

# SMOOTH FINITE ABELIAN UNIFORMIZATIONS OF PROJECTIVE SPACES AND CALABI-YAU ORBIFOLDS

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*Dedicated to Mehmet Çiftçi*

ABSTRACT. We give a classification of smooth complex manifolds with a finite abelian group action, such that the quotient is isomorphic to a projective space. The case where the manifold is a Calabi-Yau is studied in detail.

## 1. INTRODUCTION

Let  $M$  be a complex manifold with a faithful action by a finite abelian group  $G$  such that  $M/G \simeq \mathbb{P}^n$ . In this paper we give a complete classification of such pairs  $(M, G)$  and study the case where  $M$  is a Calabi-Yau manifold in detail.

A well-known example of a pair  $(M, G)$  is a cyclic multiple plane: Let

$$S : \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid P(z_0, \dots, z_n) = 0\}$$

be a smooth hypersurface in  $\mathbb{P}^n$  defined by a homogeneous polynomial  $P$  of degree  $n + 2$ . Then the hypersurface in  $\mathbb{P}^{n+1}$  defined as

$$M : \{[z_0 : \cdots : z_n : z_{n+1}] \in \mathbb{P}^{n+1} \mid z_{n+1}^{n+2} = P(z_0, \dots, z_n)\}$$

is smooth of degree  $n + 2$ , which implies that  $M$  is a Calabi-Yau variety of dimension  $n$ . In dimension two  $M$  is a smooth quartic surface and in dimension three  $M$  is a smooth quintic threefold. Let  $\omega$  be a primitive  $n + 2$ nd root of unity. Then the cyclic group  $G := \mathbb{Z}/(n + 2)$  acts on  $M$ , the action of  $i \in G$  is given by

$$[z_0 : \cdots : z_n : z_{n+1}] \in M \rightarrow [z_0 : \cdots : z_n : \omega^i z_{n+1}] \in M$$

Consider the projection

$$\varphi : [z_0 : \cdots : z_n : z_{n+1}] \in \mathbb{P}^{n+1} \rightarrow [z_0 : \cdots : z_n] \in \mathbb{P}^n,$$

its restriction to  $M$  is precisely the quotient map  $M \rightarrow M/G$ , which shows that  $M/G$  is  $\mathbb{P}^n$ . Evidently,  $G$  fixes the points  $M \cap \{z_{n+1} = 0\}$

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and the image of this set under  $\varphi$  is  $H$ . In other words,  $\varphi : M \rightarrow \mathbb{P}^n$  is a Galois covering of degree  $n + 2$ , branched along the hypersurface  $H$ .

Let  $(M, G)$  be a pair with  $M/G \simeq \mathbb{P}^n$ . The corresponding projection map  $\varphi : M \rightarrow \mathbb{P}^n$  induces an orbifold structure  $(\mathbb{P}^n, b_\varphi)$  on  $\mathbb{P}^n$ , where  $b_\varphi : \mathbb{P}^n \rightarrow \mathbb{N}$  is the map sending  $p \in \mathbb{P}^n$  to the order of the stabilizer  $G_q \subset G$ , where  $q$  is a point in  $\varphi^{-1}(p)$ . In the example given above, the induced orbifold is  $(\mathbb{P}^n, b_\varphi)$ , where

$$b_\varphi(p) := \begin{cases} n + 2 & p \in S \\ 1 & p \notin S \end{cases}$$

The *locus* of an orbifold  $(\mathbb{P}^n, b)$  is defined to be the hypersurface  $\text{supp}(b-1)$ . In our case, the locus  $(\mathbb{P}^n, b_\varphi)$  is precisely the hypersurface  $S$ . In Section 2 we give a brief introduction to orbifolds. The main Theorem 2 is proved in 2.1. The orbifold euler number of orbifolds  $(\mathbb{P}^n, b)$  with a linear locus is computed in 2.2.

An orbifold uniformized by a smooth Calabi-Yau manifold is called a *Calabi-Yau orbifold*. Section 3 is devoted to the classification of CY orbifolds admitting an abelian uniformization. In dimension one, this classification is classical. K3 orbifolds with a locus of degree  $\leq 5$  were classified in [5]. There are no K3 orbifolds with a locus of degree  $> 6$ .

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## 2. ORBIFOLDS

Here we give a brief introduction to orbifolds following Kato [3]. For details one may also consult [4]. Let  $M$  be a connected smooth complex manifold,  $G \subset \text{Aut}(M)$  a properly discontinuous subgroup and put  $X := M/G$ . Then the projection  $\varphi : M \rightarrow X$  is a branched Galois covering endowing  $X$  with a map  $b_\varphi : X \rightarrow \mathbb{N}$  defined by  $b_\varphi(p) := |G_q|$  where  $q$  is a point in  $\varphi^{-1}(p)$  and  $G_q$  is the isotropy subgroup of  $G$  at  $q$ . The pair  $(X, b_\varphi)$  is said to be uniformized by  $\varphi : M \rightarrow (X, b_\varphi)$ . An *orbifold* is a pair  $(X, b)$  of an irreducible normal analytic space  $X$  with a function  $b : X \rightarrow \mathbb{N}$  such that the pair  $(X, b)$  is locally finitely uniformizable. We shall mostly be concerned with the case where  $X$  is a smooth complex manifold. In case  $c|b$ , the orbifold  $(X, c)$  is said to be a *suborbifold* of  $(X, b)$ . Let  $\varphi : (Y, 1) \rightarrow (X, c)$  be a uniformization of  $(X, c)$ , e.g.  $b_\varphi = c$ . Then  $\varphi : (Y, b') \rightarrow (X, b)$  is called an *orbifold covering*, where  $b' := b \circ \varphi / c \circ \varphi$ . The orbifold  $(Y, b')$  is called the *lifting of  $(X, b)$  to the uniformization  $Y$  of  $(X, c)$*  and will be denoted by  $(X, b)/(X, c)$ .

Let  $X$  be a smooth manifold. Let  $(X, b)$  be an orbifold,  $H := \text{supp}(b - 1)$  be its *locus* and let  $H_1, \dots, H_r$  be the irreducible components of  $H$ . Then  $b$  is constant on  $H_i \setminus \text{sing}(H)$ ; so let  $m_i$  be this number. By abuse of language, the divisor  $\sum_i m_i H_i$  is also called the *locus* of  $(X, b)$  (In fact, the divisor  $\sum_i m_i H_i$  completely determines the  $b$ -function). The *orbifold fundamental group*  $\pi_1^{orb}(X, b)$  of  $(X, b)$  is the group defined by  $\pi_1^{orb}(X, b) := \pi_1(X \setminus H) / \langle\langle \mu_1^{m_1}, \dots, \mu_r^{m_r} \rangle\rangle$  where  $\mu_i$  is a meridian of  $H_i$  and  $\langle\langle \rangle\rangle$  denotes the normal closure. The *local orbifold fundamental group*  $\pi_1^{orb}(X, b)_p$  at a point  $p \in X$  is the orbifold fundamental group  $\pi_1^{orb}(\mathcal{O}, p)$  of the restriction of  $(X, b)$  to a sufficiently small neighborhood of  $\mathcal{O}$  of  $p$ .

Let  $(X, b)$  be an orbifold and  $\rho : \pi_1^{orb}(X, b) \twoheadrightarrow G$  be a surjection onto a finite group. Then there exists a Galois covering of  $M \rightarrow X$  branched along  $H$  with  $G$  as the Galois group. Under which conditions the covering is a uniformization of  $(X, b)$ ? The following theorem (stated in a slightly different terminology in Kato's monograph) answers this question.

**Theorem 1.** *Let  $(X, b)$  be an orbifold and  $\rho : \pi_1^{orb}(X, b) \twoheadrightarrow G$  a surjection onto a finite group. The corresponding Galois covering  $\varphi : M \rightarrow X$  is a uniformization of  $(X, b)$  if and only if for any  $p \in X$ , the composition of the maps*

$$(1) \quad \pi_1^{orb}(X, b)_p \xrightarrow{\iota_p} \pi_1(X, b) \xrightarrow{\rho} G$$

*is an injection, where  $\iota_p$  is the map induced by the inclusion  $(X, b)_p \hookrightarrow \pi_1(X, b)$ .*

If the groups  $\pi_1^{orb}(X, b)$  and  $G$  are non-abelian, it is a hopeless task to verify the condition of Theorem 1. However, assuming  $G$  to be abelian simplifies matters significantly.

**2.1. Abelian orbifolds on  $\mathbb{P}^n$ .** An *abelian orbifold* is an orbifold which admits an abelian uniformization. Let  $(\mathbb{P}^n, b)$  be an abelian orbifold,  $H := \text{supp}(b - 1)$  its locus, and let  $H_1, \dots, H_r$  be the irreducible components of  $H$ . By the following lemma, the  $b$ -function is very restricted.

**Lemma 1.** *If  $(\mathbb{P}^n, b)$  is an abelian orbifold, then*

(i) *Any singularity of the locus  $H$  is locally of the form  $\{z_1 z_2 \dots z_k = 0 : (z_1, \dots, z_n) \in \mathbb{C}^n\}$  for some  $k \leq n$ .*

(ii) *There exists an  $r$ -tuple  $(m_1, \dots, m_r) \in \mathbb{N}^r$  such that  $b$  is given by  $b(p) := \prod_{p \in H_i} m_i$ .*

(iii) *The groups  $\pi_1(\mathbb{P}^n \setminus H)$  and  $\pi_1^{orb}(\mathbb{P}^n, b)$  are abelian when  $n > 1$ .*

(iii) The hypersurfaces  $H_i$  are smooth and are in general position.

*Proof.* Suppose  $(\mathbb{P}^n, b)$  admits an abelian uniformization and let  $G$  be the corresponding abelian Galois group. By Theorem 1, for any  $x \in \mathbb{P}^n$ , the local group  $\pi_1^{orb}(X, b)_x$  injects into  $G$ . Hence all local groups are abelian and all local germs admit abelian uniformizations. An abelian covering germ  $\mathbb{C}_O^n \rightarrow \mathbb{C}_O^n$  is of the form

$$\varphi : (z_1, \dots, z_n) \in \mathbb{C}_O^n \rightarrow (z_1^{m_1}, \dots, z_n^{m_n}) \in \mathbb{C}_O^n$$

where  $m_i \in \mathbb{Z}_{\geq 1}$ . Hence the singularities of  $H$  must be of the form  $z_1 z_2 \dots z_k = 0$  for some  $k \in [1, n]$ . This proves (i). Hence locally the  $b$ -map is of the form  $b_\varphi(p) = \prod_{p \in L_i} m_i$ , where  $L_i := \{z_i = 0\}$ . This proves (ii). By the Zariski conjecture proved by Fulton-Deligne and by (i) the group  $\pi_1(\mathbb{P}^n \setminus H)$  and its quotient  $\pi_1^{orb}(\mathbb{P}^n, b)$  are abelian, hence (iii) is also proved. To prove the last claim suppose  $p$  is a singular point of  $H_1$ . For simplicity assume that  $p$  do not belong to  $\cup_2^r H_i$ . Around  $p$ , the hypersurface  $H_1$  is of the form  $z_1 \dots z_k = 0$  for some  $1 < k \leq n$ . Let  $\mu_{1,i}$  be a branch of  $z_i = 0$  for  $i \in [1, k]$ . Then

$$\pi_1^{orb}(\mathbb{P}^n, b)_p \simeq \langle \mu_{1,1}, \dots, \mu_{1,k} \mid m_1 \mu_{1,i} = 0 \text{ for } i \in [1, k] \rangle \simeq (\mathbb{Z}/(m_1))^k$$

On the other, since  $H_1$  is irreducible the meridians  $\mu_{1,i}$  are conjugate elements of the group  $\pi_1(\mathbb{P}^n \setminus B_b)$ . Since this latter group is abelian, the image of  $\iota_p$  is generated by a single meridian and is cyclic. Hence  $\iota_p$  can not be injective, contradicting Theorem 1.  $\square$

Lemma 1 does not give sufficient conditions for the existence of a uniformization. Let  $(\mathbb{P}^n, b)$  be an orbifold satisfying the conclusions of Lemma 1. Let  $H := \cup_1^r H_i \subset \mathbb{P}^n$  be its locus and let

$$(2) \quad b(p) := \prod_{p \in H_i} m_i$$

be its  $b$ -function as required by Lemma 1, where  $(m_1, \dots, m_r) \in \mathbb{N}^r$ . Put

$$f_i := \frac{m_i}{\gcd(m_i, d_i)}$$

**Theorem 2.** *The orbifold  $(\mathbb{P}^n, b)$  admits a finite abelian smooth uniformization if and only if any prime power dividing one among  $f_1, \dots, f_r$  divides at least  $n$  other  $f_i$ 's.*

*Proof.* The group  $\pi_1(\mathbb{P}^n \setminus H)$  admits the presentation

$$\pi_1(\mathbb{P}^n, b) \simeq \langle \mu_1, \dots, \mu_r, \mid \sum_1^r d_i m_i = 0 \rangle$$

where  $\mu_i$  is a meridian of  $H_i$ . Hence the group  $\pi_1^{orb}(\mathbb{P}^n, b)$  admits the presentation

$$\pi_1^{orb}(\mathbb{P}^n, b) \simeq \langle \mu_1, \dots, \mu_r, \mid m_1\mu_1 = \dots = m_r\mu_r = \sum_1^r d_i m_i = 0 \rangle$$

This group is of order

$$|\pi_1^{orb}(\mathbb{P}^n, b)| = \frac{\prod_{i \in [1, r]} m_i}{\text{lcm}\{f_i : i \in [1, r]\}}$$

Now let  $B \subset [1, r]$  with  $|B| \leq n$ . Let  $p$  be a point in  $\cap_{i \in B} H_i$ , which is not in  $\cap_{i \in C} H_i$  for any  $C \supseteq B$ , in other words,

$$p \in \bigcap_{i \in B} H_i \setminus \left( \bigcup_{C \supseteq B} \bigcap_{i \in C} H_i \right)$$

One has the isomorphism of orbifold germs

$$(\mathbb{P}^n, b)_p \simeq (\mathbb{C}, 1)_0^{n-|B|} \times \prod_{i \in [1, n]} (\mathbb{C}, b_i)_0,$$

where  $b_i(0) = m_i$  and  $b_i(z) = 1$  if  $z \neq 0$ . Hence, the local orbifold fundamental group around  $p$  is

$$\pi_1^{orb}(\mathbb{P}^n, b)_p \simeq \langle \{\mu_i : i \in B\} \mid m_i \mu_i = 0 : i \in B \rangle,$$

which is of order  $\prod_{i \in B} m_i$ . Let  $\iota_p : \pi_1^{orb}(\mathbb{P}^n, b)_p \rightarrow \pi_1^{orb}(\mathbb{P}^n, b)$  be the homomorphism induced by the inclusion. One has

$$\text{Coker}(\iota_p) \simeq \langle \{\mu_i : i \notin B\} \mid m_i \mu_i = \sum_{i \notin B} d_i \mu_i = 0 \rangle,$$

The homomorphism  $\iota_p$  is an injection if and only if  $\text{Coker}(\iota_p)$  is of order

$$\frac{|\pi_1^{orb}(\mathbb{P}^n, b)|}{|\pi_1^{orb}(\mathbb{P}^n, b)_p|} = \frac{\prod_{i \notin B} m_i}{\text{lcm}\{f_i : i \in [1, r]\}}$$

On the other hand, one has

$$|\text{Coker}(\iota_p)| = \frac{\prod_{i \notin B} m_i}{\text{lcm}\{f_i : i \notin B\}}$$

Therefore,  $\iota_p$  is injective if and only if

$$(3) \quad \text{lcm}\{f_i : i \in [1, r]\} = \text{lcm}\{f_i : i \notin B\}$$

Recall that  $B$  is a subset of the interval  $[1, r]$  with  $|B| \leq n$  elements, and  $p \in \cap_{i \in B} H_i$  is a singular point of the arrangement  $\cup_{i \in [1, r]} H_i$  of multiplicity  $|B|$ . The map  $\iota_p$  must be injective for any  $p \in \mathbb{P}^n$ , so that (3) must be valid for any  $B \subset [1, r]$  with  $|B| \leq n$ . It is easily seen that if (3) is valid for any  $B$  with  $|B| = n$ , then it is valid for any  $B$  with

$|B| \leq n$ . But if (3) is valid for any  $B$  with  $|B| = n$ , then the claim of the theorem easily follows  $\square$

How to interpret the claim of Theorem 2? Let us first discuss the case  $r = 1$ . Thus the locus  $H = H_1$  of  $(\mathbb{P}^n, b)$  is a smooth irreducible hypersurface of degree  $d_1$ , and  $f_1 = m_1 / \gcd(m_1, d_1)$ . Now if  $f_1 \neq 1$ , then it has a prime divisor, which must divide at least  $n$  other  $f_i$ 's, which is impossible (unless  $n = 0$ ). Hence  $f_1 = 1$ , which implies that  $m_1$  is a divisor of  $d_1$ . It is well known that the orbifolds  $(\mathbb{P}^n, b)$  are uniformizable in this case.

Now let us discuss the case  $r = 2$ . If  $n = 1$  then Theorem 2 implies  $f_1 = f_2$ , which in turn implies that  $m_1 = m_2$  since  $H_i$  are reduced. This is the well known orbifold  $(\mathbb{P}^1, b)$ , which is uniformized via the power map  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  of degree  $m_1$ . If  $n \geq 2$  as in the case of  $r = 1$  one has  $f_1 = f_2 = 1$ , in other words  $m_i$  divides  $d_i$ . For example, let  $H_1$  be a smooth quadric and  $H_2$  be a smooth cubic. Then  $m_1 = 2$  and  $m_2 = 3$ .

Let  $p$  be a prime,  $\alpha$  an integer  $> 0$ , and take integers  $\alpha_{n+2}, \dots, \alpha_r \in [0, \alpha]$ . Then the following  $r$ -tuple,

$$(p^\alpha, \dots, p^\alpha, p^{\alpha_{n+2}}, \dots, p^{\alpha_r})$$

as well as all its permutations, satisfies the condition of Theorem 2. Any  $r$ -tuple  $(f_1, \dots, f_r)$  satisfying the condition of Theorem 2 admits a unique factorization into a product of such  $r$ -tuples with distinct  $p$ , under the operation of componentwise multiplication.

Note that if  $H_i$  is a hyperplane then  $d_i = 1$ , so that  $f_i = m_i$ . Hence for hyperplane arrangements the conditions simplifies significantly.

**2.2. Euler numbers of abelian orbifolds with a linear locus.** Let  $(\mathbb{P}^n, b)$  be an orbifold. Then there is an induced stratification  $\mathcal{S}$  of  $\mathbb{P}^n$  with  $b$ -constant strata. Let  $b(S)$  be the constant value of the  $b$ -function on the stratum  $S \in \mathcal{S}$ . The euler number of  $(\mathbb{P}^n, b)$  is defined as

$$e_{orb}(\mathbb{P}^n, b) := \sum_{S \in \mathcal{S}} \frac{e(S)}{b(S)}$$

so that, if  $\varphi : M \rightarrow (\mathbb{P}^n, b)$  is a uniformization with  $G$  as the Galois group, then  $e(M) = |G|e_{orb}(\mathbb{P}^n, b)$ .

Now let  $(\mathbb{P}^n, b)$  be an orbifold whose locus is a hyperplane arrangement  $\mathcal{H} := \{H_1, \dots, H_r\}$ . Let  $\mathcal{L}(\mathcal{H})$  be its intersection lattice. Then the induced stratification is

$$\mathcal{S} = \{\mathcal{M}(\mathcal{H}^A) : A \in \mathcal{L}\},$$

where as usual  $\mathcal{H}^A$  denotes the restriction of  $\mathcal{H}$  to  $A$  and  $\mathcal{M}(\mathcal{H})$  denotes the complement. Let  $(m_1, \dots, m_r)$  be as in Lemma 1. Then  $b(H_i) =$

$m_i$ . The constant value assumed by the  $b$ -function along an element  $A \in \mathcal{L}$  is the product  $\prod_{A \subset H_i} b(H_i)$ . Hence one has

$$e_{orb}(\mathbb{P}^n, b) = \sum_{A \in \mathcal{L}} \frac{e(\mathcal{M}(\mathcal{H}^A))}{\prod_{A \subset H_i} m_i}$$

The number  $e(\mathcal{M}(\mathcal{H}^A))$  can be computed as follows. Suppose  $A$  is of rank  $k$  in  $\mathcal{L}$  (i.e.  $A$  is of codimension  $k$  in  $\mathbb{P}^n$ ). Then  $A$  lies in the intersection of exactly  $k$  hyperplanes  $H_i$ . By the genericity assumption,  $\mathcal{H}^A$  is an arrangement of  $r - k$  hyperplanes in general position in  $A \simeq \mathbb{P}^{r-k}$ . Let  $e(r - k, n - k)$  be the euler number of  $\mathcal{M}(\mathcal{H}^A)$ . Let  $\mathcal{H}$  be an arrangement of hyperplanes in general position and  $H \in \mathcal{H}$ . As usual, denote by  $\mathcal{H}_H$  the deletion  $\mathcal{H} \setminus \{H\}$ . The equation

$$\mathcal{M}(\mathcal{H}) = \mathcal{M}(\mathcal{H}_H) - \mathcal{M}(\mathcal{H}^H)$$

yields the recursion

$$e(r, n) = e(r - 1, n) - e(r - 1, n - 1)$$

Tabulation of  $e(r, n)$  gives the following Pascal triangle with alternating signs.

	0	1	2	3	4	5	6	7	8
$\mathbb{P}^0$	1	1	1	1	1	1	1	1	1
$\mathbb{P}^1$	2	1	0	-1	-2	-3	-4	-5	-6
$\mathbb{P}^2$	3	1	0	0	1	3	6	10	15
$\mathbb{P}^3$	4	1	0	0	0	-1	-4	-10	-20
$\mathbb{P}^4$	5	1	0	0	0	0	1	5	15
$\mathbb{P}^5$	6	1	0	0	0	0	0	-1	-6
$\mathbb{P}^6$	7	1	0	0	0	0	0	0	1
$\mathbb{P}^7$	8	1	0	0	0	0	0	0	0

Hence  $e(r, n) = (-1)^n \binom{r-2}{n}$ , in other words

$$e(\mathcal{M}(\mathcal{H}^A)) = e(r - k, n - k) = (-1)^{n-k} \binom{r-2-k}{n-k}$$

This gives

$$e_{orb}(\mathbb{P}^n, b) = \sum_{k=0}^n (-1)^{n-k} \binom{r-2-k}{n-k} \sum_{\substack{B \subset [1, r] \\ |B|=k}} \prod_{i \in B} \frac{1}{m_i}$$

The change of parameters  $s_i := 1 - 1/m_i$  gives, after recollecting terms the final formula in the following proposition.

**Proposition 1.** *The orbifold euler number of  $(\mathbb{P}^n, b)$  is*

$$(4) \quad e_{orb}(\mathbb{P}^n, b) = \sum_{j=0}^n (-1)^j (n+1-j) \sum_{\substack{B \subset [1, r] \\ |B|=j}} \prod_{i \in B} s_i$$

### 3. CALABI-YAU ORBIFOLDS

Let  $M$  be a uniformization of the orbifold  $(\mathbb{P}^n, b)$  and let  $\varphi : M \rightarrow \mathbb{P}^n$  be the corresponding covering map. Denote by  $D_\varphi$  the ramification divisor of  $\varphi$  and let  $\sum m_i H_i$  be the locus of  $(\mathbb{P}^n, b)$ . One has

$$\begin{aligned} K_M &= \varphi^* K_{\mathbb{P}^n} + D_\varphi \\ &= \varphi^* K_{\mathbb{P}^n} + \sum_i \frac{m_i - 1}{m_i} \varphi^* H_i \\ &= \varphi^* \left( K_{\mathbb{P}^n} + \sum_i \left(1 - \frac{1}{m_i}\right) H_i \right) \end{aligned}$$

Now one has  $K_m \sim -(n+1)L$  and  $H_i \sim d_i L$ , where  $L$  is the class of a hyperplane in  $\mathbb{P}^n$ . Hence  $K_M$  is trivial if and only if the equality

$$\sum_i s_i = \sum_i \left(1 - \frac{d_i}{m_i}\right) = n+1$$

is satisfied. As the following trivial lemma shows, this is a very restrictive condition on  $(\mathbb{P}^n, b)$ .

**Lemma 2.** *If  $(\mathbb{P}^n, b)$  is a Calabi-Yau orbifold with a locus  $H := \cup_1^r H_i$  of degree  $d = \sum d_i$ , then  $n+2 \leq d \leq 2n+2$ . Moreover, if  $d = 2n+2$ , then  $m_1 = m_2 = \dots = m_r = 2$ .*

Hence in any dimension there are finitely many families of abelian Calabi-Yau orbifolds on  $\mathbb{P}^n$ , which can be effectively classified. For  $n = 1$  this classification is classical. We give the details of this classification as a guide to the classifications in dimension 2 and 3, tabulated in the coming pages.

$\mathbb{P}^1$	$d$	orbifold	$e$	$ \pi_1^{orb} $	$\delta$	sub-orbifolds and coverings
<b>1</b>	2	$[\infty, \infty]$	0	$\infty$	0	$[m, m]$ <b>1</b>
<b>2</b>	3	$[2, 2, \infty]$	0	$\infty$	0	$[2, 2, 1]$ <b>1</b> , $[1, 2, 2]$ <b>2</b>
<b>3</b>	3	$[2, 3, 6]$	0	$\infty$	0	$[2, 1, 2]$ <b>5</b> , $[1, 3, 3]$ <b>6</b>
<b>4</b>	3	$[2, 4, 4]$	0	$\infty$	0	$[1, 2, 2]$ <b>6</b> , $[1, 4, 4]$ <b>6</b> , $[2, 2, 1]$ <b>4</b>
<b>5</b>	3	$[3, 3, 3]$	0	$\infty$	0	$[3, 3, 1]$ <b>5</b>
<b>6</b>	4	$[2, 2, 2, 2]$	0	$\infty$	1	$[2, 2, 1]$ <b>6</b>

Here are some remarks on how to understand the tables:



- In dimension  $n$ , the orbifold  $[\infty, \infty \dots, \infty]$  ( $n + 1$  times) is uniformized by  $\mathbb{C}^n$ , via a multi-exponential map. Strictly speaking, this is not a CY orbifold. It is included in the tables in order to complete the picture.
- $\underline{d}$ : The degree of the locus of the orbifold
- orbifold: In dimension 1, the notation  $[m, m', \dots]$  means that the locus consists of distinct points  $p, p'', \dots$ , such that  $b(p) = m, b(p') = m'$ , etc. In higher dimensions, the notation  $[m_d, m'_d, \dots]$  means that the locus consists of hypersurfaces  $H$  of degree  $d, H'$  of degree  $d'$ , etc, and the generic value the  $b$ -function takes on  $H$  is  $m$ , on  $H'$  is  $m'$ , etc. In case the corresponding hypersurface is linear,  $m_1$  is denoted simply by  $m$ .
- $\underline{e}$ : The euler number of the universal uniformization (which is simply connected). Thus  $e = e_{orb} |\pi_1^{orb}|$ .
- $|\pi_1^{orb}|$ : The order of the orbifold fundamental group. Hence this is the degree of the universal uniformization.
- $\underline{\delta}$ : The dimension of the family.
- sub-orbifolds and coverings: In dimension  $n$ , the linear orbifold  $\overline{[m, m, \dots, m]}$  ( $n + 1$  times) is uniformized by  $\mathbb{P}^n$ , for any  $m$ . Some of the CY orbifolds admits these as sub-orbifolds and consequently can be lifted to their uniformization. The lifting is another abelian CY orbifold on  $\mathbb{P}^n$ . The bold number adjacent to a sub-orbifold in the table is an internal reference to the line containing the lifting of the orbifold to the uniformization of this sub-orbifold. For example, in dimension 1, the orbifold  $[1, 3, 3]$  is a sub-orbifold of  $[2, 3, 6]$ , and the lifting of  $[2, 3, 6]$  to the uniformization of  $[1, 3, 3]$  is an orbifold of type  $[2, 2, 2, 2]$ . Another example in dimension 2 is the 3-dimensional family  $[2_2, 3, 3, 3]$ , which admit  $[1, 3, 3, 3]$  as a sub-orbifold. The lifting of this family to the uniformization of  $[1, 3, 3, 3]$  is a 3-dimensional sub-family of the 19-dimensional family  $[2_6]$ , whose loci are defined by equations  $P[x^3 : y^3 : z^3] = 0$ , the polynomial  $P$  being homogeneous of degree 2.
- The euler numbers of non-linear orbifolds which appear as coverings of linear orbifolds can be computed by using the fact that their universal uniformizations are isotopic so that their euler numbers are same.
- In dimensions  $\geq 4$ , the tables includes only CY orbifolds with linear loci. This is for sake of brevity only, there are many non-linear cases.

**3.1. A non-abelian CY orbifold.** Some abelian CY orbifolds have non-trivial automorphism groups. Their quotients sometimes yields CY orbifolds with a non-abelian uniformizing group. In dimension 2, many examples were constructed in [5]. In dimension 3 consider the CY orbifold  $[2, 2, 2, 2, 2, 2, 2, 2]$ . Suppose that its locus consists of the hyperplanes  $z_0 \pm z_1 \pm z_2 \pm z_3 = 0$ . This orbifold is invariant under the action of the group  $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$  such that the element  $(i, j, k)$  acts by

$$[z_0, z_1, z_2, z_3] \longrightarrow [z_0, (-1)^i z_1, (-1)^j z_2, (-1)^k z_3]$$

The quotient of  $\mathbb{P}^3$  under this action is again  $\mathbb{P}^3$ , via the map

$$\varphi_2([z_0, z_1, z_2, z_3]) := [z_0^2, z_1^2, z_2^2, z_3^2]$$

The quotient of  $[2, 2, 2, 2, 2, 2, 2, 2]$  under the same group action is therefore the non-abelian CY orbifold  $(\mathbb{P}^3, b)$ , whose locus is given by the equation

$$\{z_0 z_1 z_2 z_3 (\sqrt{z_0} + \sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}) = 0\} \subset \mathbb{P}^3$$

The  $b$ -function takes the generic value 2 on the locus. Computing the degree of the universal uniformization of  $(\mathbb{P}^3, b)$  we find that the group  $\pi_1^{orb}(\mathbb{P}^3, b)$  is a non-abelian group of order  $2^7 \times 2^3 = 1024$ . This latter orbifold is invariant under the symmetric group on four letters  $\{z_0, z_1, z_2, z_3\}$ . However, this time the quotient of  $\mathbb{P}^3$  is a singular space. Anyway, this shows that the uniformizing CY threefold has a group of automorphisms of order  $1024 \times 4! = 24576$ .

**3.2. CY manifolds of Enriques type.** An abelian orbifold may have several uniformizations, in other words the universal uniformization is not necessarily the only uniformization. There are also some CY orbifolds admitting intermediate uniformizations. For any  $n$ , the orbifold fundamental group of the CY orbifold  $[2, 2, \dots, 2]$  ( $2n+2$  times) admits the presentation

$$\pi_1^{orb}([2, 2, \dots, 2]) \simeq \langle \mu_1, \dots, \mu_{2n+2} \mid 2\mu_1 = \dots = 2\mu_{2n+2} = \sum_1^{2n+2} \mu_i = 0 \rangle$$

Let  $\langle \alpha \rangle$  be the subgroup generated by  $\alpha := \mu_1 + \dots + \mu_{n+1}$  and let  $G$  be the quotient by this subgroup. It has the presentation

$$G \simeq \langle \mu_1, \dots, \mu_{2n+2} \mid 2\mu_1 = \dots = 2\mu_{2n+2} = \sum_1^{2n+2} \mu_i = \sum_1^{n+1} \mu_i = 0 \rangle$$

This group satisfies the conditions of Theorem 1-therefore the corresponding uniformization exists. Since  $\alpha$  is of order 2, this is not the universal uniformization, and its fundamental group is  $\mathbb{Z}/(2)$ . In dimension 2 this uniformization is an Enriques surface. The element  $\alpha$  can be chosen in  $\frac{1}{2} \binom{2n+2}{n+1}$  different ways, hence there are as many intermediate uniformizations.

**3.3. CY orbifolds and configuration spaces.** Families of K3 surfaces frequently has a nice interpretations as quotients of symmetric spaces (see [2]). Perhaps the simplest example is the following: Consider the family  $\mathcal{F}$  of K3 orbifolds of type  $[6_2, 3, 3]$ , whose locus consists of a conic  $Q$  with two lines in general position. Now consider  $Q$  as a projective line  $\mathbb{P}^1$  and the intersection points of  $Q$  with the lines as a coloured configuration of points on  $\mathbb{P}^1$ , such that the points lying on the same line has the same color. Hence  $\mathcal{F}$  can be interpreted as a configuration space of coloured points on  $\mathbb{P}^1$ , and is isogeneous to the configuration space of 4 points on  $\mathbb{P}^1$ . The moduli space of 4-tuples on  $\mathbb{P}^1$  is a quotient of the Poincaré disc in the well-known, classical manner. Note that tangent lines to  $Q$  lifts to the uniformization of a K3 orbifold of type  $[6_2, 3, 3]$  as a pair of elliptic fibrations. In the examples we discuss below this feature is pertinent, except the last one. The fibrations corresponding to the orbifolds  $[6_2, 3, 3]$  and  $[2_2, 3, 3, 3]$  are isotrivial.

Consider now the family  $\mathcal{F}$  of K3 orbifolds of type  $[2_2, 3, 3, 3]$  (or of type  $[4_2, 2, 2, 2]$ ), whose locus consists of a conic  $Q$  with three lines in general position. This family can be interpreted as a configuration space of coloured points on  $\mathbb{P}^1$ , and is isogeneous to the configuration space of 6 points on  $\mathbb{P}^1$ . The moduli space of 6-tuples on  $\mathbb{P}^1$  is known to be a ball-quotient (see [1]). Similarly, the family of K3 orbifolds of type  $[2_2, 2, 2, 2, 2]$  is related to the moduli space of 8 points on  $\mathbb{P}^1$ , which is a ball quotient. It was shown in [6] that the family of K3 orbifolds of type  $[2, 2, 2, 2, 2, 2]$  is a quotient of a type IV-symmetric domain. It is of interest to know whether other families of Calabi-Yau orbifolds constructed in this paper admit similar interpretations.

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Abelian K3 orbifolds						
$\mathbb{P}^2$	$d$	orbifold	$e$	$ \pi_1^{orb} $	$\delta$	sub-orbifolds and coverings
<b>1</b>	3	$[\infty, \infty, \infty]$	0	$\infty$	0	$[m, m, m]$ <b>1</b>
<b>2</b>	4	$[2, 6, 6, 6]$	24	72	0	$[2, 2, 2, 1]$ <b>4</b> , $[1, 2, 2, 2]$ <b>6</b> , $[1, 3, 3, 3]$ <b>12</b> , $[1, 6, 6, 6]$ <b>14</b>
<b>3</b>	4	$[4, 4, 4, 4]$	24	64	0	$[2, 2, 2, 1]$ <b>7</b> , $[4, 4, 4, 1]$ <b>5</b>
<b>4</b>	4	$[6_2, 3, 3]$	24	18	1	
<b>5</b>	4	$[4_4]$	24	4	6	
<b>6</b>	5	$[2_2, 3, 3, 3]$	24	18	3	$[1, 3, 3, 3]$ <b>14</b>
<b>7</b>	5	$[4_2, 2, 2, 2]$	24	16	3	$[1, 2, 2, 2]$ <b>5</b>
<b>8</b>	5	$[3_3, 2_2]$	24	6	6	
<b>9</b>	6	$[2, 2, 2, 2, 2, 2]$	24	32	4	$[2, 2, 2, 1, 1, 1]$ <b>10</b>
<b>10</b>	6	$[2_2, 2, 2, 2, 2]$	24	32	5	$[1, 1, 2, 2, 2]$ <b>13</b>
<b>11</b>	6	$[2_2, 2_2, 2_2]$	24	8	7	
<b>12</b>	6	$[2_3, 2, 2, 2]$	24	8	7	$[1, 2, 2, 2]$ <b>14</b>
<b>13</b>	6	$[2_4, 2_2]$	24	4	11	
<b>14</b>	6	$[2_6]$	24	2	19	

Abelian Calabi-Yau 3-orbifolds

$\mathbb{P}^3$	$d$	orbifold	$e$	$ \pi_1^{orb} $	$\delta$	sub-orbifolds and coverings
<b>1</b>	4	$[\infty, \infty, \infty, \infty]$	0	$\infty$	0	$[m, m, m, m]$ <b>1</b>
<b>2</b>	5	$[2, 5, 10, 10, 10]$	-288	1000	0	$[1, 5, 5, 5, 5]$ <b>30</b> , $[2, 1, 2, 2, 2]$ <b>6</b>
<b>3</b>	5	$[2, 8, 8, 8, 8]$	-296	1024	0	$[1, 2, 2, 2, 2]$ <b>12</b> , $[1, 4, 4, 4, 4]$ <b>27</b> , $[1, 8, 8, 8, 8]$ <b>34</b> , $[2, 2, 2, 2, 1]$ <b>7</b>
<b>4</b>	5	$[3, 6, 6, 6, 6]$	-204	648	0	$[1, 2, 2, 2, 2]$ <b>12</b> , $[1, 3, 3, 3, 3]$ <b>22</b> , $[1, 6, 6, 6, 6]$ <b>19</b> , $[3, 3, 3, 3, 1]$ <b>16</b>
<b>5</b>	5	$[5, 5, 5, 5, 5]$	-200	625	0	$[1, 5, 5, 5, 5]$ <b>8</b>
<b>6</b>	5	$[5_2, 5, 5, 5]$	-200	125	3	
<b>7</b>	5	$[8_2, 4, 4, 4]$	-200	128	3	
<b>8</b>	5	$[5_5]$	-200	5	55	
<b>9</b>	6	$[2, 2, 3, 3, 6, 6]$	-120	216	3	$[1, 1, 3, 3, 3, 3]$ <b>28</b> , $[2, 2, 1, 1, 2, 2]$ <b>14</b>
<b>10</b>	6	$[2, 2, 4, 4, 4, 4]$	-176	256	3	$[1, 1, 2, 2, 2, 2]$ <b>26</b> , $[1, 1, 4, 4, 4, 4]$ <b>32</b> , $[2, 2, 2, 2, 1, 1]$ <b>15</b>
<b>11</b>	6	$[3, 3, 3, 3, 3, 3]$	-144	243	3	$[1, 1, 3, 3, 3, 3]$ <b>18</b>
<b>12</b>	6	$[2_2, 4, 4, 4, 4]$	-296	128	6	$[1, 4, 4, 4, 4]$ <b>34</b> , $[1, 2, 2, 2, 2]$ <b>27</b>
<b>13</b>	6	$[3_2, 3, 3, 3, 3]$	-204	81	6	$[1, 3, 3, 3, 3]$ <b>19</b>
<b>14</b>	6	$[3_2, 3_2, 3, 3]$	-120	27	9	
<b>15</b>	6	$[4_2, 4_2, 2, 2]$	-176	32	9	
<b>16</b>	6	$[6_3, 2, 2, 2]$	-204	24	13	
<b>17</b>	6	$[2_2, 4_4]$		8	28	
<b>18</b>	6	$[3_3, 3_3]$	-144	9	23	
<b>19</b>	6	$[3_6]$	-204	3	83	
<b>20</b>	7	$[4_2, 2, 2, 2, 2, 2]$		64	9	$[1, 2, 2, 2, 2]$ <b>17</b>
<b>21</b>	7	$[2_2, 4_2, 2, 2, 2]$	-176	32	12	
<b>22</b>	7	$[3_3, 2, 2, 2, 2]$	-204	24	16	$[1, 2, 2, 2, 2]$ <b>19</b>
<b>23</b>	7	$[2_2, 2_2, 3_3]$		12	22	
<b>24</b>	7	$[3_3, 2_4]$		38	6	
<b>25</b>	8	$[2, 2, 2, 2, 2, 2, 2, 2]$	-128	128	9	$[1, 1, 1, 1, 2, 2, 2, 2]$ <b>28</b>
<b>26</b>	8	$[2_2, 2_2, 2, 2, 2, 2]$	-176	32	15	$[1, 1, 2, 2, 2, 2]$ <b>32</b>
<b>27</b>	8	$[2_4, 2, 2, 2, 2]$	-296	16	31	$[1, 2, 2, 2, 2]$ <b>34</b>
<b>28</b>	8	$[2_2, 2_2, 2_2, 2_2]$	-128	16	21	
<b>29</b>	8	$[2_3, 2_3, 2, 2]$	-120	8	29	
<b>30</b>	8	$[2_5, 2, 2, 2]$	-288	8	49	
<b>31</b>	8	$[2_4, 2_2, 2_2]$		8	37	
<b>32</b>	8	$[2_4, 2_4]$	-176	4	53	
<b>33</b>	8	$[2_6, 2_2]$		4	76	
<b>34</b>	8	$[2_8]$	-296	2	164	

Some higher dimensional linear abelian Calabi-Yau orbifolds

	<i>degree</i>	<i>orbifold</i>
<hr/> $\mathbb{P}^4$ <hr/>		
$d = 6$	[2, 10, 10, 10, 10, 10]	[6, 6, 6, 6, 6, 6]
$d = 7$	[2, 2, 3, 6, 6, 6, 6]	[2, 4, 4, 4, 4, 4, 4]
$d = 10$	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2]	
<hr/> $\mathbb{P}^5$ <hr/>		
$d = 7$	[2, 7, 14, 14, 14, 14, 14]	[2, 12, 12, 12, 12, 12, 12]
		[3, 4, 12, 12, 12, 12, 12]
		[3, 9, 9, 9, 9, 9, 9]
		[4, 8, 8, 8, 8, 8, 8]
		[7, 7, 7, 7, 7, 7, 7]
$d = 8$	[2, 2, 6, 6, 6, 6, 6, 6]	[2, 3, 3, 6, 6, 6, 6, 6]
		[4, 4, 4, 4, 4, 4, 4, 4]
$d = 9$	[2, 2, 2, 3, 3, 3, 6, 6, 6]	[2, 2, 2, 4, 4, 4, 4, 4, 4]
		[3, 3, 3, 3, 3, 3, 3, 3, 3]
$d = 12$	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]	
<hr/> $\mathbb{P}^6$ <hr/>		
$d = 8$	[2, 14, 14, 14, 14, 14, 14, 14]	[8, 8, 8, 8, 8, 8, 8, 8]
$d = 9$	[2, 3, 6, 6, 6, 6, 6, 6, 6]	
$d = 10$	[2, 2, 2, 3, 3, 3, 6, 6, 6, 6]	
$d = 14$	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]	
<hr/> $\mathbb{P}^7$ <hr/>		
$d = 9$	[2, 9, 18, 18, 18, 18, 18, 18, 18]	[2, 16, 16, 16, 16, 16, 16, 16, 16]
		[3, 5, 15, 15, 15, 15, 15, 15, 15]
		[3, 12, 12, 12, 12, 12, 12, 12, 12]
		[4, 6, 12, 12, 12, 12, 12, 12, 12]
		[5, 10, 10, 10, 10, 10, 10, 10, 10]
		[9, 9, 9, 9, 9, 9, 9, 9, 9]
..	....	