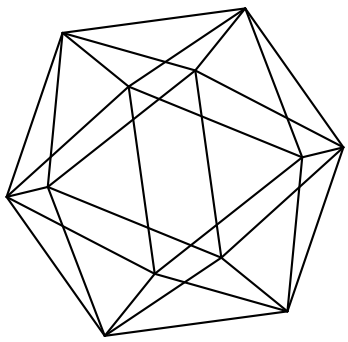
Max-Planck-Institut für Mathematik
Bonn

Quadrangulations of sphere, ball quotients and Belyi maps

by

A. Muhammed Uludag
Ayberk Zeytin



Max-Planck-Institut für Mathematik
Preprint Series 2011 (59)

Quadrangulations of sphere, ball quotients and Belyi maps

A. Muhammed Uludag
Ayberk Zeytin

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Mathematics Department
Galatasaray University
Istanbul
Turkey

MPIM 11-59

Preprint series of the Max Planck for Mathematics, Bonn No. 59

Quadrangulations of sphere, ball quotients and Belyi maps

A. Muhammed Uludağ · Ayberk Zeytin

Abstract We give a classification of sphere quadrangulations satisfying a condition of non-negative curvature, following Thurston's classification of sphere triangulations under the same condition. The generic family of quadrangulations is parametrized by the points of positive square-norm of an integral Gaussian lattice Λ' in the six-dimensional complex Lorentz space. There is a subgroup of automorphisms of Λ' acting on this lattice whose orbits parametrize sphere quadrangulations in a one-to-one manner. This group acts discretely on the corresponding five-dimensional complex hyperbolic space; is of finite co-volume; its ball quotient is the moduli space of 8 points on the Riemann sphere, and also appears in Picard-Terada-Deligne-Mostow list.

Keywords sphere quadrangulation · ramified coverings of sphere · ball quotients

1 Introduction

Let X be a closed orientable 2-manifold. A *triangulation* of X is a maximal polygonal decomposition of X. A vertex of a triangulation is *non-negatively curved* if there are at most six triangles meeting at this vertex. If all vertices of a triangulation are non-negatively curved, then the triangulation itself is said to be non-negatively curved. Let $\mathscr{T}$ be a triangulation of X and let k_i be the number of vertices of $\mathscr{T}$ adjacent to i triangles. A basic counting argument yields the formula

$$6\chi(X) = \sum_{i=1}^{\infty}(6-i)k_i \tag{1}$$

Observe that the coefficient of k_6 is zero in the above formula, so that vertices with exactly six adjacent triangles have no effect on $\chi(X)$.

A. Muhammed Uludağ
Mathematics Department, Galatasaray University, Istanbul, Turkey E-mail: muludag@gsu.edu.tr

Ayberk Zeytin
Mathematics Department, Galatasaray University, Istanbul, Turkey E-mail: azeytin@metu.edu.tr

Assume now that $\mathscr{T}$ is of non-negative curvature. This means that $k_i = 0$ unless $i \leq 6$ in the above formula, which gives $\chi(X) \geq 0$. According to Equation 1, X must be a torus if $k_i = 0$ for $i \neq 6$. In case there is a vertex of positive curvature; that is if at least one among k_1, k_2, k_3, k_4 and k_5 is non-zero, then the formula yields $\chi(X) > 0$. Therefore X must be the sphere with $\chi(X) = 2$ and the formula becomes

$$12 = \sum_{i=1}^{5} (6-i)k_i = 5k_1 + 4k_2 + 3k_3 + 2k_4 + k_5 \tag{2}$$

One can easily write down a complete list of tuples $\tau = (k_1, k_2, k_3, k_4, k_5)$ satisfying this equation, which are finite in number. In [22] Thurston indicates that each solution tuple τ is in fact realized by a family of sphere triangulations, each family being parametrized by the points of positive square-norm in an Eisenstein lattice $\Lambda(\tau)$ in the complex Lorentz space $\mathbb{C}^{1,k-3}$ with $k = k_1 + k_2 + k_3 + k_4 + k_5 > 3$. For each solution tuple, there is a group of automorphisms, $\Gamma(\tau)$, acting on the lattice $\Lambda(\tau)$, whose orbits parametrize non-negatively curved triangulations in a one-to-one manner. This group acts discretely on the related $k-3$-dimensional complex hyperbolic space $\mathbf{C}\mathbb{H}^{k-3}$; is of finite co-volume; and also appears in Picard-Terada-Deligne-Mostow list. The quotient space, $\mathscr{M}(\tau)$, is thus a complex ball-quotient of dimension $k-3$. Thurston proves these claims in detail for the solution tuple $(0,0,0,0,12)$ which yields the top dimensional (generic) case. The related ball-quotient has the moduli space of 12 points on the Riemann sphere as its underlying complex manifold. The lattice $\Lambda(0,0,0,0,12)$ will be denoted by Λ in the sequel. (see Theorem 4 for a restatement of Thurston's result).

In the paper cited above, Thurston constructs a more populous list of discrete groups acting on complex hyperbolic spaces, then those that classify triangulations. Although this list contains the list obtained in the paper by Deligne and Mostow on hypergeometric functions in several variables, the two papers are technically disjoint and our first task in the current work is to provide a link, by showing that Thurston's space of cocyles is isomorphic to the cohomology group of an associated locally constant sheaf on the sphere in Deligne-Mostow's paper (Theorem 3).

Our second aim in this paper is to carry out an analogous classification for sphere quadrangulations (Theorem 5). A vertex of a quadrangulation is *non-negatively curved* if there are at most *four* quadrangles meeting at this vertex. If all vertices of a quadrangulation are non-negatively curved, then the quadrangulation itself is said to be non-negatively curved. Let $\mathscr{Q}$ be a quadrangulation of X and let k_i be the number of vertices of $\mathscr{Q}$ adjacent to i quadrangles. A basic counting argument yields the formula

$$4\chi(X) = \sum_{i=1}^{\infty} (4-i)k_i \tag{3}$$

Observe that the coefficient of k_4 is zero in the above formula, so that vertices with exactly *four* adjacent quadrangulation have no effect on $\chi(X)$.

Assume now that $\mathscr{Q}$ is of non-negative curvature. This means that $k_i = 0$ unless $i \leq 4$ in Equation 3, which gives $\chi(X) \geq 0$. So X must be a torus if $k_i = 0$ for $i \neq 4$. In case there is a vertex of positive curvature; that is if at least one among k_1, k_2 and k_3

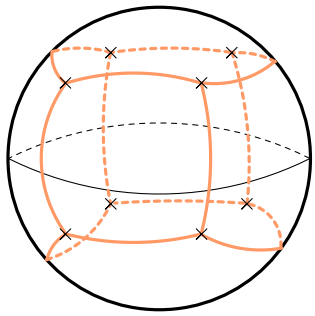


Fig. 1 The cube as a quadrangulation of S^2.

is non-zero, then the formula yields $\chi(X) > 0$. Therefore X must be the sphere with $\chi(X) = 2$ and the formula becomes

$$8 = \sum_{i=1}^{3} (4-i)k_i = 3k_1 + 2k_2 + k_3 \tag{4}$$

One can easily write down a complete list of tuples $\tau' = (k_1, k_2, k_3)$ satisfying this equation, which are finite in number. Our aim is to show that each solution tuple τ' is in fact realized by a family of sphere quadrangulations, each family being parametrized by the points of positive square-norm in a *Gaußian* lattice $\Lambda'(\tau')$ in the complex Lorentz space $\mathbb{C}^{1,k-3}$ with $k = k_1 + k_2 + k_3 > 3$. For each solution tuple, there is a group of automorphisms, $\Gamma'(\tau')$, acting on the lattice $\Lambda'(\tau')$, whose orbits parametrize quadrangulations in a one-to-one manner. This group acts discretely on the related complex hyperbolic space $\mathbf{C}\mathbb{H}^{k-3}$; is of finite co-volume; and also appears in Picard-Terada-Deligne-Mostow-Thurston list. The quotient space, $\mathscr{M}'(\tau')$, is thus a complex ball-quotient of dimension $k-3$. In the current paper we prove these claims for the solution tuple $(0,0,8)$ which yields the top dimensional (generic) case. The related ball-quotient has the moduli space of 8 points on the Riemann sphere as its underlying complex manifold. The lattice $\Lambda'(0,0,8)$ will be denoted by Λ' in the sequel. We intend to prove the remaining cases in a forthcoming paper, along with the above-mentioned claims on triangulations which are not included in Thurston's paper.

In the final part of the paper, we observe that a triangulation (or a quadrangulation) is nothing but a three-point branched covering of the sphere. This covering is determined by the embedded graph (dessin) dual to the triangulation. In this vein we may view the combinatorial Gauss-Bonnet formulas (1) and (3) as instances of the Riemann-Hurwitz formula, wherein $X \to \mathbf{P}^1$ is ramified over 0 with constant ramification index 2, over 1 with constant ramification index 3 (4 for quadrangulations) and over ∞ with ramification index i precisely at k_i points. Thereby the lattices clas-

sifying triangulations (or quadrangulations) of non-negative curvature parametrize also a family of three-point branched coverings of the sphere. Equivalently, they parametrize a family of subgroups up to conjugacy of the modular group (subgroups of the 2-4-∞ triangle group in the case of quadrangulations). Even though they are seldom congruence subgroups, there are reasons to believe that they are special and amenable to study. The second author, [27], have succeeded in determining the Belyĭ maps of a very special subfamily of triangulations, here we determine, in terms of Weierstrass' elliptic functions, the Belyĭ maps of an analogous family of quadrangulations. It seems promising to study the Galois action on this family of lattice-parametrized dessins.

We also discuss connections with Teichmüller discs along with some related arithmeticity questions and conjectures towards the end of the paper. Finally, we would like to remark that these ideas not only may pave us the road in understanding certain conjectures concerning the monster, [1], but also provides us a way to construct certain lattices in a combinatorial way, and study them likewise, [2].

2 From Cone Manifolds to Triangulations

We begin with recalling fundamental facts to be used in what follows. The terminology is borrowed from [23]. Equip the set

$$V_\theta := \{(r,t) \,|\, r \in \mathbf{R}_{\geq 0}, t \in \mathbf{R}/\theta\mathbf{Z}\}.$$

with the metric $ds_\theta^2 = dr^2 + r^2 dt^2$.

Definition 1 For X an orientable topological surface and S a finite set of points on X, we will say a metric, c, on X is a *Euclidean cone metric*, whenever every element $x \in X - S$ has an open neighbourhood, U_x, isometric to $\mathbb{E}^2 := (\mathbf{C}, ds^2 = |dz|^2)$, and every element $p \in S$ has a neighbourhood, U_p such that there is an isometry, φ_p, between U_p and V_θ with $\varphi_p(p) = (0,0)$.

We will write cone metric for short instead of Euclidean cone metric. Elements of the set $S = S_c$ will be called *singular points* and the elements of $X - S_c$ will be called *regular points*. We will call the pair (X,c) a Euclidean cone manifold of dimension 2, or a cone surface, in short. The real number $\beta = \frac{\theta}{2\pi} - 1$ is referred to as the *residue*. Notice that, when $\theta = 2\pi$ our local model is nothing but $\mathbf{C}$ with its flat metric. We will call $\kappa = 2\pi - \theta$ the *concentrated curvature* at $(0,0)$.

Let (X,c) be a cone surface. Regardless of a point p on X being regular or singular around p one always has a local analytic chart around p, i.e a cone metric induces a complex structure on the surface X. In fact, any given Riemannian metric on $X_{g,N}$ induces a conformal structure, and the conformal structure induced by the metric is the same as the conformal structure induced by its arbitrary positive function multiples.

Let us recall:

Theorem 1 (Singular Gauß-Bonnet, [23, Proposition 3]) *Let X be a cone surface where the points $p_1, \ldots, p_N \in X$ are singular with concentrated curvatures $\kappa_1, \ldots, \kappa_N$,*

respectively. Then:

$$\sum_{i=1}^{N} \kappa_i = 2\pi\chi(X);$$

where $\chi(X)$ is the topological Euler characteristic of X.

And, in fact, this is the only restriction, known as the *Gauss - Bonnet restriction*. In other words, we have the following:

Theorem 2 ([23, §5, Théorème]) *Let X be as above, $p_1, \cdots, p_N$ are points in X and $\kappa_1, \ldots, \kappa_N$ are rational numbers such that*

$$\sum_{i=1}^{N} \kappa_i = 2\pi\chi(X)$$

Then, X admits a cone metric, c, with concentrated curvature $\kappa_i = 2\pi - \theta_i$, at the point p_i, $i = 1, \ldots, N$. Moreover, this metric is unique up to normalization.

Definition 2 *A polygonal decomposition* $\mathscr{P}$ of a surface X is a finite set of subsets of X, U_i together with homeomorphisms, $f_i \colon U_i \longrightarrow P_i$, where P_i is a polygon in $\mathbf{R}^2$ such that:

i. $\bigcup_i U_i = X$, and
ii. whenever $U_i \cap U_j \neq \emptyset$, for $i \neq j$, then the intersection is a subset of the union of set of edges, $e(\mathscr{P})$, of $\mathscr{P}$ and the set of vertices, $v(\mathscr{P})$ of $\mathscr{P}$;

where we define the **set of vertices**, **edges** of $\mathscr{P}$, to be the set of inverse images under f_i of all vertices, edges, of the polygons P_i. We define the **set of faces**, $f(\mathscr{P})$, to be the set $\{U_i\}$.

In particular, a triangulation is a polygonal decomposition in which every face is a triangle. Now let (X, c) be a cone surface. If, a polygonal decomposition of X satisfies the following two properties, then it is called a *Euclidean polygonal decomposition* of (X, c):

iii. P_is are subspaces of $\mathbb{E}^2$ with f_is being isometries and
iv. for every pair of distinct faces $U_i \cap U_j$ which intersect in an edge, $e \in e(\mathscr{P})$, there exists an element, $\gamma_{i,j}$ in the group of isometries of the Euclidean plane, $\mathrm{Isom}(\mathbb{E}^2)$, such that $\gamma_{ij}\big(f_i(e)\big) = f_j(e)$.

Let $\mathscr{T}$ be a finite metric triangulation of $X = S^2$. Let $p \in v(\mathscr{T})$ be a vertex at which the faces $T_1 = U_{i_1}, \ldots, T_n = U_{i_n}$ meet. The curvature concentrated at p is defined as:

$$2\pi - \sum_{k=1}^{n} \alpha_k;$$

where α_k is the angle at the point p inside the triangle $f_{i_k}(T_i)$, for $i \in \{1, \ldots, n\}$ see Figure 2. Then, Theorem 2 provides us a cone metric, say $c_{\mathscr{T}}$, associated to $\mathscr{T}$.

Conversely, let c be a euclidean cone metric. Fix any singular point $p_1 \in S_c$, and order the elements of S_c with respect to their distance to p_1. By re-indexing if necessary, we may assume that the distance of p_1 to p_i is less than or equal to p_j if and only

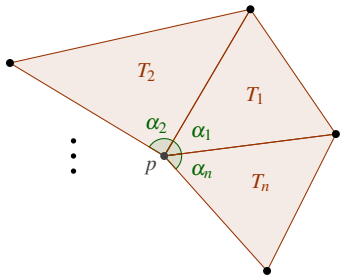


Fig. 2 Property iv. allows us to glue euclidean triangles.

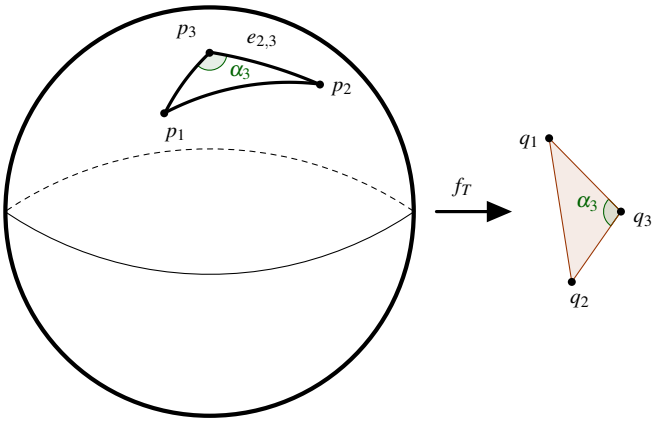


Fig. 3 A euclidean triangle in (S^2, c) determines a euclidean triangle in $\mathbb{E}^2$

if $i \leq j$. Let T be the geodesic triangle with vertices at p_1, p_2, p_3; $e_{i,j}$ be the geodesic between p_i and p_j; α_j denote the angle between $e_{i,j}$ and $e_{j,k}$; where $i, j \in \{1,2,3\}$.

Observe that edges, $e_{i,j}$ must have empty intersection with the set $S_c \setminus \{p_1, p_2, p_3\}$. The triangle T determines, in $\mathbb{E}^2$, a geodesic triangle with the property that the angle at $q_i = f_T(p_i)$ is exactly α_i, and the length of the edge $f_{i,j} = f_T(e_{i,j})$ is equal to that of $e_{i,j}$; where $f_T : T \longrightarrow \mathbb{E}^2$ is the induced isometry in between, see Figure 3.

More generally, fix an element $p_1 \in S_c$ and enumerate the remaining singular points so that there is a continuous path, $\gamma : [0,1] \longrightarrow S^2$, joining p_1 to p_N with the following properties:

- there is a sequence of numbers $t_1 = 0 < t_2 < \ldots < 1 = t_N$ satisfying $\gamma(t_i) = p_i$, $i = 1, 2, \ldots, N$,
- $\gamma|_{[t_i, t_{i+1}]}$ is a geodesic with respect to c, for $i = 1, \ldots, N-1$,
- γ is one-to-one.

Proposition 1 ([19, Theorem 10.1], [22, Proposition 3.1]) *Let c be a cone metric on S^2. Then c induces a geodesic metric triangulation denoted by $\mathscr{T}_c$, on S^2 with the property that the set of vertices of $\mathscr{T}_c$ is exactly the set of singular points, S_c, of c.*

Proof Cut the sphere open along γ. where γ is as above. Since all the singular points of c are along $\gamma([0,1])$, one can write a map, ψ, from S^2 to a polygon, P in $\mathbb{E}^2$ so that ψ is an isometry when restricted to $S^2 \setminus \gamma$. Moreover, ψ maps any geodesic $\gamma|_{[t_i,t_{i+1}]}$ to an edge of P, for all $i = 1,\ldots,n-1$, and in fact twice. Notice also that the polygon P is uniquely determined, up to $\mathrm{Isom}(\mathbb{E}^2)$. In order to obtain a euclidean triangulation on S^2, it is enough to draw the necessary diagonals of P. As every diagonal is a geodesic in $\mathbb{E}^2$, we obtain a metric triangulation.

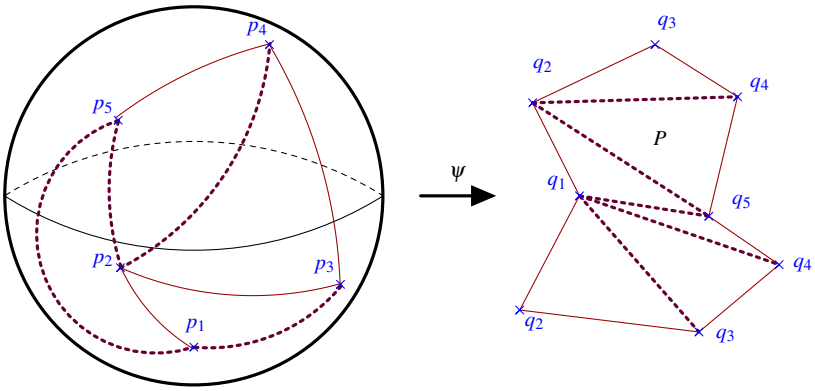


Fig. 4 From a cone metric to a geodesic triangulation

One has to note however that geodesic metric triangulation associated to a cone metric is **not** unique, even if scertain normalizations are chosen in the very beginning. Nevertheless, there are finitely many such choices, up to orientation preserving similarity. One might call triangulations arising from Proposition 1 *minimal*.

2.1 Universal Branched Cover of a Cone Surface

Before we proceed, we to introduce some notation: for the curvature parameters $\kappa = (\kappa_1,\ldots,\kappa_N)$ by $C(\kappa) = C(\kappa_1,\kappa_2,\ldots,\kappa_N)$, we will denote the set of all cone metrics which has N singular points with concentrated curvatures $\kappa_i \neq 0$, $i = 1,2,\ldots,N$; $N \geq 3$, up to *orientation preserving similarity*. Let $c \in C(\kappa)$ be a cone metric, with an induced triangulation $\mathscr{T}_c$ on S^2, see Proposition 1.

Definition 3 Let $\gamma : [0,1] \longrightarrow S^2$ be a piecewise smooth path in S^2, such that $\gamma\big((0,1)\big) \subset S^2 \setminus S_c$. We will say that γ is **admissible** if $\gamma([0,1])$ intersects the edges, $e(\mathscr{T}_c)$, of $\mathscr{T}_c$ finitely many times. We will call a homotopy $\gamma_t(s)$, $t,s \in [0,1]$, of piecewise smooth paths γ,γ' with $\gamma_0 = \gamma$, and $\gamma_1 = \gamma'$ **admissible** if γ_t is an admissible path for every $t \in [0,1]$.

As in the classical case of fundamental groups, we define two admissible curves to be *homotopic* if and only if there is an admissible homotopy from one to another.

If one fixes a base point in $S^2 \setminus S_c$, the set of homotopy classes of admissible paths form a group, which is isomorphic to the fundamental group of $S^2 \setminus S_c$, as a result of the following lemma:

Lemma 1 *Let γ: $[0,1] \longrightarrow S^2$ be a continuous, piecewise differentiable path, with $\gamma(0,1) \subset S^2 \setminus S_c$. Assume further that γ is not admissible, i.e. there is an edge $e \in e(\mathscr{T}_c)$ which intersects γ infinitely many times. Then the homotopy class, $[\gamma]$, of γ contains an admissible path.*

Proof Suppose that γ intersects $e_o \in e(\mathscr{T}_c)$ infinitely many times. We exclude the case when γ, or a part of γ follows a portion of e, as in that case we may perturb γ so that it intersects e in only two points. There is, then, an increasing sequence, r_n, of elements of $(0,1)$, not necessarily non-constant, with the property that $e_o \cap \gamma(0,1) = \{\gamma(r_n) \,|\, n \in \mathbf{N}\}$. One can find a sufficiently large $M \in \mathbf{N}$ such that for every $n > M$ the restriction of the path γ to the closed interval $[r_n, r_{n+1}]$ is homotopic to the path that follows e with initial point $\gamma(r_n) \in e$ and terminal point $\gamma(r_{n+1}) \in e$. So, γ is homotopic to the path that follows e from γ_{r_M} to γ_{r_∞}; where γ_∞ denotes the limit of the sequence $(\gamma(r_k))_{k\in\mathbf{N}}$. As noted above, this last path is homotopic to a path which intersects e in only two points.

Corollary 1 *The group of homotopy classes of admissible paths is independent of the chosen triangulation.*

Now, regard $\mathscr{T}_c$ as a simplicial complex on S^2 and fix a base point $p_I \in S^2 \setminus \big(S_c \cup e(\mathscr{T}_c)\big)$. By $\widehat{\mathscr{T}_c}$ denote the set of all pairs $(\sigma, [\gamma])$; where σ is a $0,1$ or 2-simplex of $\mathscr{T}_c$, and $[\gamma]$ is *the* admissible homotopy class of an admissible curve $\gamma : [0,1] \longrightarrow \mathbf{P}^1$ which connects p_I to a point, call p_F, in σ. Note that $\widehat{\mathscr{T}_c}$ is by definition a simplicial complex. Let $\widehat{X}$ denote the geometric realization of $\widehat{\mathscr{T}_c}$. Note that $\widehat{X}$ comes together with a projection map:

$$\begin{aligned} \widehat{\pi} : \qquad \widehat{X} &\longrightarrow S^2 \\ (\sigma, [\gamma]) &\mapsto p_F = \gamma(1) \end{aligned}$$

Let us denote the set $S^2 \setminus S_c$ by $\mathbf{P}_c$. We also assume that $\kappa_i \in \pi\mathbf{Q} \cap (0, 2\pi)$ and $N \geq 3$. As is well-known there is a torsion-free subgroup $\Gamma_c \leq \mathrm{PSL}_2(\mathbf{R})$ with cusps. By adding the set of cusps of Γ_c to the upper half plane $\mathbb{H}$, see [21, Chapter 1], we obtain a map $\widetilde{\pi} : \mathbb{H}_\flat := \mathbb{H} \cup \{\text{cusps of } \Gamma_c\} \longrightarrow \mathbf{P}^1$. Pull back the triangulation $\mathscr{T}_c$ by $\widetilde{\pi}$, to obtain a triangulation on $\mathbb{H}_\flat$, denoted by $\widetilde{\mathscr{T}_c}$. Then we have:

Proposition 2 *The geometric realization $\widehat{X}$ of $\widehat{\mathscr{T}_c}$ is nothing but $\mathbb{H}_\flat$.*

Proof Call the triangle, in $\mathscr{T}_c$ which contains p_I, the *base triangle* and denote it by T_I. Choose a point, $\widetilde{p}_I$, in the set $\pi^{-1}(p_I)$. Consider the map from $\widehat{\mathscr{T}_c}$ to $\widetilde{\mathscr{T}_c}$ described as follows: take an element $(\sigma, [\gamma]) \in \widehat{X}$, let $p_F \in \sigma$ denote the endpoint of γ. We may lift the path γ to a path in $\mathbb{H} \subset \mathbb{H}_\flat$ in a unique way as we chose already an initial

$$\begin{tikzcd} (\mathbb{H}_\flat, \widetilde{\mathscr{T}_c}) \arrow[rr, leftrightarrow] \arrow[dr, "\widetilde{\pi}"'] & & (\widehat{X}, \widehat{\mathscr{T}_c}) \arrow[dl, "\widehat{\pi}"] \\ & (\mathbf{P}^1, \mathscr{T}_c) & \end{tikzcd}$$

point, $\widetilde{p_I}$. Therefore the final point $\widetilde{p_F}$ is already determined, which we define to be the image of the pair $(\sigma, [\gamma])$.

For the inverse map, take any point $\widetilde{x}$ in $\mathbb{H}_\flat$, and any piecewise smooth path, $\gamma_{\widetilde{p_I},\widetilde{x}}$ from $\widetilde{p_I}$ to $\widetilde{x}$ so that it has empty intersection with $v(\widetilde{\mathscr{T}_c})$ for every $t \in (0,1)$. Then the path $\gamma_{p_I,x} = \pi(\gamma_{\widetilde{p_I},\widetilde{x}})$ is a path in $\mathbf{P}_c$, which does not pass through the vertices of $\mathscr{T}_c$ except possibly at endpoints. Then there is a 0, 1 or 2-simplex, σ, of $\mathscr{T}_c$ to which $x = \pi(\widetilde{x})$ belongs. Map this element to the pair $(\sigma, \gamma_{p_I,x})$. The map is easily seen to be well-defined.

Definition 4 The pair $(\mathbb{H}_\flat, \widetilde{\mathscr{T}_c})$, together with the locally flat metric obtained by lifting c is called the **universal branched cover** of $(\mathbf{P}^1, \mathscr{T}_c)$.

2.2 Two Representations

In this section, our aim is to compare two well-known representations of the fundamental group of $S^2 \setminus S_c$ for a cone metric c with curvature parameters $\kappa = (\kappa_1, \ldots, \kappa_N)$. More precisely, we will show that the holonomy representation factors through the monodromy representation.

Definition 5 We will call the image of $hol\,(\pi_1(\mathbf{P}_c, p_I))$ in $O(2)$, the orthogonal group, under the natural projection the **orthogonal part** of the holonomy representation and denote by h_o the composition, $h_o : \pi_1(\mathbf{P}_c, p_I) \longrightarrow O(2)$.

Take a vertex $p \in v(\mathscr{T}_c)$, i.e. a singular point $p \in S_c$. Let U_p be an open neighbourhood of p so that U_p contains no other singular point of c. Suppose that there are l triangles having p as a vertex with angles at p being $\alpha_1, \ldots, \alpha_l$, see Figure 5. If by θ_p we denote the cone angle at p, then we have

$$\theta_p = \sum_{i=1}^{l} \alpha_i.$$

Recall that the generating set for the fundamental group $\pi_1(\mathbf{P}_c, p_I)$ may be chosen as positively oriented simple closed curves that rotates once around every element of S_c. Let γ_p denote the positively oriented simple closed curve which rotates *once* around p. Without loss of generality we may assume that $p_I \in U_p$. It follows that, γ_p is a generator of the *local* fundamental group $\pi_1(U_p, p_I) \cong \mathbf{Z}$. Then, the pair (T_1, id) is send to $(T_1, \gamma_p) \in \mathbb{H}_\flat$. And hence, the element induced by γ_p, $h_{[\gamma_p]}$ is then nothing but rotation by an angle of θ_p, r_{θ_p}.

So, we proved:

Proposition 3 *The image of $h_o : \pi_1(\mathbf{P}_c, p_I) \longrightarrow O(2)$ is generated by rotations of angle θ_p, for $p \in S_c$.*

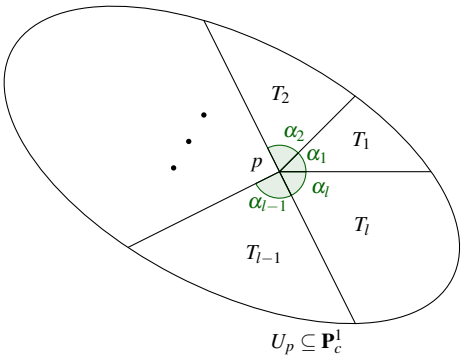


Fig. 5 Neighborhood of p

Now we are ready to prove the result:

Proposition 4 *The orthogonal part of the holonomy representation and the monodromy representation associated to c are isomorphic.*

Proof Consider $f : h_0(\pi_1(\mathbf{P}_c, p_I)) \longrightarrow \mathrm{GL}_1(\mathbf{C}) \cong \mathbf{C}^\times$ defined as $f(r_{\theta_p}) = e^{\theta_p\sqrt{-1}}$, for every $p \in S_c$. As generators are mapped to generators taking into account the orders, f is an isomorphism.

2.3 Combinatorics and Cohomology

In this section, we will begin with introducing a vector space, which is closely related to the space of cone metrics. This vector space is closely related to a certain cohomology of a locally constant sheaf. Throughout we fix the curvature parameters $\kappa = (\kappa_1, \ldots, \kappa_N) \in \pi \cdot \mathbf{Q}^N$, which are, assumed to be elements of the open interval $(0, 2\pi)$ with $N \geq 3$, and satisfy the Gauss-Bonnet condition, see Theorem 1.

2.3.1 Cone Metrics as Cocycles

For $c \in C(\kappa)$ the developing map, denoted by $\widetilde{\varphi}$, may be utilized to associate two complex numbers to each edge, namely the difference between the endpoints. Denote this association by $Z_c : e(\mathscr{T}_c) \times \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{C}$, where the group $\mathbf{Z}/2\mathbf{Z}$ is used for keeping track of the orientation. Observe the following two properties of Z_c:

i. $Z_c(e,+) + Z_c(e,-) = 0$, for every edge e of $\mathscr{T}_c$
ii. if $(e_1,+),(e_2,+),(e_3,+)$ denote the oriented boundary of some triangle in $\mathscr{T}_c$, then $\sum_i Z_c(e_i,+) = 0$.

These properties encourage us to call Z_c a *cocycle*. Depending on κ, such *cocycles* form a $\mathbf{C}$ vector space, say H_κ. Define the following *hermitian form* on H_κ:

$$A(c) := \frac{1}{4} \sum_{\text{triangles} \in \mathscr{T}_c} Z_c(e_1)\overline{Z_c(e_2)} - \overline{Z_c(e_1)}Z_c(e_2); \tag{5}$$

where e_is denote the positively oriented edges of each triangle. Note that the form A defined above is nothing but a measure of the area of a given cone metric on the sphere. We also have:

Proposition 5 ([22, Propositions 3.2, 3.3]) *The hermitian form A on the vector space H has signature $(1, N-3)$, where N is the number of singular points of c. In particular, the complex dimension of the vector space of cocycles is $N-2$.*

Before the proof, we would like to make:

Definition 6 For a cone metric $c \in C(\kappa)$, a subset $F \subseteq \mathbb{E}^2$ will be called a **euclidean (or flat) fundamental region**, whenever the followings hold:

- F is connected,
- the developing map $\widetilde{\varphi}_c$ has a well defined inverse when restricted to F,
- $\widetilde{\varphi}_c(\mathbf{P}^1_c) = \overline{F}$.

Proof [Proposition 5].

$$H \cong H^1(X_c, \mathbf{C})^{\chi} \tag{6}$$

where χ denotes the tautological character of the Galois group of the abelian cover of $\mathbf{P}_c$ ramified only over the singular points with *compatible* orders, see [15, §4] for details. In that case [5] tells us that it is of signature $(1, N-3)$.

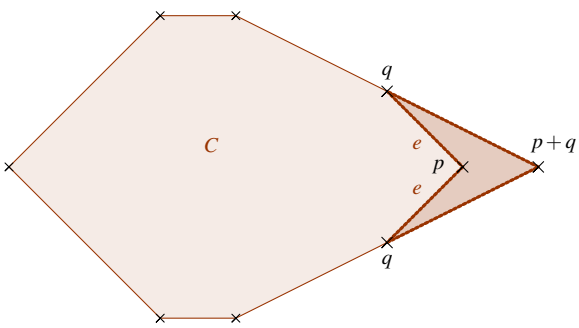


Fig. 6 Obtaining $(\mathbf{P}^1, c')$ from $(\mathbf{P}^1, c)$.

The projectivization of the set of elements in H_κ which are of positive norm with respect to a hermitian form of signature $(1, N-3)$ form a complex ball of dimension $N-3$, which can be regarded as a model for the complex hyperbolic space, $\mathbf{C}\mathbb{H}^{N-3}$, together with a negatively curved hermitian metric induced by the form A.

2.3.2 Cohomology and H_κ

Recall that on the Riemann surface $\mathbf{P}_c$ there exists a locally constant sheaf of rank one, say $\mathscr{F}_\kappa = \mathscr{F}_{\kappa_1,\ldots,\kappa_N}$, whose monodromy around the vertex $p_i \in S_c$ is rotation by θ_i, see [6] for details. In this setting the vector space H_κ has also the following nice interpretation proof of which relies on the facts coming after it:

Theorem 3 *The vector space H_κ of cocycles and $H^1(\mathbf{P}_c, \mathscr{F}_{\kappa_1,\ldots,\kappa_N})$ are isomorphic.*

For $c \in C(\kappa)$, by X_c let us denote the normalization of the plane algebraic curve given by affine equation:

$$y^\eta = \prod_{i=1}^{N-1} (x - p_i)^{\delta_i}; \tag{7}$$

where δ_is are chosen so as to satisfy $\frac{\kappa_i}{2\pi} = \frac{\delta_i}{\eta}$ with the property that the greatest common divisor of $\delta_1, \ldots \delta_N$ and η is 1. The projection map $(x,y) \mapsto x$ from X_c to $\mathbf{P}^1$ is then ramified precisely over the singular set S_c. Let $\mathrm{Gal}(pr_x)$ denote the Galois group of the covering $X_c \longrightarrow \mathbf{P}^1$ and let $\chi\colon \mathrm{Gal}(pr_x) \longrightarrow \mathbf{C}^\times$ denote the tautological character. In this setup χ acts on Ω_{X_c} and we have

$$H^1(X_c, \mathbf{C})^\chi \cong H^1(\mathbf{P}_c, \mathscr{F}_{\kappa_1,\ldots,\kappa_N}); \tag{8}$$

where $H^1(X_c, \mathbf{C})^\chi$ is the set of classes of 1-forms on X_c which are invariant under χ.

Lemma 2 **([7, Proposition 2.3.1])** $\dim_{\mathbf{C}}(H^1(\mathbf{P}_c, \mathscr{F}_{\kappa_1,\ldots,\kappa_N})) = N-2$.

On the other hand, χ acts on $\overline{\Omega_{X_c}}$ and we have a canonical identification

$$\overline{\Omega_{X_c}}^\chi \cong \Omega_{X_c}^{\overline{\chi}} = \Omega_{X_c}^{\chi^{\eta-1}}$$

via complex conjugation. To compute the dimension of $\Omega_{X_c}^{\chi^{\eta-1}}$ it is enough to note that only 1-forms of type $f(x)\frac{dx}{y}$ can be an eigenform; where $f(x) \in \mathbf{C}[x]$ of degree less than $N-2$.

Lemma 3 $\dim_{\mathbf{C}} \Omega_{X_c}^{\chi^{\eta-1}} = N-3$.

Proof We will prove our claim only for the case where $\delta_i = 1$ for each $i = 1, \ldots N-1$. The general case follows the same line of arguments, only somewhat more complicated. To prove, let us show that the set $\mathscr{B} := \{\frac{dx}{y}, x\frac{dx}{y}, \ldots, n^{N-2}\frac{dx}{y}\}$ is a basis. $\mathbf{C}$-linear independence of $\mathscr{B}$ is clear. Let $(x)_0 = D$, $(x)_\infty = D' = \sum_{j=1}^{\eta} q_j$ denote the zero and pole divisor of x, respectively. At any ramification point p_i of the projection pr_x, the function $x - p_i$ is locally of order η, hence dx is of order $\eta - 1$. On the other hand at each pole, q_j, of x, the function $x - q_j$ is locally of the form $\frac{1}{z_j}e^h$; where h is a holomorphic function. Thus $(dx)_\infty = 2D'$. The zero divisor of the function y is

nothing but the ramification divisor of pr_x, i.e. $(y)_0 = R = \sum_{i=1}^{N-1} p_i$. As $\deg y = N-1$ we must have $(y)_\infty = \frac{N-1}{\eta} D'$. So:

$$\left(x^k \frac{dx}{y^{\eta-1}}\right) = kD - kD' + (\eta-1)R - 2D' - (\eta-1)(R - \frac{N-1}{\eta} D')$$
$$= kD - \left((\eta-1)\frac{N-1}{\eta} - k - 2\right) D'.$$

So, we must have $k \geq 0$ and $k < N-3$. Hence the claim follows.

Theorem 3 can be considered not only as an explanation of the comment "This turns out to be closely related to work of Picard and Mostow and Deligne." made in [22], but also as a combinatorial description of some cohomology groups.

3 Quadrangulations of the Sphere as a Lattice

In this section, our aim is to obtain a classification of a family of quadrangulations of the sphere satisfying certain curvature conditions. That is: we will obtain a lattice in a specific space of cone metrics whose points parametrize quadrangulation s of non-negative curvature. The result we obtain is an analogue of [22, Theorem 0.1]. As before, we assume that our curvature parameters $\kappa_1, \ldots, \kappa_N$ are rational multiples of π lying in the open interval $(0, 2\pi)$. The result of Thurston and ours may be regarded as classification of certain subgroups of the modular group, $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ and of the group $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/4\mathbf{Z}$, respectively.

3.1 Quadrangulations...

As a particular case of Definition 2 let us choose each P_i as a quadrangle to obtain a quadrangulation. We fix X to be the 2-sphere. Just as in the case of a metric triangulation, there exists a flat metric on a given euclidean quadrangulation , which in turn induces a complex structure on the sphere, S^2. Hence, we are allowed to consider the sphere with a euclidean quadrangulation as the projective line $\mathbf{P}^1$.

Let $\mathcal{Q}$ be a sphere quadrangulation. Recall from the introduction that $\mathcal{Q}$ is said to be *non-negatively curved* whenever $\mathcal{Q}$ has no vertex at which more than four quadrangles meet. Analogous to Proposition 1 we have:

Lemma 4 *Let c be a cone metric on the sphere. Then, there is an associated metric quadrangulation of the sphere.*

Proof Suppose we are given an element $c \in C(\kappa)$. Let F_c denote the *euclidean fundamental region* corresponding to c, see Definition 6. Without loss of generality, we assume that the singular vertices, $S_c = \{p_1, \ldots, p_N\}$, appear on ∂F and the boundary segments connecting singular vertices are geodesics with respect to the cone metric c, hence they are, possibly broken, straight lines. We will use induction on the cardinality of S_c. If $N = 3$ then the euclidean fundamental region is itself a quadrangle. For the

general case, take 4 consecutive singular points, call p_1, p_2, p_3 and p_4 so that there are no other elements of the singular set on the path from p_1 to p_4 along the boundary of F, which is a $2(N-1)$-gon, see Figure 7. Note that possible identifications of the chosen vertices do not pose any problems, for we are only interested in the existence of a quadrangulation . We now connect p_1 to p_4 with a straight line to obtain the first quadrangle. The remaining is now a $2(N-2)$-gon, which, by induction assumption, can be divided into quadrangles finishing the proof.

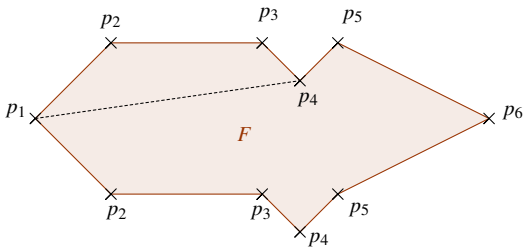


Fig. 7 Induction step for the case $N = 6$.

3.1.1 Shapes of Quadrangulations in $\mathbb{E}^2$

Let $\mathbf{Z}[\sqrt{-1}]$ be the ring of Gaußian integers considered as as subset of $\mathbb{E}^2$, or equivalently $\mathbf{C}$. In this section, we will analyze Gaußian lattice quadrangles whose sides are parallel to the sides of a standard quadrangle and whose vertices are at Gaußian integers, to which we will refer simply as a lattice quadrangle, see Figure 8 for an example.

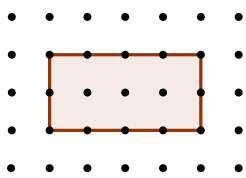


Fig. 8 A Lattice Quadrangle

Such an object is given by two parameters, the number of quadrangles in the vertical direction to which we will refer as n_1, and number of quadrangles in the horizontal direction to which we will refer as n_2. Moreover, a quadrangle having n_1 many vertical and n_2 many horizontal has $A(n_1,n_2) = n_1 n_2$ many quadrangles. In this coordinates, however, our area form is not diagonal with respect to this basis. There

is a geometric way of achieving this, see Figure 9. Given any n_1 and n_2 we consider the following area form:

$$A(n_1,n_2) := \frac{1}{4}\left((n_1+n_2)^2-(n_1-n_2)^2\right),$$

which measures the area of a lattice quadrangulation in terms of number of quadrangles. Note that the area form is of signature $(1,1)$. One may extend our definition to the case where n_1 are n_2 are real numbers. In that case, of course, the real parameters do not lead to a lattice quadrangulation. So one obtains an $\mathbf{R}$-vector space with a form of signature $(1,1)$. As $n_1,n_2 \geq 0$, forms a cone, say $\mathscr{C}$, the possible *shapes* of lattice quadrangulations are elements of the projective image of $\mathscr{C}$.

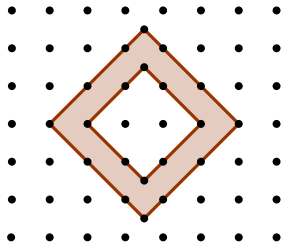


Fig. 9 Geometry of diagonalization of the area form($n_1 = 1$, $n_2 = 4$)

Remark 1 Possible shapes of lattice hexagons, i.e. hexagons whose vertices are at the Eisenstein integers, $\mathbf{Z}[e^{2\pi\sqrt{-1}/3}]$, sides are parallel to the sides of a standard hexagon are analyzed in [22, §1]. In the case when $\mathbf{R}$ is replaced by $\mathbf{C}$, the space that one obtains is a hermitian form on $\mathbf{C}^{1,1}$.

3.2 ...as a Lattice

In this section we will generalize the results of Section 3.1.1 to shapes of quadrangulations of the sphere. We are going to prove that quadrangulations of the sphere are given by a lattice inside a complex Lorentzian vector space. The complex hyperbolic space $\mathbf{C}\mathbb{H}^n$ is defined to be the set of all positive *lines* in the projectivization, $\mathbb{P}(\mathbf{C}^{1,n})$, of the vector space $\mathbf{C}^{1,n}$. One can give a complex manifold structure to $\mathbb{P}(\mathbf{C}^{1,n})$ if one considers the usual quotient map $\mathbf{C}^{1,n} \setminus \{\overrightarrow{0}\} \longrightarrow \mathbb{P}(\mathbf{C}^{1,n})$. Thus we may regard $\mathbf{C}\mathbb{H}^n$ as a complex manifold. Consider $\mathbf{C}^n$ with its usual Hermitian form:

$$\langle\langle z,w\rangle\rangle := \sum_{i=1}^{n} z_i\overline{w_i}.$$

By $\mathbb{B}^n$ denote the set of all elements, $z \in \mathbf{C}^n$, with $\langle\langle z,z\rangle\rangle < 1$. The map $\Xi\colon \mathbf{C}^n \longrightarrow \mathbf{C}\mathbb{H}^n$ as:

$$z = (z_1,\ldots,z_n) \mapsto [z_1 : \ldots : z_n : 1].$$

settles an embedding $\mathbf{C}^n$ onto the subset of $\mathbb{P}(\mathbf{C}^{1,n})$ defined by $Z_{n+1} \neq 0$ and we deduce $\mathbb{B}^n$ and $\mathbf{C}\mathbb{H}^n$ are complex analytically isomorphic.

Example 1 This situation has already appeared in Section 3.1.1 where it is proved that possible shapes of quadrangulations of lattice quadrangles in $\mathbb{E}^2$ are parametrized by a cone $\mathscr{C}$ inside the projectivization of $\mathbf{R}^{1,1}$. It is, in fact, an example of the above machinery except the base field was $\mathbf{R}$ instead of $\mathbf{C}$.

3.2.1 Non-negatively Curved Quadrangulations of the Sphere

A lattice, Λ, in a vector space V is a free $\mathbf{Z}$-module together with a symmetric bilinear form, $\langle \cdot, \cdot \rangle$. More generally, an Eisenstein(respectively Gaußian) lattice is a free $\mathbf{Z}[e^{2\pi\sqrt{-1}/3}]$-module(respectively $\mathbf{Z}[\sqrt{-1}]$-module) with a Hermitian form. Λ is called integral whenever the Hermitian form takes values in $\mathbf{Z}[e^{2\pi\sqrt{-1}/3}]$(respectively in $\mathbf{Z}[\sqrt{-1}]$).

We state now the following:

Theorem 4 ([22, Theorem 0.1]) *There is an integral **Eisenstein** lattice Λ in $\mathbf{C}^{1,9}$ such that $\Lambda_+/Aut(\Lambda)$ parametrizes non-negatively curved triangulations, i.e. at every vertex meets at most 6 triangles, of the sphere which have 5 triangles meeting at 12 marked vertices; where Λ_+ is the set of lattice points with positive square-norm, denoting the number of triangles in the triangulation. The quotient of $\mathbf{C}\mathbb{H}^9$ by the action of $Aut(\Lambda)$ has finite volume.*

Analogously, we have:

Theorem 5 *There is an integral **Gaußian** lattice, Λ' in $\mathbf{C}^{1,5}$ such that $\Lambda'_+/Aut(\Lambda')$ parametrizes non-negatively curved quadrangulations of the sphere having 3 quadrangles that meet at 8 marked vertices; where Λ'_+ is the set of lattice points with positive square-norm, which is the number of quadrangles in the quadrangulation. The quotient of $\mathbf{C}\mathbb{H}^5$ by the action of $Aut(\Lambda')$ has finite volume.*

Given a cone metric c, Lemma 4 provides us with a metric quadrangulation. And given a metric quadrangulation, $\mathscr{Q}$, of the sphere, for every quadrangle in $\mathscr{Q}$ we draw one of the two diagonals so as to obtain a triangulation, $\mathscr{T}_{\mathscr{Q}}$. There are $2^{|f(\mathscr{Q})|}$ distinct choices for $\mathscr{T}_{\mathscr{Q}}$; where $f(\mathscr{Q})$ denotes the set of faces of a quadrangulation. We, however, have:

Lemma 5 *$A(\mathscr{T}_{\mathscr{Q}})$ is independent of the choice of $\mathscr{T}_{\mathscr{Q}}$; where A is the hermitian form defined on the vector space of cocycles, see Equation 5 for the definition of A, and by abuse of notation we write $A(\mathscr{T}_{\mathscr{Q}})$ to denote the area of the cocycle associated to the metric triangulation $\mathscr{T}_{\mathscr{Q}}$.*

Proof It is enough to concentrate on one quadrangle. Let ω_1, ω_2, ω_3, ω_4 denote the edges and d_1, d_2 denote the two possible diagonals of a single quadrangle $q \in$

$f(\mathscr{Q})$, see Figure 10. Let us denote by A_i the value of the hermitian form obtained by subdividing q using d_i, $i = 1,2$ and write:

$$\begin{aligned}
A_1 - A_2 &= [\omega_1(\overline{-\omega_2}) - \overline{\omega_1}(-\omega_2) + \omega_4(\overline{-\omega_3}) - \overline{\omega_4}(-\omega_3)] - \\
&\quad [\omega_3(\overline{-\omega_1}) - \overline{\omega_3}(-\omega_1) + \omega_2(\overline{-\omega_4}) - \overline{\omega_2}(-\omega_4)] \\
&= -\omega_1\overline{\omega_2} + \overline{\omega_1}\omega_2 - \omega_4\overline{\omega_3} + \overline{\omega_4}\omega_3 + \\
&\quad \omega_3\overline{\omega_1} - \overline{\omega_3}\omega_1 + \omega_2\overline{\omega_4} - \overline{\omega_2}\omega_4 \\
&= -\omega_1(\overline{\omega_2} + \overline{\omega_3}) + \overline{\omega_4}(\omega_2 + \omega_3) + \overline{\omega_1}(\omega_2 + \omega_3) - \omega_4(\overline{\omega_2} + \overline{\omega_3}) \\
&= (\omega_2 + \omega_3)(\overline{\omega_1} + \overline{\omega_4}) - (\overline{\omega_2} + \overline{\omega_3})(\omega_1 + \omega_4) \\
&= -(\omega_1 + \omega_4)(\overline{\omega_1 + \omega_4}) + (\overline{\omega_1 + \omega_4})(\omega_1 + \omega_4), \text{ as } \sum_{i=1}^{4} \omega_i = 0. \\
&= 0.
\end{aligned}$$

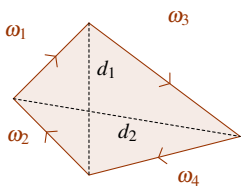


Fig. 10 A quadrangle, q, may be divided into two triangles using both d_1 and d_2

Proof (Theorem 5) Let us choose $\kappa_i = \pi/2$ for $i \in \{1,2,\ldots,8\}$ as curvatures, see Theorem 2. The vector space of cocycles, H_κ, associated to chosen curvature parameters has signature $(1, 8-3)$ by Proposition 5. We would like to note at this point that as a consequence of Lemma 5, we may and will write $A(\mathscr{Q})$ for the value of the hermitian form on a euclidean quadrangulation , instead of a euclidean triangulation. Now, let $\mathscr{Q}$ be a non-negatively curved quadrangulation of the sphere having 8 marked vertices at which exactly 3 quadrangles meet. To $\mathscr{Q}$ we associate the cone metric, $c_{\mathscr{Q}}$, on S^2 obtained by declaring that every quadrangle of $\mathscr{Q}$ is a unit square. The cocycle, $Z_{c_{\mathscr{Q}}}$, associated to $c_{\mathscr{Q}}$ is by its very definition an element of H. Moreover, as every $q \in f(\mathscr{Q})$ is a unit square, the difference between the endpoints of the edges under the developing map are naturally elements of $\mathbf{Z}[\sqrt{-1}]$. Let Λ' denote the set of all cocycles. Multiplying and $c \in \Lambda'$ by an element of $\mathbf{Z}[\sqrt{-1}]$ produces an element of Λ'. Finally, any two elements of Λ', say c_1 and c_2 gives us the following sum:

$$A(c_1, c_2) = \sum_i Z_{c_1}(e_i)\overline{Z_{c_2}(e_i)} - \overline{Z_{c_1}(e_i)}Z_{c_2}(e_i)$$

each of whose elements are in $\mathbf{Z}[\sqrt{-1}]$, hence the sum is an element of $\mathbf{Z}[\sqrt{-1}]$.

Proof (Theorem 4, Sketch) Following the same lines of the proof of Theorem 5, we choose $\kappa_i = \pi/3$, $i = 1, \ldots, 12$ as curvature parameters and consider the vector space of cocycles, H, associated to these parameters. For any given triangulation, $\mathscr{T}$, we declare that each triangle is equilateral of unit side length in order to obtain a euclidean triangulation. We then consider the associated cone metric, $c_{\mathscr{T}}$, which is by construction an element of H. The *Eisenstein* lattice, Λ, is comprised of all such triangulations inside H which is of signature $(1, 12-3)$.

Let us now concentrate on the lattice Λ. One has the following inclusion relations:

$$\begin{array}{ccc} \Lambda_+ & \subseteq & \mathbf{C}^{1,9} \\ \downarrow & & \downarrow \\ \mathbb{P}\Lambda_+ & \subseteq & \mathbb{P}\mathbf{C}^{1,9} \end{array}$$

Let now $\overline{Z}$ be a cocycle in $\mathbb{P}\Lambda_+$. Then the elements above $\overline{Z}$ may be obtained by subdivision, see Figure 11 for an example.

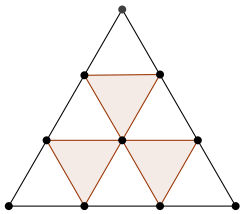


Fig. 11 Sub-dividing edges of a triangle

Remark 2 As in the case of quadrangulations of lattice quadrangles, the possible *shapes* of quadrangulations of the *sphere* is obtained via taking the quotient of the vectors of positive norm by the action of $\mathbf{C}^\times$ on H.

We end this section with two aspects of Theorem 4, and Theorem 5, both of which are related to the absolute Galois group, $\mathrm{Gal}(\mathbf{Q})$. The first one is that, by dualizing the triangulation, or quadrangulation, one obtains a bipartite graph on S^2. This way, each point of Λ_+ and Λ'_+ may be considered as a covering of the thrice punctured sphere, or a genus zero subgroup of $\mathrm{PSL}_2(\mathbf{R})$.

To demonstrate another aspect we make a little pause, and introduce origamis and Veech groups, see [20] or [14] for further details:

Definition 7 An origami is defined to be a finite set of Euclidean squares of side length one that are glued according to following set of rules:

i. every left edge is identified with a right edge(by a translation),

ii. every upper edge is identified with a lower one(by a translation),
iii. the closed surface obtained after the identifications is oriented and connected.

The simplest origami, which we call E^*, is obtained by considering only one unit square. The above rules leaves us no choice but to glue the upper edge with the lower one, and left edge with the right one. Hence, if we mark a vertex of the square, then every other vertex of the square has the same marking, and we get a punctured, or marked, torus, see Figure 12.

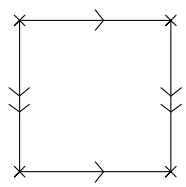


Fig. 12 The simplest origami, E^*

For an arbitrary origami, if one marks the vertices considering the identifications then, one gets a ramified covering of E^*, which is unramified away from the vertices. A surface together with a complex atlas whose every transition function is a translation is called a *translation surface*. If one identifies $\mathbb{E}^2$ with $\mathbf{C}$ then to every origami, one associated a translation surface, which becomes a Riemann surface under $\mathbb{E}^2 \cong \mathbf{C}$. For a translation surface, call X, we define the associated *affine group* as:

$$Aff(X) := \{\sigma \colon X \longrightarrow X \,|\, \sigma \text{ is an affine diffeomorphism preserving orientation}\}. \tag{9}$$

In other words, σ can be locally written as $Az+t$, for some $A \in \mathrm{GL}_2(\mathbf{R})$ and $t \in \mathbf{C}$. When X is of finite volume, the matrix $A \in \mathrm{SL}_2(\mathbf{R})$. Also, for any matrix $B \in \mathrm{SL}_2(\mathbf{R})$ one gets another Riemann surface structure, which is essentially the same structure whenever $B \in \mathrm{SO}_2(\mathbf{R})$. Hence the embedding $\mathrm{SL}_2(\mathbf{R}) \hookrightarrow \mathscr{T}_{g,N}$ factors through the quotient $\mathbb{H} \cong \mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R}) \hookrightarrow \mathscr{T}_{g,N}$; where X is a surface of genus g with N punctures and $\mathscr{T}_{g,N}$ stands for the Teichmüller space of genus g surfaces with N punctures. The embedding is an isometry with respect to the Poincaré metric on $\mathbb{H}$ and Teichmüller metric on $\mathscr{T}_{g,N}$, and the image is called a *Teichmüller disc*, which is geodesic, see [9]. In view of the above constructions, every point of Λ'_+ represents an origami hence we conclude that Λ'_+ parametrizes a certain family of Teichmüller discs in $\mathscr{T}_{g,N}$ for any g and N corresponding to curves having exactly 8 points at which meets 3 squares instead of 4.

4 Two Applications

As a consequence of the theory we have developed so far, lattices Λ_+ and Λ'_+ parametrize a family of dessins. On the other hand, every lattice point determines a shape param-

eter in the corresponding moduli space. We believe that these shape parameters are $\overline{\mathbf{Q}}$-rational points, see Conjecture 1.

4.1 Shape parameters on Moduli of Pointed Rational Curves

Let $c \in C(\kappa)$. If one labels the singular vertices, then one obtains a finite covering of $C(\kappa)$, call $P(\kappa)$, whose fundamental group is the pure braid group of the sphere on N strands. The group $\pi_1(C(\kappa))$ depends solely on the curvature parameters, κ_i.

On the other hand if we focus only on the complex structure, then we obtain a mapping from $C(\kappa)$ to the moduli space of smooth N-pointed rational curves, $\mathscr{M}_{0,N}$. However, for obvious reasons the mapping cannot be injective. Nevertheless the target is a finite cover of $\mathscr{M}_{0,N}$:

Theorem 6 ([22, Theorem 8.1]) *The map from $C(\kappa)$ to $\mathscr{M}_{0,N}$, denoted by $\mathfrak{S}$, described above is a homeomorphism. In particular, when $\kappa_i = \kappa_j$ for each $i, j \in \{1,\dots,N\}$ we have an isomorphism.*

There is an inverse to the map $\mathfrak{S}$, denoted by $\mathfrak{S}^{-1}$, explained in the proof of [22, Theorem 8.1]]. We, on the other hand, already know an inverse to $\mathfrak{S}$. Any element of $\mathscr{M}_{0,N}$ comes with a distinguished set of points which forms the singular set. The metric, which is unique up to normalization, is the provided by Theorem 2. For further details see [24].

Similarly, we have $\mathfrak{T}\colon C(\kappa_1,\dots,\kappa_N) \longrightarrow X(\kappa_1,\dots,\kappa_N)$; where $X(\kappa_1,\dots,\kappa_N)$ denotes a finite covering of $C_N\mathbf{P}^1$. The map sends every euclidean cone metric c to its singular set S_c. And as in the case of $\mathfrak{S}$, the degree of the covering depends on the curvature parameters. Summing up, we have:

$$\begin{tikzcd}
S(C(\kappa_1,\dots\kappa_N)) \arrow[d] & C(\kappa_1,\dots,\kappa_N) \arrow[l, "S"'] \arrow[r, bend left, "\mathfrak{S}"] & \mathfrak{S}(C(\kappa_1,\dots\kappa_N)) \arrow[l, bend left, "\mathfrak{S}^{-1}"] \arrow[d] \\
C_N\mathbf{P}^1/\mathrm{PGL}_2(\mathbf{C}) \arrow[dr, "\mathfrak{S}ch"'] & & \mathscr{M}_{0,N} \arrow[dl, "\cong"'] \\
& \mathbb{B}^{N-3}/\Delta(\kappa_1,\dots,\kappa_N) &
\end{tikzcd}$$

Fig. 13 Configuration spaces, moduli of cone metrics and pointed rational curves

Example 2 A classical case of the above phenomenon occurs when one considers the configuration space of 4 points on $\mathbf{P}^1$, in other words when one chooses the parameters as $\frac{2\pi}{2}, \frac{2\pi}{2}, \frac{2\pi}{2}, \frac{2\pi}{2}$. One obtains the following diagram:

$$\begin{array}{ccc} \mathbf{P}^1 \setminus \{0,1,\infty\} & \longrightarrow & \Delta_{\infty,\infty,\infty}\backslash\mathbb{H} \\ \downarrow & & \downarrow \\ C_4\mathbf{P}^1 & \longrightarrow & \mathrm{PSL}_2(\mathbf{Z})\backslash\mathbb{H} = \mathscr{M}_1 \end{array}$$

One, on the other hand, has a natural quadrangulation of each such configuration consisting of two quadrangles. As we did in the proof of Theorem 5, let us set each quadrangle to be a euclidean unit square. Then the curve in $\mathscr{M}_1$ one gets has the affine equation $y^2 = x^3 - x$, which is defined over $\mathbf{Q}$. As the map between $\mathscr{M}_1$ and $\mathscr{M}_{0,4}$ is algebraic the corresponding pointed rational curve is also defined over $\mathbf{Q}$. Alternatively, for the curve corresponding to the quadrangulation consisting of two unit squares glued from their boundary, one may take the fourth ramification point to be defined over $\mathbf{Z}[\sqrt{-1}]$, hence the Jacobian has complex multiplication, [17, Theorem 12.8], thus we get an algebraic point.

Conjecture 1 (Shape parameters are algebraic) The elements of $\mathbb{P}\Lambda_+$ and $\mathbb{P}\Lambda'_+$ are $\overline{\mathbf{Q}}$-rational points on $\mathscr{M}_{0,12}$ and $\mathscr{M}_{0,8}$, respectively.

4.2 Graphs on Surfaces

In this section we demonstrate an application of the lattice Λ' on embedded graphs (dessins of Grothendieck in [11]). An *embedded graph or a map* is a graph, Γ, embedded into a topological surface, X, i.e. a **closed**, **oriented**, two dimensional topological manifold so that (*) edges intersect only at vertices, and (**) each connected component of $X \setminus \{\text{image of } \Gamma\}$ is homeomorphic to a disc. The embedding of the graph into X is denoted by ι.

It is common to regard graphs as cell complex comprised only of 0 and 1 cells, and hence embedded graphs as an *injection* $\iota : \Gamma \longrightarrow X$ satisfying $X \setminus \iota(\Gamma)$ is a union of open sets each of which is homeomorphic to a disc. Each connected component of $X \setminus \iota(\Gamma)$ is called a **face** of Γ.

We have the following well-known result.

Theorem 7 ([3]) *An algebraic curve X may be defined over the field of algebraic numbers, $\overline{\mathbf{Q}}$, if and only if X admits a meromorphic function (or a Belyĭ morphism), $f : X \longrightarrow \overline{\mathbb{C}}$, ramified at most over* 3 *points which may be chosen to be* 0, 1 *and* ∞.

So, given an arithmetic curve, X, one has a corresponding Belyĭ morphism, β, and inverse image of the closed real interval $[0,1] \subset \mathbf{P}^1$ is an embedded graph on X, and vice versa, [25, Proposition 1]. One further has the following:

Theorem 8 *The following categories are equivalent:*

- *finite topological covers of* $\mathbf{P}^1_{\mathbf{C}} \setminus \{0,1,\infty\}$,
- *finite connected étale covers of* $\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0,1,\infty\}$,
- *finite sets equipped with the action of* $\pi_1^{alg}(\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0,1,\infty\})$,
- *subgroups of* $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus \{0,1,\infty\})$ *up to conjugation,*

– *embedded graphs*

Observe that since X is oriented, around each vertex of Γ there is a canonical orientation of the edges of Γ coming out of this vertex. Keeping in mind these observations we define two embedded graphs to be equivalent if there is a map between vertices and edges respecting orientation.

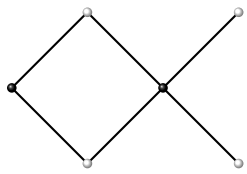


Fig. 14 A Graph Embedded in the Riemann Sphere

4.2.1 Division values of Elliptic Functions and Belyĭ Morphisms

There is a natural family of curves, say Y_n, each of which is defined over a number field by Theorem 7, whose n^{th} element can be constructed as follows:

1. Take a unit Euclidean square,
2. Divide the edges of the square into n equal parts,
3. Connect the possible edges by *new* lines parallel to edges of the square, call the resulting square Q_n,
4. Mark the midpoints of the squares with a black vertex, and connect the black vertices lying in neighboring squares,
5. Put a white vertex at every point where lines connecting black vertices and *new* lines intersects,
6. Identify the top edges with bottom and left edge with right to get a torus, see Figure 15.
7. Use the inclusion relation between $\mathbf{Z}[\sqrt{-1}]$ and $\Delta_{2,4,4}$ to project down to $\mathbf{P}^1$, see Figure 16 for a geometric description, and obtain Y_n.

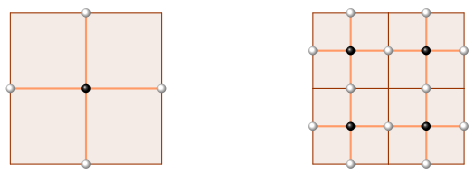


Fig. 15 First two tori with embedded graphs.

The embedded graph defining the curve Y_n will be referred to as Γ_n. Figure 17 displays the curve Y_3 together with Γ_3. To every Γ_n, one may associate a quadrangulation of the sphere by connecting the midpoint of each face by the white vertices lying on the boundary of the face. Observe that such a quadrangulation is an element of the compactification of the space in which the lattice Λ' found in Theorem 5 lies.

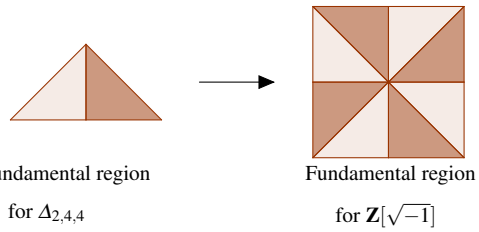


Fig. 16 Geometric description of the natural projection between $\Delta_{2,4,4}$ and $\mathbf{Z}[\sqrt{-1}]$

The computation uses the following commutative diagram: where the functions

$$\begin{array}{ccc} \mathbf{Z}[\sqrt{-1}]\backslash\mathbf{C} & \xrightarrow{m_n} & \mathbf{Z}[\sqrt{-1}]\backslash\mathbf{C} \\ \downarrow & & \downarrow \\ H_n\backslash\mathbf{C} & \longrightarrow & \Delta(2,4,4)\backslash\mathbf{C} \xrightarrow{\frac{\eta_1}{\eta_2}} \mathbf{P}^1 \end{array}$$

η_1 and η_2 refers to the solutions of the hypergeometric differential equation for $\Delta_{2,4,4}$, H_i corresponds to the subgroup of $\mathrm{PSL}_2(\mathbf{R})$ making the square commutative, m_i refers to the multiplication by i self-morphism of the elliptic curve $\mathbf{Z}\sqrt{-1}\backslash\mathbf{C}$, which has Weierstraß form $y^2 = 4x^3 - x$.

The corresponding Belyĭ morphisms in this case are composition of the arrows on the bottom. However, we know the ramification points are the i-division values of a particular elliptic function, where by an i-division value we mean the value of an elliptic function at points $x \in \mathbf{Z}\sqrt{-1}\backslash\mathbf{C}$ so that $x \notin \mathbf{Z}[\sqrt{-1}]$ however $n \cdot x \in \mathbf{Z}[\sqrt{-1}]$. Our aim is thus to find the ramification points of the Belyĭ morphism. For our purposes it is enough to consider the elliptic function, $w = \varepsilon(z)$,

$$\begin{aligned} z \mapsto w &= \frac{1}{(\wp(\omega_3) - \wp(\omega_1))\,(\wp(\omega_3) - \wp(\omega_1))}\,(\wp(z) - \wp(\omega_1))\,(\wp(z) - \wp(\omega_2)) \\ &= -4\wp^2(z) + 1, \end{aligned}$$

where ω_1 is the real and ω_2 is the purely imaginary period of $y^2 = 4x^3 - x$, and $\omega_3 = \frac{1}{2}(\omega_1 + \omega_2)$. The last equality is a result of the fact that $\wp(\omega_1) = \frac{1}{2} = -\wp(\omega_2)$,

which implies $\wp(\omega_3) = 0$. Then the Belyĭ morphism corresponding to Y_n, which we call the **Gauss-Chebyshev** function, has the following general form (up to a constant)

$$g_n(w) := c_n \frac{\prod_{z\in \text{ white vertices}} (w - \varepsilon(z)^{\text{ord}(z)})}{\prod_{z\in \text{ poles}} (w - \varepsilon(z))^{\text{ord}(z)}}$$

where by poles we mean midpoints of faces, and by order the valency of corresponding vertex or face, and c_n is a constant which will be described in Example 3.

Example 3 We would like to demonstrate the case $n = 3$, whose dessin can be found in Figure 17. The list of ramification points may be found in Table 2. Thus, g_3 is:

$$c_3 \frac{\left[\prod_{p\in P_3}(w-\varepsilon(p))\prod_{q\in Q_3}(w-\varepsilon(q))^2\right]}{(w-\varepsilon(0))(w-\varepsilon(\frac{2\omega_1}{3}))^2(w-\varepsilon(\frac{2\omega_3}{3}))^2(w-\varepsilon(\frac{4\omega_1}{3}))^2}$$

where c_3 is the normalization constant and $P_3 = \{\frac{\omega_1}{3}, \omega_1, \frac{5\omega_1}{3}\}$, $Q_3 := \{\frac{2\omega_1}{3}) + \frac{\omega_2}{3}, \omega_1 + \frac{2\omega_2}{3}, \frac{4\omega_1}{3}) + \frac{\omega_2}{3}\}$. As 1 is a ramification value, we choose $c_3 = \frac{1}{g_3(\omega_3)/3}$, and in general, $c_n = \frac{1}{g_n(\omega_3/n)}$.

The well-known formula

$$\wp(z+z') = \frac{1}{4}\left[\frac{\wp'(z)-\wp'(z')}{\wp(z)-\wp(z')}\right] - \wp(z) - \wp(z')$$

together with the fact that $\wp(\omega_1), \wp(\omega_2) \in \mathbf{Q}$ implies that for every n the values of ε are algebraic. However as n assumes larger values the degree of the algebraic number gets larger, too. Numerical data for the ramification points of g_3 may be found in Table 2.

white vertices (inverse image of 0)	black vertices (inverse image of 1)	poles (inverse image of ∞)
$\frac{1}{3}\omega_1$	$\frac{1}{3}\omega_3$	0
$\frac{2}{3}\omega_1 + \frac{1}{3}\omega_2$	$\omega_1 + \frac{1}{3}\omega_2$	$\frac{2}{3}\omega_1$
ω_1	ω_3	$\frac{2}{3}\omega_3$
$\omega_1 + \frac{2}{3}\omega_2$	$\frac{5}{3}\omega_1 + \frac{1}{3}\omega_2$	$\frac{4}{3}\omega_1$
$\frac{4}{3}\omega_1 + \frac{1}{3}\omega_2$		$\frac{4}{3}\omega_1 + \frac{2}{3}\omega_2$
$\frac{5}{3}\omega_1$		$2\omega_1 = 0 \bmod \mathbf{Z}[\sqrt{-1}]$

Table 1 Points on E whose values give ramification data of g_3

Remark 3 A similar family for the lattice Λ appeared in Theorem 4 may be defined. The description of the family and corresponding calculations of Belyĭ morphisms as well as an application to curves of higher genera may be found in [27].

Conjecture 2 The Galois action on the dessin related to a lattice point and the Galois action on the corresponding shape parameter are compatible.

Acknowledgements We are grateful to Prof. Hurşit Önsiper for his guidance during the conception of this work. This work was supported by Tubitak grant 110T690 and Galatasaray University research grant 09.504.001 and was finalized during the first named author's stay in Max Planck Institute at Bonn.

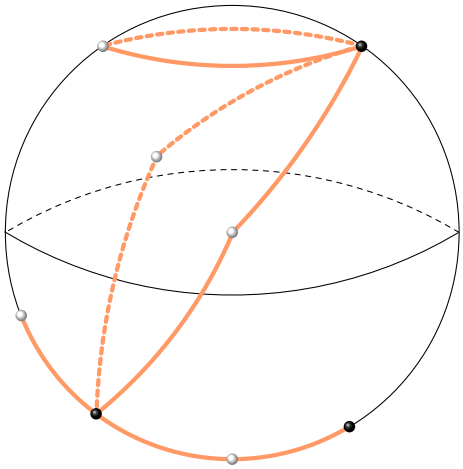


Fig. 17 The curve Y_3

zeros of g_3
$\varepsilon(\frac{1}{3}\omega_1) = -26.8204616940335$
$\varepsilon(\frac{2}{3}\omega_1 + \frac{1}{3}\omega_2) = 0.9282032302755 + 0.9974192818755\sqrt{-1}$
$\varepsilon(\omega_1) = 0$
$\varepsilon(\omega_1 + \frac{2}{3}\omega_2) = 0.9640552334825$
$\varepsilon(\frac{4}{3}\omega_1 + \frac{1}{3}\omega_2) = 0.9282032302755 - 0.9974192818755\sqrt{-1}$
$\varepsilon(\frac{5}{3}\omega_1) = -26.8204616940335$
poles of g_3
$\varepsilon(0) = \infty$
$\varepsilon(\frac{2}{3}\omega_1) = -1.15470053837925$
$\varepsilon(\frac{2}{3}\omega_3) = 1.15470053837925$
$\varepsilon(\frac{4}{3}\omega_1) = -1.15470053837925$
$\varepsilon(\frac{4}{3}\omega_1 + \frac{2}{3}\omega_2) = +1.15470053837925$

Table 2 Zeros and poles of g_3

References

1. Allcock, D.:A monstrous proposal. In: Groups and symmetries, pp. 17–24. CRM Proc. Lecture Notes (2009)
2. Allcock, D.: The Leech lattice and complex hyperbolic reflections. Invent. Math. **140**, 283–301 (2000)
3. Belyi, G.: On galois extensions of a maximal cyclotomic field. Math. USSR Izv. **14, No.2**, 247–256 (1980)
4. Birman, J.S.: Braids, links, and mapping class groups. Princeton University Press, Princeton, N.J. (1974). Annals of Mathematics Studies, No. 82
5. Chevalley, C., Weil, A., Hecke, E.: Über das verhalten der integrale 1. gattung bei automorphismen des funktionenkörpers. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **10**, 358–361 (1934). URL http://dx.doi.org/10.1007/BF02940687. 10.1007/BF02940687
6. Deligne, P.: Equations Différentielles à points Singuliers Réguliers. Lecture Notes in Mathematics. Springer-Verlag (1970)
7. Deligne, P., Mostow, G.D.: Monodromy of hypergeometric functions and nonlattice integral monodromy. Publications Mathématiques de l'IHES **63**, 5–89 (1986)

8. Fillastre, F.: From spaces of polygons to spaces of polyhedra following Bavard, Ghys and Thurston Enseign. Math. (2) **57**, 23–56 (2011)
9. Gardiner, F.P.: Teichmüller theory and quadratic differentials. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York (1987). A Wiley-Interscience Publication
10. Goldman, W.M.: Complex hyperbolic geometry. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York (1999). Oxford Science Publications
11. Grothendieck, A.: Esquisse d'un programme. In: Geometric Galois Actions, 1. Around Grothendieck's Esquisse d'un Programme, pp. 5–48. Cambridge University Press (1997)
12. Hurwitz, A., Courant, R.: Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer (1929)
13. Ihara, Y.: Braids, Galois groups, and some arithmetic functions. In: Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), pp. 99–120. Math. Soc. Japan, Tokyo (1991)
14. Lochak, P.: On arithmetic curves in the moduli spaces of curves. J. Inst. Math. Jussieu **4**(3), 443–508 (2005). DOI 10.1017/S1474748005000101. URL http://dx.doi.org/10.1017/S1474748005000101
15. Looijenga, E.: Uniformization by Lauricella functions—an overview of the theory of Deligne-Mostow. In: Arithmetic and geometry around hypergeometric functions, *Progr. Math.*, vol. 260, pp. 207–244. Birkhäuser, Basel (2007)
16. Milne, J.S.: Lectures on etale cohomology (v2.10) (2008). Available at www.jmilne.org/math/
17. Milne, J.S.: Modular functions and modular forms (v1.20) (2009). Available at www.jmilne.org/math/
18. Oort, F.: The algebraic fundamental group. In: Geometric Galois actions, 1, *London Math. Soc. Lecture Note Ser.*, vol. 242, pp. 67–83. Cambridge Univ. Press, Cambridge (1997)
19. Rivin, I.: Euclidean structures on simplicial surfaces and hyperbolic volume. Ann. of Math. (2) **139**(3), 553–580 (1994). DOI 10.2307/2118572. URL http://dx.doi.org/10.2307/2118572
20. Schmithüsen, G.: Veech groups of origamis. Ph.D. thesis, Universität Karlsruhe (TH) (2005)
21. Shimura, G.: Introduction to the arithmetic theory of automorphic functions, *Publications of the Mathematical Society of Japan*, vol. 11. Princeton University Press, Princeton, NJ (1994). Reprint of the 1971 original, Kanô Memorial Lectures, 1
22. Thurston, W.: Shapes of polyhedra and triangulations of the sphere. In: The Epstein Birthday Schrift, Geometry and Topology Monogrpahs: Volume 1, pp. 511–549. Geom. Topol. Publ., Coventry (1998)
23. Troyanov, M.: Les surfaces euclidiennes à singularités coniques. Enseign. Math. (2) **32**(1-2), 79–94 (1986)
24. Troyanov, M.: Prescribing curvature on compact surfaces with conical singularities. Trans. Amer. Math. Soc. **324**(2), 793–821 (1991). DOI 10.2307/2001742. URL http://dx.doi.org/10.2307/2001742
25. Wolfart, J.: Kinderzeichnungen und unifomisierungstheorie (2001). URL http://www.math.uni-frankfurt.de/ wolfart/Artikel/kizei.pdf
26. Yoshida, M.: Hypergeometric functions, my love. Modular interpretations of configuration spaces. Friedr. Vieweg and Sohn, Braunschweig (1997)
27. Zeytin, A.: An explicit method to write belyĭ morphisms (2010). URL http://arxiv.org/abs/1011.5644. Submitted