Progress in Mathematics

Volume 260

Arithmetic and Geometry Around Hypergeometric Functions

Lecture Notes of a CIMPA Summer School held at Galatasaray University, Istanbul, 2005

Rolf-Peter Holzapfel
A. Muhammed Uludağ
Masaaki Yoshida
Editors

Birkhäuser
Basel • Boston • Berlin

Editors:

Rolf-Peter Holzapfel
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin
e-mail: holzapfl@mathematik.hu-berlin.de

Masaaki Yoshida
Department of Mathematics
Kyushu University
Fukuoka 810-8560
Japan
e-mail: myoshida@math.kyushu-u.ac.jp

A. Muhammed Uludağ
Department of Mathematics
Galatasaray University
34357 Besiktas, Istanbul
Turkey
e-mail: muhammed.uludag@gmail.com

2000 Mathematics Subject Classification 14J10, 14J28, 14J15, 11F06, 11S80, 33C65, 22E40, 11F55, 11G15,11K22, 33C70, 32Q30, 33C05

Library of Congress Control Number : 2006939568

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 978-3-7643-8283-4 Birkhäuser Verlag AG, Basel – Boston – Berlin

ISBN-10: 3-7643-8283-X
ISBN-13: 978-3-7643-8283-4
e-ISBN-10: 3-7643-7449-7
e-ISBN-13: 978-3-7643-8284-1

9 8 7 6 5 4 3 2 1
www.birkhauser.ch

Preface

This volume comprises the Lecture Notes of the CIMPA Summer School *Arithmetic and Geometry around Hypergeometric Functions* held at Galatasaray University, Istanbul during June 13-25, 2005. In the Summer School there were fifteen lectures forming an impressive group of mathematicians covering a wide range of topics related to hypergeometric functions. The full schedule of talks from the workshop appears on the next page. In addition to the lecture notes submitted by its lecturers, this volume contains several research articles.

A group of fourty graduate students and young researchers attended the school. Among the participants there were 2 Algerian, 3 American, 1 Armenian, 1 Bulgarian, 1 Canadian, 3 Dutch, 2 Georgian, 7 German, 1 Indian, 2 Iraqi, 1 Iranian, 1 Italian, 1 Russian, 5 Japanese, 23 Turkish and 1 Ukrainian mathematicians, including the lecturers.

We would like to thank the *Centre des Mathématiques Pures et Appliquées*, for their financial support and Professor Michel Jambu for organizational help. We could support participants from across the region thanks to the generous financial help provided by the *International Center for Theoretical Physics* (ICTP) and the *International Mathematical Union* (IMU). The local participants has been supported by the *Scientific and Technological Research Councel of Turkey* (TÜBİTAK).

This summer school has been realized not only by financial support from its sponsors but also thanks to the generousity of its lecturers, who all agreed to finance their travel from their own personal grants. Some of them did so also for the accomodation.

The proposal for the AGAHF Summer School was submitted to CIMPA in February 2004. During the long preparatory process and during the summer school, Ayşegül Ulus, Özgür Ceyhan, and Özgür Kişisel contributed at various levels to the organization. We are grateful to them.

Sabine Buchmann is a French artist living in Istanbul, who likes to draw Ottoman-style miniatures of the boats serving across the bosphorus; these boats are an inseparable part of the city panorama. When asked, she liked the idea of a boat full of mathematicians and drew it for the conference poster — with the names of all the lecturers hidden inside, written in minute letters. Her miniature helped us much in attracting the audience of the summer school.

We are thankful to the student team hired by the university comprising Anet İzmitli, Egemen Kırant, Günce Orman, Haris Saybaşılı and Eylem Şentürk for turning this summer school into a pleasant experience.

Finally we would like to thank warmly Prof. Dr. Duygun Yarsuvat, the rector of the Galatasaray University for offering us the great location and financial support of the university.

The second named editor was supported by TÜBİTAK grant Kariyer 103T136 during the summer school and during the preperation of this volume.

Rolf-Peter Holzapfel, A. Muhammed Uludağ and Masaaki Yoshida, Editors

PROGRAM

Daniel Allcock: Real hyperbolic geometry in moduli problems

Igor Dolgachev: Moduli spaces as ball quotients (followed by Kondo's lectures)

Rolf Peter Holzapfel: Orbital Varieties and Invariants

Michel Jambu: Arrangements of Hyperplanes

A. Kochubei: Hypergeometric functions and Carlitz differential equations over function fields

Shigeyuki Kondo: Complex ball uniformizations of the moduli spaces of del Pezzo surfaces

Edward Looijenga: (first week) Introduction to Deligne-Mostow theory

Edward Looijenga: (second week) Hypergeometric functions associated to arrangements

Keiji Matsumoto: Invariant functions with respect to the Whitehead link

Hironori Shiga: Hypergeometric functions and arithmetic geometric means (followed by Wolfart's lectures)

Jan Stienstra: Gel'fand-Kapranov-Zelevinsky hypergeometric systems and their role in mirror symmetry and in string theory

Toshiaki Terada: Hypergeometric representation of the group of pure braids.

A. Muhammed Uludağ: Geometry of Complex Orbifolds

Alexander Varchenko: Special functions, KZ type equations, and representation theory

Jürgen Wolfart: Arithmetic of Schwarz maps (preceded by Shiga's lectures)

Masaaki Yoshida: Schwarz maps (general introduction)

Contents

Preface v

Daniel Allcock, James A. Carlson and Domingo Toledo
Hyperbolic Geometry and the Moduli Space of Real Binary Sextics .. 1

Frits Beukers
Gauss' Hypergeometric Function 23

Igor V. Dolgachev and Shigequki Kondō
Moduli of K3 Surfaces and Complex Ball Quotients 43

Amir Džambić
Macbeaths Infinite Series of Hurwitz Groups 101

Rolf-Peter Holzapfel
Relative Proportionality on Picard and Hilbert Modular Surfaces 109

Anatoly N. Kochubei
Hypergeometric Functions and Carlitz Differential Equations over Function Fields 163

Shigeyuki Kondō
The Moduli Space of 5 Points on $\mathbb{P}^1$ and K3 Surfaces 189

Eduard Looijenga
Uniformization by Lauricella Functions — An Overview of the Theory of Deligne–Mostow 207

Keiji Matsumoto
Invariant Functions with Respect to the Whitehead-Link 245

Thorsten Riedel
On the Construction of Class Fields by Picard Modular Forms 273

Hironori Shiga and Jürgen Wolfart
Algebraic Values of Schwarz Triangle Functions 287

Jan Stienstra
GKZ Hypergeometric Structures 313

A. Muhammed Uludağ
Orbifolds and Their Uniformization 373

Masaaki Yoshida
From the Power Function to the Hypergeometric Function 407

Celal Cem Sarıoğlu (ed.)
Problem Session 431

Progress in Mathematics, Vol. 260, 1–22

Hyperbolic Geometry and the Moduli Space of Real Binary Sextics

Daniel Allcock, James A. Carlson and Domingo Toledo

Abstract. The moduli space of real 6-tuples in $\mathbb{C}P^1$ is modeled on a quotient of hyperbolic 3-space by a nonarithmetic lattice in $\operatorname{Isom} H^3$. This is partly an expository note; the first part of it is an introduction to orbifolds and hyperbolic reflection groups.

Keywords. Complex hyperbolic geometry, hyperbolic reflection groups, orbifolds, moduli spaces, ball quotients.

These notes are an exposition of the key ideas behind our result that the moduli space $\mathcal{M}_s$ of stable real binary sextics is the quotient of real hyperbolic 3-space H^3 by a certain Coxeter group (together with its diagram automorphism). We hope they can serve as an aid in understanding our work [3] on moduli of real cubic surfaces, since exactly the same ideas are used, but the computations are easier and the results can be visualized.

These notes derive from the first author's lectures at the summer school "Algebra and Geometry around Hypergeometric Functions", held at Galatasary University in Istanbul in July 2005. He is grateful to the organizers, fellow speakers and students for making the workshop very rewarding. To keep the flavor of lecture notes, not much has been added beyond the original content of the lectures; some additional material appears in an appendix. The pictures are hand-drawn to encourage readers to draw their own.

Lecture 1

Hyperbolic space H^3 is a Riemannian manifold for which one can write down an explicit metric, but for us the following model will be more useful; it is called the upper half-space model. Its underlying set is the set of points in $\mathbb{R}^3$ with

First author partly supported by NSF grant DMS 0231585. Second and third authors partly supported by NSF grants DMS 9900543 and DMS 0200877.