Q} \to \mathbf{R},$$

which is involutive (e.g. $\mathbf{J} \circ \mathbf{J} = Id$) and sends normal numbers to "sparse numbers" (i.e. numbers with partial quotients equal to 1 with frequency 1). To define $\mathbf{J}(x)$ for irrational $x = [0, n_1, n_2, \dots]$ in the unit interval $[0, 1]$, assume first $n_1, n_2, \dots \geq 2$. Then

$$\mathbf{J}([0, n_1, n_2, \dots]) := [0, 1_{n_1-1}, 2, 1_{n_2-2}, 2, 1_{n_3-2}, \dots], \tag{1}$$

where $1_k$ denotes the sequence $1, 1, \dots, 1$ of length $k$. To extend this definition for $n_1, n_2, \dots \geq 1$, we eliminate the $1_{-1}$'s emerging in (1) by the rule $[\dots m, 1_{-1}, n, \dots] \to [\dots m{+}n{-}1, \dots]$ and $1_0$'s by the rule $[\dots m, 1_0, n, \dots] \to [\dots m, n, \dots]$. These rules are applied once at a time. See the next section for more details on the definition and properties of $\mathbf{J}$.

Recall that the degree of an algebraic number $x \in \overline{\mathbf{Q}}$ is defined to be the degree of the polynomial of minimal degree satisfied by $x$. Our aim here is to experimentally confirm the following conjecture:

**Transcendence conjecture:** If $x$ is a real algebraic number of degree $> 2$, then $\mathbf{J}(x)$ is transcendental.

Here is the chain of reasonings which led us to this conjecture:

$$x \in \overline{\mathbf{Q}}, \deg(x) > 2 \implies x \text{ is typical (belief)} \tag{2}$$

$$x \text{ is typical} \implies \lim_{k\to\infty} \frac{n_1 + \cdots + n_k}{k} = \infty \tag{3}$$

$$\lim_{k\to\infty} \frac{n_1 + \cdots + n_k}{k} = \infty \implies y = \mathbf{J}(x) \text{ is sparse} \tag{4}$$

$$y \text{ is sparse} \implies y \text{ is not algebraic (belief)} \tag{5}$$

In this reasoning, the statement (2) is based on the numerical evidence mentioned in the introduction and it is widely believed to be true. The statement (3) is Khinchine's theorem, (4) is an easily observation (see Lemma 1 below) and finally (5) is a contrapositive instance of (2).

Although there are some recent results in the literature, pertaining to the transcendence of sparse continued fractions, (see the works of Adamczewski [1] and Bugeaud [2]) we don't know how this conjecture can be proven. There is also a much bolder version of the transcendence conjecture. Recall that the $\mathrm{PGL}_2(\mathbf{Z})$ is the group of invertible linear fractional transformations of dimension 2, i.e.

$$\mathrm{PGL}_2(\mathbf{Z}) := \left\{ \frac{ax+b}{cx+d} \,:\, a,b,c,d \in \mathbf{Z}, \quad ad-bc=\pm 1 \right\}$$

where the group operation is the functional composition. $\mathrm{PGL}_2(\mathbf{Z})$ acts naturally on the extended real line $\mathbf{R}\cup\{\infty\}$.

**Strong transcendence conjecture:** In addition to the transcendence conjecture, any set of algebraically related numbers in the set

$$S := \{\mathbf{J}(x) \,:\, x \in \overline{\mathbf{Q}}, \deg(x) > 2\}$$

are in the same $\mathrm{PGL}_2(\mathbf{Z})$-orbit.

As an example, $\mathbf{J}(x)$ and $\mathbf{J}(\frac{ax+b}{cx+d})$ are (provably) algebraically dependent for any $x \in \mathbf{R}$ and $\frac{ax+b}{cx+d} \in \mathrm{PGL}_2(\mathbf{Z})$, whereas if $x \in \overline{\mathbf{Q}}$ is non-quadratic, then $\mathbf{J}(x)$, $\mathbf{J}(x^2)$ and $\mathbf{J}(2x)$ are (conjecturally) not. A challenge might be to find some *real* number $x$ which is not rational nor a quadratic irrational, and such that $\mathbf{J}(x)$, $\mathbf{J}(2x)$ and $\mathbf{J}(x^2)$ are algebraically dependent.

The next section of the paper is devoted to the involution $\mathbf{J}$. Section 3 gives a theoretical study of frequencies of partial quotients of $\mathbf{J}(x)$ for general $x$ and compare them with the experimentally found frequencies for $\mathbf{J}$-values for algebraic $x$. We also consider some transcendental $x$ such as the number $\pi$.

Computational results presented are produced using Matlab and Python environments. For arbitrary precision arithmetic, on Matlab we use Symbolic Math Toolbox, and on Python we use SymPy [11] which in turn uses mpmath [7] library. It must be stressed that the numerical transcendence tests of this paper are with high confidence though not with certainty.

# 2 The involution J

The involution $\mathbf{J}$ originates from the outer automorphism group of $\mathrm{PGL}_2(\mathbf{Z})$ and can be viewed as an automorphism of the Stern-Brocot tree of continued fractions. These descriptions immediately shows that $\mathbf{J}$ is involutive. It has some very peculiar analytic, arithmetic and dynamical properties.

Here we give an overview of its definition and some of its properties. For details we refer to [17], [16]and [15].

## 2.1 Definition of the involution

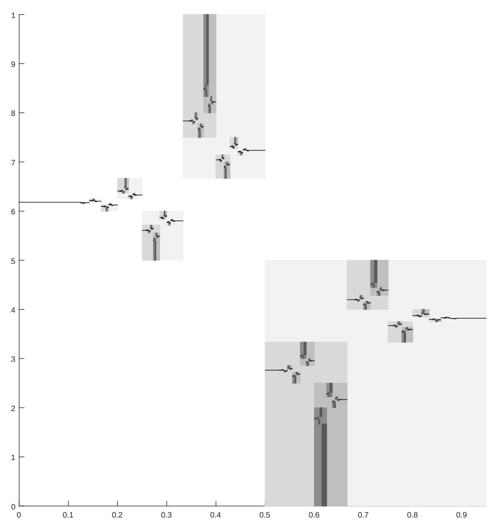


Figure 1: The plot of the involution $\mathbf{J}$ on the unit interval.

To illustrate the definition of $\mathbf{J}$ given in the introduction, consider the following example:

**Example 1.** One has

$$\mathbf{J}([1,1,1,1,1,13,\dots]) = [1_0, \underbrace{2, 1_{-1}, 2}, 1_{-1}, 2, 1_{-1}, 2, 1_{-1}, 2, 1_{11}, \dots]$$
$$[\underbrace{3, 1_{-1}, 2}, 1_{-1}, 2, 1_{-1}, 2, 1_{11}, \dots]$$
$$[\underbrace{4, 1_{-1}, 2}, 1_{-1}, 2, 1_{11}, \dots]$$
$$[\underbrace{5, 1_{-1}, 2}, 1_{11}, \dots]$$
$$[6, 1_{11}, \dots]$$

The next example should convince the reader that $\mathbf{J}$ is involutive:

**Example 2.** One has

$$\begin{aligned}\mathbf{J}([6, 1_{11}, \dots]) &= [1_5, 2, 1_{-1}, 2, \dots, 1_{-1}, 2, \dots] \\ &= [1, 1, 1, 1, 1, 13, \dots]\end{aligned}$$

**J on rationals.** Since every nonzero rational number admits two simple continued fraction representations (one ending with a 1 and the other not), and since the defining rule of $\mathbf{J}$ produces two different values when applied to these representations, our involution is not well-defined on $\mathbf{Q}$. In fact, there is a way to extend $\mathbf{J}$ to the set of rationals as well, see [17]. In Appendix I, we provide a maple code which evaluates $\mathbf{J}$ at a given rational.

**J on noble numbers.** A number is said to be *noble* if its continued fraction terminates with $1_\infty$. If $x$ is noble, then $\mathbf{J}(x)$ is rational. In fact

$$\mathbf{J}([n_0, \dots, n_k, 1_\infty]) = [1_{n_0-1}, 2, \dots, 2, 1_{n_k-2}] = \mathbf{J}([n_0, \dots, n_k - 2, 2, 1_\infty]),$$

so that $\mathbf{J}$ is 2-1 on the set of noble numbers.

**J on quadratic irrationals.** It is well known that ultimately periodic continued fractions are precisely real quadratic irrationals. Since by definition $\mathbf{J}$ preserves the periodicity of continued fractions, $\mathbf{J}$ sends quadratic irrationals to quadratic irrationals (setwise). As an example, $\sqrt{2} = [1, \overline{2}] \implies \mathbf{J}(\sqrt{2}) = [\overline{2}] = 1 + \sqrt{2}$. In general the situation is not so simple; we give a list of

**J**-transforms of some quadratic surds in Appendix II below. This action respects the Galois conjugation, i.e.

$$\mathbf{J}(a+\sqrt{b}) = c+\sqrt{d} \implies \mathbf{J}(a-\sqrt{b}) = c-\sqrt{d} \tag{6}$$
$$\mathbf{J}(a+\sqrt{b}) = c-\sqrt{d} \implies \mathbf{J}(a-\sqrt{b}) = c+\sqrt{d}, \tag{7}$$

where $a, b, c, d \in \mathbf{Q}$ with $c, d > 0$ being non-squares. This fact, together with the functional equations (see Section 2.1 below) below implies that if $x = \sqrt{q}$ where $q \in \mathbf{Q}$ is a positive non-square, with $\mathbf{J}(x) = a \pm \sqrt{b}$, then $\mathrm{Norm}(x) := a^2 - b = -1$. There is a host of such correspondences of quadratic irrationals under $\mathbf{J}$, for example, $\mathrm{Norm}(x) = 1 \iff \mathrm{Norm}(\mathbf{J}(x)) = 1$. See [15] for details.

**J on other numbers.** To finish, let us give the **J**-transform of two numbers that we used in our experiments:

$$\begin{aligned}\mathbf{J}(\sqrt[3]{2}) &= \mathbf{J}([1;3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,12,2,3,2,1,\dots]) \\ &= [2,1,3,1,1,1,4,1,1,4,1_6,3,1_{12},3,1_8,2,3,1,1,2,1_{10},2,2,1,2,\dots] \\ &= 2.784731558662723\dots, \\ \mathbf{J}(\pi) = \mathbf{J}([3,7,15,1,292,1,1,1,2,1,3,\dots]) &= [1_2,2,1_5,2,1_{13},3,1_{290},5,3,\dots] \\ &= 1.723770792548027\dots.\end{aligned}$$

## 2.2 Further properties of J

**Functional equations.** We have defined **J** on the unit interval. We may extend it to **R** via the equation $\mathbf{J}(-x) = -1/\mathbf{J}(x)$. This extension satisfies the functional equations (see [15])

$$\mathbf{J}(1/x) = 1/\mathbf{J}(x), \quad \mathbf{J}(1-x) = 1-\mathbf{J}(x), \quad \mathbf{J}(-x) = -1/\mathbf{J}(x). \tag{8}$$

In fact, these equations characterize **J**. Given the continued fraction representation of $x$, one can use these functional equations to compute $\mathbf{J}(x)$.

**Harmonic numbers and Beatty partitions.** Using the second and the third equations one obtains

$$\frac{1}{x}+\frac{1}{y} = 1 \iff \frac{1}{\mathbf{J}(x)}+\frac{1}{\mathbf{J}(y)} = 1;$$

so **J** preserves harmonic pairs of real numbers. This implies that **J** acts on the Beatty partitions (see [13]) of the set of natural numbers.

**Modularity.** Another consequence of the functional equations is that, if the continued fractions of $x$ and $y$ have the same tail, then the same is true for $\mathbf{J}(x)$ and $\mathbf{J}(y)$. This shows that $\mathbf{J}$ sends $\mathrm{PGL}_2(\mathbf{Z})$-orbits to $\mathrm{PGL}_2(\mathbf{Z})$-orbits, i.e. it defines an involution of $\mathbf{R}/\mathrm{PGL}_2(\mathbf{Z})$, the "moduli space of degenerate rank-2 lattices". As such, we may consider it as a kind of "modular form". It is easy to see that $\mathbf{J}$ is continuous on $\mathbf{R} \setminus \mathbf{Q}$ and with jumps at rationals; and we were able to prove that $\mathbf{J}$ is differentiable almost everywhere with a derivative vanishing almost everywhere [14].

**Dynamics.** Recall that the celebrated Gauss continued fraction map $\mathbb{T}_G : [0,1] \to [0,1]$ is the map which forgets the first partial quotient:

$$\mathbb{T}_G : x = [0, n_1, n_2, n_3, \dots] = [0, n_2, n_3, n_4, \dots] \tag{9}$$

Our involution $\mathbf{J}$ conjugates the Gauss continued fraction map to the so-called Fibonacci map $\mathbb{T}_F : [0,1] \to [0,1]$, defined as

$$\mathbf{J} \circ \mathbb{T}_G \circ \mathbf{J} : \mathbb{T}_F : [0, 1_k, n_{k+1}, n_{k+2}, \dots] \to [0, n_{k+1} - 1, n_{k+2}, \dots] \tag{10}$$

where it is assumed that $n_{k+1} > 1$ and $0 \le k < \infty$. Dynamical properties of $\mathbb{T}_F$ and $\mathbb{T}_G$ are tightly related, (see [6] and [16]). For example the eigenfunctions of their transfer operators (see [10]) satisfies the same three-term functional equation studied in [9].

Our hope is that, due to these rich properties of the involution $\mathbf{J}$, especially the functional equations (8) and its effect on quadratic irrationals, it might be possible to infer the transcendence of $\mathbf{J}(x)$ directly from the knowledge of algebraicity of $x$; by-passing the "beliefs" in our chain of reasonings which led us to the transcendence conjecture.

# 3 The transcendence conjectures

Let us explain the theoretical basis for the conjecture. If $X \in [0,1]$ is a uniformly distributed random variable, then by Gauss-Kuzmin's theorem [8], the frequency of an integer $k > 0$ among the partial quotients of $X$ equals almost surely

$$p(k) = \frac{1}{\log 2} \log\left(1 + \frac{1}{k(k+2)}\right).$$

Moreover, the arithmetic mean of its partial quotients tends almost surely to infinity (see [8]), i.e. if $X = [0, n_1, n_2, \dots]$ then

$$\lim_{k\to\infty} \frac{n_1 + \cdots + n_k}{k} = \infty \qquad \text{(a.s.)} \tag{11}$$

In other words, the set of numbers in the unit interval such that the above limit exists and is infinite, is of full Lebesgue measure. Denote this set by $\mathcal{A}$. Since the first $k$ partial quotients of $X$ give rise to at most $n_1 + \cdots + n_k - k$ partial quotients of $\mathbf{J}(X)$ and at least $n_1 + \cdots + n_k - 2k$ of these are 1's, one has

$$\frac{n_1 + \cdots + n_k - k}{n_1 + \cdots + n_k - 2k} \to \frac{\frac{n_1+\cdots+n_k}{k} - 1}{\frac{n_1+\cdots+n_k}{k} - 2} \to 1$$

This proves that $\mathbf{J}(x)$ is almost surely 'sparse' in the following sense:

**Lemma 1** *The frequency of* $1$*'s among the partial quotients of* $\mathbf{J}(X)$ *equals 1 a.s.. In particular the the partial quotient averages of* $\mathbf{J}(X)$ *tend to 1 a.s. and* $\mathbf{J}(\mathcal{A})$ *is a set of zero measure.*

Note that $x$ and $\mathbf{J}(x)$ can be simultaneously sparse, consider e.g. $x = [0, 1_2, 2^2, 1_{2^3}, 2^4, 1_{2^5} \dots]$.

Below is a partial quotient statistics for the numbers $\mathbf{J}(\pi)$ and $\mathbf{J}(\sqrt[3]{2})$.

The proof of the Lemma applies to any number $x$ with partial quotient averages tending to infinity. Since for $x$ algebraic of degree $> 2$, it is widely believed that this is the case, we see that the average partial quotient of $\mathbf{J}(x)$ is very likely to tend to 1 for such $x$. Since this is far from being unbounded, it is natural to believe that the transcendence conjecture is true.

# 4 Testing the transcendence conjecture

In order to asses the transcendence of $\mathbf{J}(x)$ for non-quadratic algebraic numbers, we performed a computational search to find a polynomial of degree $> 2$ having root $\mathbf{J}(x)$ using the PSLQ integer relation algorithm[4, 3].

For computations we used Matlab environment with its Symbolic Toolbox, and the Python programming environment along with SymPy[11] package for symbolic computations which itself uses mpmath[7] package for arbitrary precision numeric calculations.

We conducted a search for a polynomial for a given $x \in \mathbb{R}$ represented at very high precision, by successively running the PSLQ algorithm with the

| partial quotient | frequency |
|---|---|
| 1 | 95.160 |
| 2 | 2.636 |
| 3 | 1.418 |
| 4 | 0.471 |
| 5 | 0.186 |
| 6 | 0.078 |
| 7 | 0.033 |
| 8 | 0.009 |
| 9 | 0.004 |
| 11 | 0.002 |
| 10 | 0.001 |
| 13 | 0.001 |

(a) Statistics of $\mathbf{J}(\pi)$.

| partial quotient | frequency |
|---|---|
| 1 | 94.761 |
| 2 | 2.891 |
| 3 | 1.535 |
| 4 | 0.476 |
| 5 | 0.207 |
| 6 | 0.073 |
| 7 | 0.034 |
| 8 | 0.013 |
| 9 | 0.004 |
| 11 | 0.001 |
| 10 | 0.000 |
| 13 | 0.000 |

(b) Statistics of $\mathbf{J}(\sqrt[3]{2})$.

Table 1: Both statistics have been made for the continued fraction expansions of $\pi$ of length $10^5$ and of $\sqrt[3]{2}$ of length $2 \times 10^4$. For mentioned expansion sizes, the length of $\mathbf{J}(\pi)$ is $> 1.4 \times 10^6$ terms whereas that of $\mathbf{J}(\sqrt[3]{2})$ is $> 2.2 \times 10^5$.

set of input variables $\{x^n : 0 \leq n < m\}$ where $m$ is increased by one at each iteration, until a set error tolerance constraint or maximum degree constraint is reached. PSLQ algorithm allows to efficiently search integer coefficients $c_i$ satisfying $|c_1x_1 + c_2x_2 + ... + c_nx_n| < tol$ for a given tolerance limit $tol$.

With integer relations algorithms it is always possible to find a better algebraic approximation to any irrational either by increasing the maximum allowed degree for the polynomial, or by allowing larger coefficients; on the other hand the increase in complexity of the polynomial as the tolerance is decreased provides us stronger lower bounds for the degree and coefficients.

## 4.1 Algebraics in the neighborhoods of $\mathbf{J}(\sqrt[3]{2})$

In agreement with the conjecture, PSLQ algorithm do not find any polynomial having $\mathbf{J}(\sqrt[3]{2})$ as a root when bounded with degree up to 20 and coefficients up to $\pm 100$. For the computations only the first $10^5$ terms of continued fraction expansion is used which yields $21,975$ decimal digits of accuracy in the case of $\mathbf{J}(\sqrt[3]{2})$, which provides a safe margin considering the error tolerance of 13 decimal digits requested by the last iteration. As expected, it can be observed that as the error tolerance constraint getting tighter, the polynomial satisfying it gets more complex.

| Tolerance | Polynomial | Error |
|---|---:|---:|
| $10^{-5}$ | $-18x^3+29x^2+38x+58$ | $-0.000172115$ |
| $10^{-6}$ | $3x^3+6x^2-45x+14$ | $-1.81707\times 10^{-5}$ |
| $10^{-7}$ | $15x^3-4x^2-98x-20$ | $2.08021\times 10^{-6}$ |
| $10^{-8}$ | $23x^4-61x^3-10x^2+6x-5$ | $1.27915\times 10^{-8}$ |
| $10^{-9}$ | $23x^4-61x^3-10x^2+6x-5$ | $1.27915\times 10^{-8}$ |
| $10^{-10}$ | $-20x^5+41x^4+55x^3-28x^2-19x-34$ | $-3.13035\times 10^{-9}$ |
| $10^{-11}$ | $-22x^5+29x^4+64x^3+54x^2+26x+67$ | $9.1381\times 10^{-10}$ |
| $10^{-12}$ | $-12x^5+64x^4-68x^3-33x^2-66x+69$ | $-8.09244\times 10^{-11}$ |
| $10^{-13}$ | None | |

Table 2: Searching for a polynomial with $\mathbf{J}(\sqrt[3]{2})$ as root, by iteratively decreasing error tolerance

As the constraint on maximum permitted degree wasn't reached on the previous experiment, we relax the constraint on the absolute value of coefficients to $\pm 10^6$, while all other settings remain the same. Notice that PSLQ algorithm prefers using larger coefficients instead of higher degrees, hence up to the case of error tolerance of $10^{-20}$ it only returns polynomials up to degree 3; but for any tighter error bounds it couldn't find any polynomial in the allowed bounds.

# 5 Statistics

Let $X \in [0,1]$ be a uniformly distributed random variable. Lemma 1 implies that the density of any $k>1$ among the partial quotients of $\mathbf{J}(X)$ is a.s. zero. What if we ignore the 1's in the partial quotients? To be more precise, define the "collapse of continued fraction map" as

$$\mathsf{P} : [0, n_1, n_2, \dots,] \to [0, n_1 - 1, n_2 - 1, \dots],$$

where the vanishing partial quotients are simply ignored. For example,

$$\mathsf{P}([0,2,4,1,1,1,9,5,\dots]) = [0,1,3,8,4,\dots].$$

| Tolerance | Polynomial | Error |
|---|---|---|
| $10^{-5}$ | $9667x - 26920$ | $-2.44287 \times 10^{-5}$ |
| $10^{-6}$ | $-328023x + 913456$ | $1.36212 \times 10^{-6}$ |
| $10^{-7}$ | $-1514x^2 + 6087x - 5210$ | $-6.01215 \times 10^{-7}$ |
| $10^{-8}$ | $-36x^2 - 1845x + 5417$ | $-4.22778 \times 10^{-8}$ |
| $10^{-9}$ | $-561x^2 - 5135x + 18650$ | $5.24536 \times 10^{-9}$ |
| $10^{-10}$ | $-19147x^2 + 27520x + 71844$ | $-4.85943 \times 10^{-11}$ |
| $10^{-11}$ | $-19147x^2 + 27520x + 71844$ | $-4.85943 \times 10^{-11}$ |
| $10^{-12}$ | $18066x^2 - 205093x + 431032$ | $2.65181 \times 10^{-12}$ |
| $10^{-13}$ | $140x^3 - 3215x^2 + 3525x + 12092$ | $-1.48887 \times 10^{-12}$ |
| $10^{-14}$ | $-1909x^3 - 3414x^2 + 17605x + 18674$ | $-1.99434 \times 10^{-13}$ |
| $10^{-15}$ | $-52555x^3 + 135385x^2 + 21681x + 24667$ | $5.61226 \times 10^{-15}$ |
| $10^{-16}$ | $-58238x^3 + 128952x^2 + 91442x + 3011$ | $2.27444 \times 10^{-17}$ |
| $10^{-17}$ | $-58238x^3 + 128952x^2 + 91442x + 3011$ | $2.27444 \times 10^{-17}$ |
| $10^{-18}$ | $-58238x^3 + 128952x^2 + 91442x + 3011$ | $2.27444 \times 10^{-17}$ |
| $10^{-19}$ | $302431x^3 - 805912x^2 - 389516x + 803378$ | $-8.3815 \times 10^{-19}$ |
| $10^{-20}$ | None | |

Table 3: Search for a polynomial with $\mathbf{J}(\sqrt[3]{2})$ as root, with constraint on coefficients relaxed to $\pm 10^6$

(We don't care about the collapse of the continued fractions terminating with $1_\infty$ as these are countable in number.) Then we can determine the partial quotient statistics of $\mathsf{P}\mathbf{J}(X)$, as follows. It is known that the frequency of $1_i$ in the continued fraction expansion of $X$ equals (see [5])

$$k(i) = \frac{(-1)^i}{\log 2} \log \left( 1 + \frac{(-1)^i}{F_{i+2}^2} \right)$$

This counting involves a certain repetitiveness in that, if $j \leq i$, then the string $\ldots n, 1_i, m, \ldots$ $(n, m > 1)$ in the continued fraction contributes $k - l + 1$ to the census. Denoting by $m(i)$ the frequency of the string $1_i$ occur, but not as a substring of a longer string of 1's, we have

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \ldots \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ 0 & 1 & 2 & \ldots & n-1 \\ 0 & 0 & 1 & \ldots & n-2 \\ & & & \ldots & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \ldots \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \dots \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ & & & & & \dots & & \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \dots \end{bmatrix}$$

Hence, we get $m_i = k_i - 2k_{i+1} + k_{i+2}$. Note that

$$\sum_{i=0}^{\infty} m(i) = 0.584962500, \quad \sum_{i=1}^{\infty} im(i) = p(1) = 0.41503749927,$$

and that $m(0)$ is the frequency of the string $\dots, n, m, \dots$ with $n, m > 1$. This string transforms under $\mathbf{J}$ to the string $1_{n-1}, 2, 1_{m-1}$, which after the collapse map yields a partial quotient 1 in $\mathsf{P}\mathbf{J}(X)$. The desired frequency of $i$ among the partial quotients of $\mathsf{P}\mathbf{J}(X)$ is the number $u(i)$ given by

$$u(i) = \frac{m(i-1)}{\sum_{j=0}^{\infty} m(i)}$$

Above we tabulate these theoretical values of the frequencies Table 4, followed by the experimental values obtained by computation for $\mathsf{P}\mathbf{J}(\pi)$ and $\mathsf{P}\mathbf{J}(\sqrt[3]{2})$ Table 5.

| $i$ | $k(i)$ | $m(i)$ | $m(i)/\sum m_i$ | $p(i)$(Gauss-Kuzmin) |
|---|---|---|---|---|
| 0 | 1.0 | 0.3219280948 | 0.5503397134 | |
| 1 | 0.4150374993 | 0.1699250016 | 0.2904887087 | 0.4150374989 |
| 2 | 0.1520030934 | 0.0565835283 | 0.0967301805 | 0.1699250015 |
| 3 | 0.0588936890 | 0.0227200765 | 0.0388402273 | 0.09310940485 |
| 4 | 0.0223678130 | 0.0085115001 | 0.0145505056 | 0.05889368952 |
| 5 | 0.0085620135 | 0.0032751312 | 0.0055988737 | 0.04064198510 |
| 6 | 0.0032677142 | 0.0012474677 | 0.0021325602 | 0.02974734293 |
| 7 | 0.0012485461 | 0.0004770014 | 0.0008154393 | 0.02272007668 |
| 8 | 0.0004768458 | 0.0001821241 | 0.0003113433 | 0.01792190800 |
| 9 | 0.0001821469 | 0.0000695769 | 0.0001189425 | 0.01449956955 |
| 10 | 0.0000695722 | 0.0000265739 | 0.0000454285 | 0.01197264119 |

Table 4: Theoretical values for the expected frequencies

If we consider $\mathsf{P}\mathbf{J}$ as a kind of derivation, then it is possible to compute the statistics of higher derivatives, by using results of ([12]) where the Gauss-Kuzmin statistics for $n_k$ $(n, k = 1, 2, \dots)$ have been computed.

| partial quotient | frequency |
|---|---|
| 1 | 54.891 |
| 2 | 29.250 |
| 3 | 9.703 |
| 4 | 3.854 |
| 5 | 1.391 |
| 6 | 0.541 |
| 7 | 0.229 |
| 8 | 0.088 |
| 9 | 0.023 |
| 10 | 0.018 |
| 11 | 0.003 |
| 12 | 0.003 |

(a) Statistics of $\mathsf{P}\mathbf{J}(\pi)$.

| partial quotient | frequency |
|---|---|
| 1 | 55.202 |
| 2 | 29.304 |
| 3 | 9.099 |
| 4 | 3.952 |
| 5 | 1.408 |
| 6 | 0.657 |
| 7 | 0.256 |
| 8 | 0.085 |
| 9 | 0.008 |
| 10 | 0.025 |
| 11 | 0.000 |
| 12 | 0.000 |

(b) Statistics of $\mathsf{P}\mathbf{J}(\sqrt[3]{2})$.

Table 5: Digit frequencies observed by numerical computation

## 5.1 Conjectures concerning algebraic operations

Recall that $\mathcal{A} \subset [0,1]$ is the set of real numbers whose partial quotient averages tend to infinity and that it is of full measure. It includes the set of "normal" numbers where by "normal" we mean that $x$ and $y$ obeys all predictions of the Gauss-Kuzmin statistics.

**Conjecture 3.** If $x, y \in \mathcal{N}$, then $\mathbf{J}(x) + \mathbf{J}(y)$ and $\mathbf{J}(x)\mathbf{J}(y)$ are normal a.s..

Note that if $x$ is normal then so is $1 - x$, whereas $\mathbf{J}(x) + \mathbf{J}(1 - x) = 1$ is surely not normal. Also note that the set $\mathbf{J}(\mathcal{A})$ and therefore the sets $\mathbf{J}(\mathcal{A}) + \mathbf{J}(\mathcal{A})$ and $\mathbf{J}(\mathcal{A}) \times \mathbf{J}(\mathcal{A})$ are of zero measure.

On Figure 2 you can find the continued fraction term frequencies for the sums and multiplications between $\mathbf{J}(\pi), \mathbf{J}(\sqrt[3]{2})$ and $\mathbf{J}$-transformed Euler-Mascheroni constant $\mathbf{J}(\gamma)$, along with the Gauss-Kuzmin distribution as a reference for the expected term distribution for normal numbers. A stem plot instead of the line plot would better suit for displaying term frequencies as these are only defined for integers; but as we want to portray the overlap between many distributions stem plot would result on a too occluded graph, hence the use of line graph instead. It can be seen that even though the $\mathbf{J}$ transformed numbers are 'sparse' on themselves, any algebraic operation

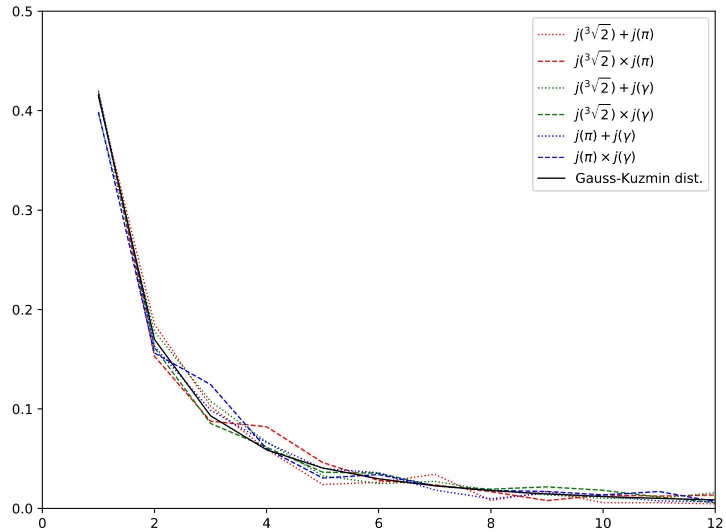


Figure 2: Continued fraction term frequencies of some presumably normal numbers under algebraic operations ($\gamma$ is Euler-Mascheroni constant)

between them result with a term distribution which is characteristic of normal numbers.

**Conjecture 4.** $q\mathbf{J}(X)$ obeys a certain law for every $q \in \mathbf{Q}$.

Figure 3 and Figure 4 display a few cases cases of the form $q\mathbf{J}(X)$ with $q \in \mathbf{Z}$ and $q \in \mathbf{Q}$ respectively. In both figures the distributions of $\mathbf{J}$ transformed (presumably) normal numbers and golden ratio are in agreement under multiplication with the same constant.

## 5.2 Algebraic independence

Another property to investigate is whether the algebraic dependence (or independence thereof) is a property that $\mathbf{J}$ transformation preserves. In this regard we conducted a series of experiments which search for possible relations between $\mathbf{J}$ transformation of some algebraically related numbers.

PSLQ is an integer relation algorithm that searches an integer relation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ given some real numbers $x_1, x_2, \cdots, x_n$. Implementation on mpmath extends the search to allow for exponentiation, logarithm, rational functions, and $a_n \in \mathbf{Q}$ instead of $\mathbf{Z}$. We restrict the numerator and denominator of rational constants used on search by $10^3$;

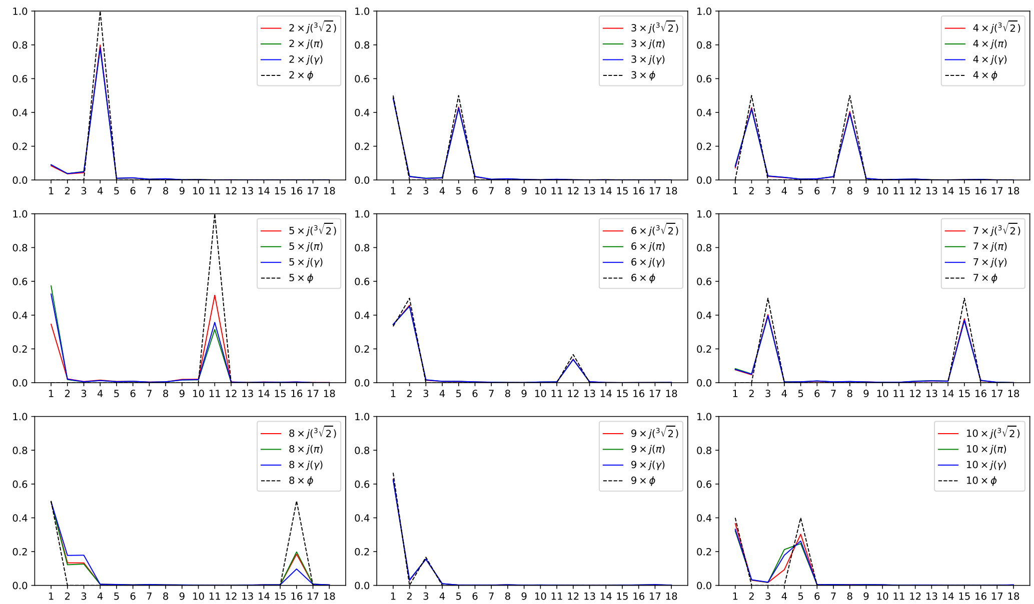


Figure 3: Continued fraction term frequencies of some presumably normal numbers under algebraic operations ($\gamma$ is Euler-Mascheroni constant)

otherwise it is always possible to approach arbitrarily close to any irrational by using rationals of greater complexity.

**Experiment 1.** In this experiment we search for a possible algebraic relation between $\mathbf{J}(\sqrt[3]{2})$ and $\mathbf{J}(\sqrt[3]{4})$, by progressively increasing the allowed error tolerance in order to observe the complexity of proposed relation as a function of allowed error.

It can be seen on Table 6 that as the allowed error tolerance is reduced, the simplest expression relating $\mathbf{J}(\sqrt[3]{2})$ and $\beta = \mathbf{J}(\sqrt[3]{4})$ gets more complicated, up to the point where for error tolerance of $10^{-12}$ no relations can be found, even with a precision of 150 decimal digits.

**Experiment 2.** As a second investigation, we search whether the relation between $\sqrt[3]{2}$ and $2 \times \sqrt[3]{2}$ is mapped to another algebraic relation under $\mathbf{J}$ transformation?

Table 7 presents results similar to those of Table 6, which allows us to conclude that even if there still is a relation between $\mathbf{J}$ transforms of those

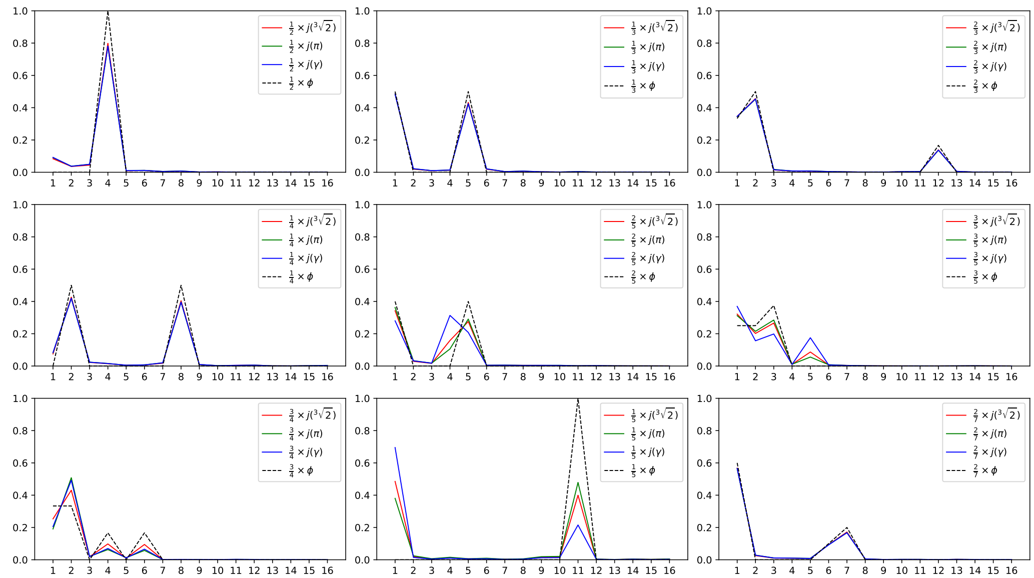


Figure 4: Continued fraction term frequencies of some presumably normal numbers under algebraic operations ($\gamma$ is Euler-Mascheroni constant)

two numbers, the expression for it would be highly non-trivial.

**Experiment 3.** Finally, we search for a possible algebraic relation between the triple of numbers consisting of $(\alpha = \mathbf{J}(\sqrt[3]{2}), \beta = \mathbf{J}(2 \times \sqrt[3]{2}), \gamma = \mathbf{J}(\frac{\sqrt[3]{2}}{2}))$

Once again we see on Table 8 that the complexity of a possible relation between the triple of numbers grows with the allowed error tolerance up to the point where the space of expressions defined by constraints on allowed primitives is exhausted.

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| Tolerance | Relation Expression |
|---|---|
| $10^{-5}$ | $\alpha = 136 - 30\beta$ |
| $10^{-6}$ | $\alpha = \frac{56}{701} + \frac{427}{701}\beta$ |
| $10^{-7}$ | $\alpha = \frac{565}{289} + \frac{54}{289}\beta$ |
| $10^{-8}$ | $\alpha = \frac{\beta}{\log(59/11 - 13/132\beta)}$ |
| $10^{-9}$ | $\alpha = \frac{3^{36/73}5^{2/73}7^{25/73}\beta^{33/73}}{2^{95/73}}$ |
| $10^{-10}$ | $\alpha = \frac{2^{3/4}3^{11/18}\beta^{47/36}}{5^{11/36}7^{5/6}}$ |
| $10^{-11}$ | $\alpha = \frac{2^{108/91}5^{48/91}7^{9/13}\beta^{5/91}}{3^{172/91}}$ |
| $10^{-12}$ | *None* |

Table 6: Algebraic relations between $\alpha = \mathbf{J}(\sqrt[3]{2})$ and $\beta = \mathbf{J}(\sqrt[3]{4})$ up to various error levels

| Tolerance | Relation Expression |
|---|---|
| $10^{-5}$ | $\alpha = \frac{-14}{19} + \frac{55}{19}\beta$ |
| $10^{-6}$ | $\alpha = \frac{101}{135} + \frac{226}{135}\beta$ |
| $10^{-7}$ | $\alpha = \frac{\sqrt{553 + \sqrt{234969}/110}}{\sqrt{\beta}}$ |
| $10^{-8}$ | $\alpha = \frac{2^{2/7}3^{29/14}\beta^{10/7}}{5^{9/14}7^{5/14}}$ |
| $10^{-9}$ | $\alpha = \frac{2^{4/9}3^{32/9}}{5^{5/9}7^{8/9}\beta^{26/9}}$ |
| $10^{-10}$ | $\alpha = \frac{2^{49/38}3^{63/19}}{5^{15/19}7^{33/38}\beta^{107/38}}$ |
| $10^{-11}$ | $\alpha = \frac{7^{35/39}\beta^{12/13}}{2^{43/39}3^{2/39}5^{2/39}}$ |
| $10^{-12}$ | $\alpha = \frac{5^{29/37}7^{20/111}\beta^{140/111}}{2^{64/111}3^{44/111}}$ |
| $10^{-13}$ | $\alpha = \frac{2^{199/95}7^{99/95}\beta^{10/19}}{3^{107/95}5^{78/95}}$ |
| $10^{-14}$ | $\alpha = \frac{2^{56/405}3^{4/405}7^{233/405}\beta^{379/405}}{5^{97/405}}$ |
| $10^{-15}$ | *None* |

Table 7: Algebraic relations between $\alpha = \mathbf{J}(\sqrt[3]{2})$ and $\beta = \mathbf{J}(2 \times \sqrt[3]{2})$ up to various error levels

| Tolerance | Relation Expression |
|---|---|
| $10^{-5}$ | $\alpha = \frac{1}{2} + \frac{13}{5}\beta - \frac{39}{10}\gamma$ |
| $10^{-6}$ | $\alpha = \frac{7}{11} + \frac{37}{22}\beta + \frac{5}{11}\gamma$ |
| $10^{-7}$ | $\alpha = \frac{128}{227} + \frac{392}{227}\beta + \frac{121}{227}\gamma$ |
| $10^{-8}$ | $\alpha = \frac{117}{74} + \frac{153}{74}\beta - \frac{431}{74}\gamma$ |
| $10^{-9}$ | $\alpha = \frac{-64}{77} + \frac{694}{231}\beta - \frac{40}{231}\gamma$ |
| $10^{-10}$ | $\alpha = \frac{-88}{433} + \frac{908}{433}\beta + \frac{840}{433}\gamma$ |
| $10^{-11}$ | $\alpha = \frac{\gamma}{-{}^{88}/_{243} + {}^{61}/_{243}\beta + {}^{11}/_{18}\gamma}$ |
| $10^{-12}$ | $\alpha = \frac{7^{31/5}\gamma^{4/5}}{2^{7/5}3^{7/5}5^{4}\beta^{23/5}}$ |
| $10^{-13}$ | *None* |

Table 8: Algebraic relations between $\alpha = \mathbf{J}(\sqrt[3]{2}), \beta = \mathbf{J}(2 \times \sqrt[3]{2})$, and $\gamma = \mathbf{J}(\frac{\sqrt[3]{2}}{2})$ up to various error levels

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**Appendix-I: Maple code to evaluate Jimm on Q.**

```
>with(numtheory)
>jimm := proc (q) local M, T, U, i, x;
T := matrix([[1, 1], [1, 0]]);
U := matrix([[0, 1], [1, 0]]);
M := matrix([[1, 0], [0, 1]]);
x := cfrac(q, quotients);
if x[1] = 0 then for i from 2 to nops(x) do
M := evalm(‘&*‘(‘&*‘(M, T^x[i]), U)) end do;
return M[2, 2]/M[1, 2] else for i to nops(x) do
M := evalm(‘&*‘(‘&*‘(M, T^x[i]), U)) end do;
return M[1, 2]/M[2, 2] end if
end proc;
```

**Appendix-II** **J**-transforms of some quadratic surds

| $N$ | $\mathbf{J}(\sqrt{N})$ | $N$ | $\mathbf{J}(\sqrt{N})$ |
|---|---|---|---|
| 3 | $\frac{1}{2}(\sqrt{13}+3)$ | 21 | $\frac{1}{307}(\sqrt{113570}+139)$ |
| 5 | $\frac{1}{3}(\sqrt{10}+1)$ | 22 | $\frac{13}{307}(\sqrt{677}+142)$ |
| 6 | $\frac{1}{14}(\sqrt{221}+5)$ | 23 | $\frac{1}{24}(\sqrt{697}+11)$ |
| 7 | $\frac{1}{6}(\sqrt{37}+1)$ | 24 | $\frac{1}{50}(\sqrt{3029}+23)$ |
| 8 | $\frac{1}{4}(\sqrt{17}+1)$ | 26 | $\frac{1}{49}(\sqrt{3026}+25)$ |
| 10 | $\frac{1}{7}(\sqrt{65}+4)$ | 27 | $\frac{1}{194}(\sqrt{47437}+99)$ |
| 11 | $\frac{1}{26}(\sqrt{901}+15)$ | 28 | $\frac{1}{139}(\sqrt{24362}+71)$ |
| 12 | $\frac{1}{34}(\sqrt{1517}+19)$ | 29 | $\frac{1}{495}(\sqrt{308026}+251)$ |
| 13 | $\frac{1}{3}(\sqrt{13}+2)$ | 30 | $\frac{1}{238}(\sqrt{71285}+121)$ |
| 14 | $\frac{1}{5}(\sqrt{34}+3)$ | 31 | $\frac{1}{17226}(\sqrt{376748101}+8945)$ |
| 15 | $\frac{1}{18}(\sqrt{445}+11)$ | 32 | $\frac{1}{94}(\sqrt{11237}+49)$ |
| 17 | $\frac{1}{19}(\sqrt{442}+9)$ | 33 | $\frac{1}{101}(\sqrt{12905}+52)$ |
| 18 | $\frac{1}{78}(\sqrt{7453}+37)$ | 34 | $\frac{1}{130}(\sqrt{21389}+67)$ |
| 19 | $\frac{1}{730}(\sqrt{656101}+351)$ | 35 | $\frac{1}{64}(\sqrt{5185}+33)$ |
| 20 | $\frac{1}{23}(5\sqrt{26}+11)$ | 37 | $\frac{1}{129}(\sqrt{20737}+64)$ |