ON BRANCHED COVERINGS OF $\mathbb{P}^n$ BY PRODUCTS OF DISCS

A. MUHAMMED ULUDAĞ
Galatasaray University, Department of Mathematics, 80840 Ortaköy, İstanbul, Turkey
muludag@gsu.edu.tr

For any $n > 1$, we construct examples of branched Galois coverings $M \to \mathbb{P}^n$ where M is one of $(\mathbb{P}^1)^n$, $\mathbb{C}^n$ and $(\mathbb{B}_1)^n$, where $\mathbb{B}_1$ is the 1-ball. In terms of orbifolds, this amounts to giving examples of orbifolds over $\mathbb{P}^n$ uniformized by M. We also discuss the related "orbifold braid groups".

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1. Introduction

In contrast with the considerable literature on the orbifolds over $\mathbb{P}^2$ uniformized by the 2-ball $\mathbb{B}_2$ (see [17], [8], [10] [16] and references therein), not much is known about which orbifolds over $\mathbb{P}^n$ are uniformized by the product of 1-balls $(\mathbb{B}_1)^n$. The aim of the present article is to construct some orbifolds over the projective space $\mathbb{P}^n$ uniformized either by $(\mathbb{P}^1)^n$, $\mathbb{C}^n$ or $(\mathbb{B}_1)^n$ and prove the following result.

Theorem 1.1. *Let (n, b) be a pair of coprime integers with $n \geq 2$. There exists a Galois covering $\big(D^{(b)}_{n,1}\big)^n \to \mathbb{P}^n$ of degree $n!b^{n^2-n}$ branched along an irreducible degree-$2b(n-1)$ hypersurface $D^{(b)}_n \subset \mathbb{P}^n$ where $D^{(b)}_{n,1} \subset D^{(b)}_n$ is a curve of euler number $e = b^{n-1}(n + 1 + b - nb)$.*

For $b = 1$, the hypersurface $D^{(1)}_n$ is the discriminant hypersurface, and $D^{(1)}_{n,1} \simeq \mathbb{P}^1$ is a rational normal curve. In this case one obtains the well-known branched Galois covering $(\mathbb{P}^1)^n \to \mathbb{P}^n$. The subvarieties $D^{(b)}_n$ and $D^{(b)}_{n,1}$ are the liftings respectively of $D^{(1)}_n$ and $D^{(1)}_{n,1}$ by an abelian branched self-covering $[Z_0, \dots, Z_n] \in \mathbb{P}^n \to [Z^b_0, \dots, Z^b_n] \in \mathbb{P}^n$. For $(n, b) \in \{(3, 2), (2, 3)\}$ one has $e\big(D^{(b)}_{n,1}\big) = 0$, and the universal covering of $\big(D^{(b)}_{n,1}\big)^n$ is $\mathbb{C}^n$. The curve $D^{(3)}_2 = D^{(3)}_{2,1}$ is a nine-cuspidal sextic, dual of a smooth cubic. For $b > 1$ and $(n, b) \notin \{(3, 2), (2, 3)\}$ one has $e\big(D^{(b)}_{n,1}\big) < 0$, and the universal covering of $\big(D^{(b)}_{n,1}\big)^n$ is $(\mathbb{B}_1)^n$.

In case $(n, b) = (2, 3)$, the claim of Theorem 1.1 was proved in [12]. The case $n = 2$ was established in [16]. In this case, $D^{(b)}_2$ coincides with $D^{(b)}_{n,1}$, which is a curve of genus $\frac{1}{2}(b^2-3b+2)$ with $3b$ cusps of type $x^2 = y^b$ and no other singularities, see Appendix for a proof. Irreducibility of $D^{(b)}_n$ is proved in Proposition 4.2. The remaining assertions of Theorem 1.1 are proved in Theorem 4.4. Our construction leads naturally to the definition of orbifold braid groups of the sphere $\mathbb{P}^1$ with punctures, which we dicuss in Section 4. These groups were already introduced by Allcock [1] in the "braid-picture" setting for some basic cases.

2. Orbifolds

Let M be a connected complex manifold, $G \subset \mathrm{Aut}(M)$ a properly discontinuous subgroup and put $N := M/G$. Then the projection $\phi : M \to N$ is a branched Galois covering endowing N with a map $\beta_\phi : N \to \mathbb{N}$ defined by $\beta_\phi(p) := |G_q|$ where q is a point in $\phi^{-1}(p)$ and G_q is the isotropy subgroup of G at q. In this setting, the pair (N, β_ϕ) is said to be uniformized by $\phi : M \to (N, \beta_\phi)$. An *orbifold* is a pair (N, β) of an irreducible normal analytic space N with a function $\beta : N \to \mathbb{N}$ such that the pair (N, β) is locally finitely uniformizable. A covering $\phi : (N', \beta') \to (N, \beta)$ of orbifolds is a branched Galois covering $N' \to N$ with $\beta' = (\beta \circ \phi)/\beta_\phi \circ \phi$. Note that the restriction $(N', 1) \to (N, \beta_\phi)$ is a uniformization of (N, β_ϕ). Conversely, let (N, β) and (N, γ) be two orbifolds with $\gamma|\beta$, and let $\phi : (N', 1) \to (N, \gamma)$ be a uniformization of (N, γ), e.g. $\beta_\phi = \gamma$. Then $\phi : (N', \beta') \to (N, \beta)$ is a covering, where $\beta' := \beta \circ \phi / \gamma \circ \phi$. The orbifold (N', β') is called the *lifting of* (N, β) *to the uniformization* N' *of* (N, γ).

Let (N, b) be an orbifold, $B_\beta := \mathrm{supp}(\beta - 1)$ and let $B_1, \dots, B_n$ be the irreducible components of B_β. Then β is constant on $B_i \backslash \mathrm{sing}(B_\beta)$; so let b_i be this number. The *orbifold fundamental group* $\pi_1^{orb}(N, \beta)$ of (N, β) is the group defined by $\pi_1^{orb}(N, \beta) := \pi_1(N \backslash B_\beta)/\langle\langle \mu_1^{b_1}, \dots, \mu_n^{b_n} \rangle\rangle$ where $\mu_i^{b_i}$ is a meridian of B_i and $\langle\langle\rangle\rangle$ denotes the normal closure. An orbifold (N, β) is said to be *smooth* if N is smooth. In case (N, β) is a smooth orbifold the map β is determined by the numbers b_i; in fact $\beta(p)$ is the order of the local orbifold fundamental group at p. Since the orbifolds to be considered in this article are exclusively smooth, we shall adopt the convention that such orbifolds are defined to be the pairs (N, B) where $B := b_1 B_1 + \cdots + b_n B_n$ is a divisor with $b_i \geq 1$. We shall also allow b_i to take infinite values, meaning that the corresponding hypersurface B_i is removed from the base space N. If $\mathcal{O} := (N, B)$ is an orbifold and C a hypersurface in N, then we shall use the notation $(\mathcal{O}, bC)$ to denote the orbifold $(N, B + bC)$.

3. Discriminants

For a recent treatment of discriminant varieties, see Katzs' article [13] or [9]. Let $n \geq 1$ be an integer and consider the action of the symmetric group Σ_n on $(\mathbb{P}^1)^n$. Let $p_i = [u_i, v_i] \in \mathbb{P}^1$ and let σ_j $(j \in [0, n])$ be the homogeneous elementary symmetric polynomial

$$\sigma_j(p_1, \dots, p_n) := \sum_{A \subset [1,n],\, |A| = j} \left(\prod_{\alpha \in A} x_\alpha \prod_{\beta \in [1,n] \backslash A} y_\beta \right)$$

It is well known that the map $\phi_n : (\mathbb{P}^1)^n \to \mathbb{P}^n$ given by

$$\phi_n : (p_1, \dots, p_n) := [\sigma_0(p_1, \dots, p_n) : \cdots : \sigma_n(p_1, \dots, p_n)]$$

is Σ_n- invariant and gives an isomorphism $(\mathbb{P}^1)^n / \Sigma_n \simeq \mathbb{P}^n$.

Let $\pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$ be the *ith* projection map, q a point in $\mathbb{P}^1$, and put $F_q^i := \pi_i^{-1}(q)$. Let $\tau_{ij} \in \Sigma_n$ be the transposition exchanging the ith and jth coordinates of $(p_1, \dots, p_n) \in (\mathbb{P}^1)^n$. Since $\tau_{1i} F_q^1 = F_q^i$, the hypersurface $H_q := \phi_n(F_q^i)$ does not depend on i.

Lemma 3.1. *For any $q \in \mathbb{P}^1$, the hypersurface H_q is a hyperplane in $\mathbb{P}^n$. For any set $\{q_0, \dots, q_m\} \subset \mathbb{P}^1$ of distinct points, the hyperplanes $H_{q_0}, \dots, H_{q_m}$ are in general position.*

Proof. Suppose without loss of generality that $i = 1$. Then H_q is parametrized as $H_q = [X_0 : X_1 : \cdots : X_n] \in \mathbb{P}^n$, where $X_j = \sigma_j(q, p_2, \dots, p_n)$ and $p_i \in \mathbb{P}^1$ $(i \in [2, n])$. If $q = [u_1 : v_1] = [x : y]$ and $p_i = [u_i : v_i]$ $(i \in [2, n])$ then one has the identity

$$P(A, B) := \sum_{j \in [0,n]} (-1)^{n-j} \sigma_j(q, p_2, \dots, p_n) A^j B^{n-j} = \prod_{i \in [1,n]} (u_i A - v_i B) \tag{3.1}$$

Substitute $[A : B] = [y : x]$ in (3.1). Since the right-hand side of (3.1) vanish at the point $(q, p_2, \dots, p_n)$, so does the middle term, and thus H_q satisfies the linear equation

$$\sum_{j \in [0,n]} (-1)^{n-j} y^j x^{n-j} X_j = 0 \tag{3.2}$$

Let $\{q_i = [x_i : y_i] : i \in [0, n]\}$ be a set of $n + 1$ points. Since the determinant of the projective Vandermonde matrix $\mathcal{V}an(q_0, \dots, q_n)$ given by

$$\mathcal{V}an_{i,j}(q_0, \dots, q_n) := (-1)^{n-j} y_i^j x_i^{n-j} \quad i, j \in [0, n]$$

vanish if and only if $q_i = q_j$ for some $i, j \in [0, n]$, the hyperplanes $H_{q_0}, \dots, H_{q_n}$ are always in general position. □

The hypersurface $\Delta_n := \{(p_1, \dots, p_n) \in (\mathbb{P}^1)^n : p_i = p_j \text{ for some } 1 \le i \ne j \le n\}$ of $(\mathbb{P}^1)^n$ consists of points fixed by an element of Σ_n, so that the covering ϕ_n is branched along the hypersurface $D_n := \phi(\Delta_n)$, which is called the *discriminant hypersurface* since it is defined by the discriminant of the homogeneous polynomial $P(A, B)$. In terms of orbifolds, this means that there is an orbifold covering

$$\phi_n : \big((\mathbb{P}^1)^n, a\Delta_n\big) \to (\mathbb{P}^n, 2aD_n) \tag{3.3}$$

Let $\{q_0, \dots, q_m\} \subset \mathbb{P}^1$ be $m + 1$ distinct points, $b_0, \dots, b_m$ numbers in $\mathbb{N} \cup \{\infty\}$ and consider the orbifold

$$\mathcal{F}(b_0, \dots, b_m) := (\mathbb{P}^1, b_0 q_0 + \cdots + b_m q_m)$$

Let $n \ge 1$ be an integer and consider the orbifold $\mathcal{F}(b_0, \dots, b_m)^n$. Let $\mathcal{G}_n$ be the orbifold

$$\mathcal{G}_n(a; b_0, \dots, b_m) := \big(\mathcal{F}(b_0, \dots, b_m)^n, a\Delta_n\big)$$

and define the orbifold $\mathcal{H}_n(a; b_0, \dots, b_m)$ as

$$\mathcal{H}_n(a; b_0, \dots, b_m) := (\mathbb{P}^n, aD_n + b_0 H_{q_0} + \cdots + b_m H_{q_m})$$

By the covering in (3.3) and Lemma 3.1 one has the fact

Lemma 3.2. *There is an orbifold covering of degree $n!$*

$$\phi : \mathcal{G}_n(a; b_0, \dots, b_m) \to \mathcal{H}_n(2a; b_0, \dots, b_m)$$

In particular, for $a = 1$ one has the orbifold covering

$$\phi : \mathcal{F}(b_0, \dots, b_m)^n \simeq \mathcal{G}_n(1; b_0, \dots, b_m) \to \mathcal{H}_n(2; b_0, \dots, b_m)$$

The following facts are well known (see [15]):

Theorem 3.3. *[Bundgaard-Nielsen,Fox] The orbifold $\mathcal{F}(b_0,\dots,b_m)$ admits a finite uniformization if $n>1$, $b_i<\infty$ $(1\le i\le m)$ and if $n=2$, then $b:=b_0=b_1$. Let $R\to\mathcal{F}(b_0,\dots,b_m)$ be a finite uniformization.*
(i) $R\simeq\mathbb{P}^1$ if $n=2, b_0=b_1<\infty$ or $n=3$, $b_0^{-1}+b_1^{-1}+b_2^{-1}>1$. In this case , $\mathbb{P}^1$ is also the universal uniformization. The groups $\pi_1^{orb}\big(\mathcal{F}(b,b)\big)$ and $\pi_1^{orb}\big(\mathcal{F}(b_0,b_1,b_2)\big)$ are finite of orders b and $2\big[b_0^{-1}+b_1^{-1}+b_2^{-1}-1\big]^{-1}$ respectively.
(ii) R is of genus 1 if $n=3$, $b_0^{-1}+b_1^{-1}+b_2^{-1}=1$ or $n=4$, $b_0=b_1=b_2=b_3=2$. Hence, $\mathbb{C}$ is the universal uniformization of these orbifolds. Moreover, $\mathcal{F}(\infty,\infty)$ and $\mathcal{F}(2,2,\infty)$ are uniformized by $\mathbb{C}$. The corresponding orbifold fundamental groups are infinite solvable.
(iii) R is of genus >1 otherwise, and the universal uniformization is $(\mathbb{B}_1)^n$, where $\mathbb{B}_1$ is the unit disc in $\mathbb{C}$. The corresponding orbifold fundamental groups are big (i.e. they contain non-abelian free subgroups).

In virtue of the covering $\phi:\mathcal{F}(b_0,\dots,b_m)^n\to\mathcal{H}_n(2;b_0,\dots,b_m)$ one has the result

Corollary 3.4. *Let $n>1$, $b_i<\infty$ $(1\le i\le m)$ and if $n=2$, then $b_0=b_1$. Then the orbifold $\mathcal{H}_n(2;b_0,\dots,b_m)$ admits a finite uniformization by R^n, where R is the uniformization of $\mathcal{F}(b_0,\dots,b_m)$ given in Theorem 3.3. The orbifolds $\mathcal{H}(2;\infty,\infty)$ and $\mathcal{H}(2;2,2,\infty)$ are uniformized by $\mathbb{C}^n$. Moreover, $\pi_1^{orb}\big(\mathcal{H}(2;b,b)\big)$ is a finite group of order $n!b^n$ and $\pi_1^{orb}\big(\mathcal{H}(2;b_0,b_1,b_2)\big)$ is a finite group of order $n!2^n\big[b_0+b_1+b_2-1\big]^{-n}$ if $b_0^{-1}+b_1^{-1}+b_2^{-1}>1$.*

4. Another covering of $\mathcal{H}(a;b_0,\dots,b_m)$

Let $b\in\mathbb{N}$ be an integer and consider the orbifold $\mathcal{K}_n(b):=(\mathbb{P}^n,bH_{q_0}+\dots+bH_{q_n})$. By Lemma 3.1, the hyperplanes $H_{q_0},\dots,H_{q_n}$ are in general position. It is well known that the universal uniformization of this orbifold is $\mathbb{P}^n$. Applying a projective transformation one may assume that the hyperplanes H_{q_i} are given by the equations $Y_i=0$ where $[Y_0:\dots:Y_n]\in\mathbb{P}^n$. In this case the uniformization $\psi_b:\mathbb{P}^n\to\mathcal{K}_n(b)$ is nothing but the map

$$[Y_0:\dots:Y_n]=\psi_b([Z_0:\dots:Z_n])=[Z_0^b:\dots:Z_n^b]$$

It is clear that the orbifold $\mathcal{H}_n(a;bb_0,\dots,bb_n,b_{n+1},\dots,b_m)$ lifts to the uniformization of $\mathcal{K}_n(b)$. Put $D_n^{(b)}:=\psi_b^{-1}(D_n)$, denote $M_{q_i}:=\psi^{-1}(H_{q_i})$ and define the orbifold

$$\mathcal{L}_n^{(b)}(a;b_0,\dots,b_m):=(\mathbb{P}^n,aD_n^{(b)}+b_0M_{q_0}+\dots b_nM_{q_m})$$

to be this lifting. In case $n=2$ these liftings were studied in [16]. For $n>2$ the following proposition is valid:

Proposition 4.1. *For $n>2$ and $b\ge2$ the orbifolds $\mathcal{L}_n^{(b)}(2;b_0,\dots,b_m)$ are uniformized by $(\mathbb{B}_1)^n$ except the orbifold $\mathcal{L}_3^{(2)}(2)$, which is uniformized by $\mathbb{C}^3$.*

Proof. There is an orbifold covering $\mathcal{L}_n^{(b)}(2;b_0,\dots,b_m)\longrightarrow\mathcal{H}_n(a;bb_0,\dots,bb_n)$. The claim follows, since by Corollary 3.4 the latter orbifold is uniformized by $\mathbb{C}^3$ if $b=2$, $n=3$, $b_0=\dots=b_n=1$ and by $(\mathbb{B}_1)^n$ otherwise. □

For $k\in[1,n]$, define the k-dimensional subvarietiy $\Delta_{n,k}$ of Δ_n by

$$\Delta_{n,k}:=\{(p_1,p_2,\dots,p_n)\in(\mathbb{P}^1)^n\ :\ p_k=p_{k+1}=\dots=p_n\}\simeq(\mathbb{P}^1)^k$$

Thus, $\Delta_{n,n-1}$ is an irreducible component of Δ_n and $\Delta_{n,1}$ is the diagonal in $(\mathbb{P}^1)^n$. The subgroup of Σ_n acting on $\Delta_{n,k}$ is a symmetric group Σ_{k-1}, so that $D_{n,k} := \mathbb{P}^1 \times \mathbb{P}^{k-1}$. These varieties admits the parametrizations

$$D_{n,k} : [X_0 : \cdots : X_n] \in \mathbb{P}^n \quad X_j = \sigma_j(p_1, \ldots, p_n), \quad p_k = \cdots = p_n \tag{4.1}$$

In particular, the curve $D_{n,1}$ is a rational normal curve parametrized as

$$\left[\binom{n}{0} v^n, \binom{n}{1} uv^{n-1}, \ldots, \binom{n}{n} u^n\right] \qquad ([u : v] \in \mathbb{P}^1)$$

Applying the projective transformation $\mathcal{V}an(q_0, \ldots, q_n)$ to the parametrizations (4.1) gives the parametrization $D_{n,k} : [Y_0 : \cdots : Y_n] \in \mathbb{P}^n$, where

$$\sum_{j\in[0,n]} (-1)^{n-j} y_i^j x_i^{n-j} \sigma_j(p_1, \ldots, p_n), \quad p_k = \cdots = p_n \tag{4.2}$$

Let $p_i = [u_i : v_i]$ and let $[u : v] = [u_k : v_k] = \cdots = [u_n : v_n]$. In virtue of the identity (3.1) one has the parametrizations $D_{n,k} : [Y_0 : \cdots : Y_n] \in \mathbb{P}^n$ where

$$Y_j = (uy_j - vx_j)^{n-k+1} \prod_{i\in[1,k-1]} (u_i y_j - v_i x_j) \tag{4.3}$$

In particular, the curve $D_{n,1}$ is parametrized as

$$D_{n,1} : \left[(uy_0 - vx_0)^n : \cdots : (uy_n - vx_n)^n\right] \tag{4.4}$$

The varieties $D^{(b)}_{n,k}$ are parametrized as

$$D^{(b)}_{n,k} : [Z_0 : \cdots : Z_n] \quad Z_j^b = (uy_j - vx_j)^{n-k+1} \prod_{i\in[1,k-1]} (u_i y_j - v_i x_j) \tag{4.5}$$

Note that the parametrizations (4.3) and (4.5) are not generically one-to-one unless $k \leq 2$, since (4.3) is a map $(\mathbb{P}^1)^k \to D_{n,k}$.

Proposition 4.2. *(i) The curve $D^{(b)}_{n,1}$ is irreducible if and only if* $\gcd(n, b) = 1$. *Hence, the subvarieties $D^{(b)}_{n,k}$ are irreducible if* $\gcd(n, b) = 1$.

Definition 4.3. Let $t \in \mathbb{Z}$ and ψ_t be the map

$$\psi_t : [Z_0 : \cdots : Z_n] \in \mathbb{P}^n \to [Z_0^t : \ldots Z_n^t] \in \mathbb{P}^n$$

Let $V \subset \mathbb{P}^n$ be a subvariety and $r, s \in \mathbb{Z}$ such that $s > 1$. Then $V^{(r/s)}$ is the subvariety of $\mathbb{P}^n$ defined as

$$V^{(r/s)} := (\psi_r^{-1} \circ \psi_s)(V)$$

In particular, $V^{(r/r)}$ is the orbit of V under the $(\mathbb{Z}/(r))^n$-action on $\mathbb{P}^n$.

Proof of the Proposition. The parametrization (4.4) shows that $D_{n,1} \simeq L^{1/n}$, where L is a line $\subset \mathbb{P}^n$ in general position with respect to ψ_n, in other words L intersects the hyperplane arrangement $Z_0 \ldots Z_n = 0$ transversally at smooth points. Hence there is a surjection of fundamental groups

$$\pi_1(L\backslash\{\tilde{q}_0, \ldots, \tilde{q}_n\}) \twoheadrightarrow \pi_1(\mathbb{P}^n\backslash\{Z_0, \ldots, Z_n\}) \tag{4.6}$$

where $\tilde{q}_i := Z_i \cap L$. Let $\mathcal{M}(b)$, $\mathcal{K}(b)$ be the orbifolds

$$\mathcal{M}(b) := (L, b\tilde{q}_0 + \cdots + b\tilde{q}_n), \quad \mathcal{K}(b) := (\mathbb{P}^n, bZ_0 + \cdots + bZ_n)$$

Then (4.6) induce a surjection of orbifold fundamental groups

$$\pi_1^{orb}\big(\mathcal{M}(b)\big) \twoheadrightarrow \pi_1^{orb}\big(\mathcal{K}(b)\big)$$

(one may say: $\mathcal{M}(b)$ is a sub-orbifold of $\mathcal{K}(b)$). This shows that the curve $L^{(b)}$ is irreducible and is a uniformization of $\mathcal{M}(b)$. Since $\gcd(n, b) = 1$, one has $D^{(b)}_{n,1} = L^{(b/n)}$, showing that $D^{(b)}_{n,1}$ is irreducible. Note that $D^{(b)}_{n,1}$ is the maximal abelian orbifold covering of $\mathcal{M}(b)$. Irreducibility of $D^{(b)}_{n,k}$ follows since $D^{(b)}_{n,1}$ is a subvariety of $D^{(b)}_{n,k}$. □

Let $\mathcal{O}(b)$ be the orbifold $\mathcal{O}(b) := (D_{n,1}, b\bar{q}_0 + \cdots + b\bar{q}_n)$, where $\bar{q}_i := Y_i \cap D_{n,1}$. The orbifold $\mathcal{O}(b)$ is identified via the covering ϕ with the orbifold $\mathcal{P}(b) := (\Delta_{n,1}, bq'_0 + \cdots + bq'_n)$, where this time $q'_i := \phi^{-1}(\bar{q}_i)$. In turn, $\mathcal{O}(b)$ is identified with the orbifold $\mathcal{F}(b, \ldots, b)$ via the coordinate projection. By the proof of Proposition 4.2, these orbifolds are identified with the orbifold $\mathcal{M}(b)$ in case $(n, b) = 1$.

Theorem 4.4. *Let* $\gcd(n, b) = 1$. *Then there is a finite uniformization* $\xi_n : (D^{(b)}_{n,1})^n \longrightarrow \mathcal{L}^{(b)}_n(2)$ *which is of degree* $n!b^{n^2-n}$.

Proof. One has the diagram

$$\begin{array}{ccc} \mathcal{L}^{(b)}_n(2) & \overset{\xi_n}{\longleftarrow} & (D^{(b)}_{n,1})^n \\ \psi_b \big\downarrow & & \big\downarrow \zeta_b \\ \mathcal{H}_n(2; b, \ldots, b) & \overset{\phi_n}{\longleftarrow} & \mathcal{O}(b)^n \end{array}$$

where $\zeta_b : (D^{(b)}_{n,1})^n \to \mathcal{O}(b)^n$ is the maximal abelian orbifold covering and ξ_n is to be shown to be a branched Galois covering of degree $n!b^{n^2-n}$. It suffices to show that the group $H := (\phi_n \circ \zeta_b)_* \pi_1\big((D^{(b)}_{n,1})^n\big)$ is a normal subgroup of $K := (\psi_b)_* \pi_1^{orb}\big(\mathcal{L}^{(b)}_n(2)\big)$. Let σ be a meridian of D_n. Then since $\pi_1^{orb}\big(\mathcal{H}_n(2, b, \ldots, b)\big)/\langle\langle\sigma\rangle\rangle \simeq \pi_1^{orb}\big(\mathcal{K}_n(b)\big) \simeq \big(\mathbb{Z}/(b)\big)^n$ is the Galois group of ψ_b, the group K is the normal subgroup of $\pi_1^{orb}\big(\mathcal{H}_n(2, b, \ldots, b)\big)$ generated by σ, i.e. $K \simeq \langle\langle\sigma\rangle\rangle$. The group $\pi_1^{orb}\big(\mathcal{H}_n(2, b, \ldots, b))\big)/K$ being abelian, one has $[\tau_i, \tau_j] \in K$ for $i, j \in [0, n]$. On the other hand one has

$$\pi_1^{orb}\big(\mathcal{H}_n(2; b, \ldots, b)\big)/\langle\langle\tau_0, \ldots, \tau_n\rangle\rangle \simeq \pi_1^{orb}\big(\mathcal{H}_n(2)\big) \simeq \Sigma_n$$

Since Σ_n is the Galois group of ϕ_n, one has $\phi_* \mathcal{O}(b)^n \simeq \langle\langle\tau_0, \ldots, \tau_n\rangle\rangle$. Since ζ_n is the maximal abelian orbifold covering, one has $H \simeq \langle\langle[\tau_i, \tau_j]\rangle\rangle$. This shows that H is a normal subgroup of K. Since $\deg(\zeta_b) = b^{n^2}$, $\deg(\phi_n) = n!$ and $\deg(\psi_b) = b^n$, one has

$$\deg(\xi_n) = \frac{\deg(\zeta_b)\deg(\phi_n)}{\deg(\psi_b)} = n!b^{n^2-n}$$

The euler number of $D^{(b)}_{n,1}$ is easily computed by the Riemann-Hurwitz formula. □

5. Braid Groups

Following and generalizing Allcock [1], let us call the groups

$$\mathbf{P}_n(a; b_0, \dots, b_m) := \pi_1^{orb}\big(\mathcal{G}_n(a; b_0, \dots, b_m)\big)$$

the *pure braid groups of* $\mathcal{F}(b_0, \dots, b_m)$ *on* n *strands*, and the groups

$$\mathbf{B}_n(a; b_0, \dots, b_m) := \pi_1^{orb}\big(\mathcal{H}_n(a; b_0, \dots, b_m)\big)$$

the *braid groups of* $\mathcal{F}(b_0, \dots, b_m)$ *on* n *strands*. Obviously, the group $\mathbf{B}_n(a; b_0, \dots, b_m)$ is a quotient of $\mathbf{B}_n(a'; b'_0, \dots, b'_m)$ provided $a|a'$ and $b_i|b'_i$ for $0 \le i \le n$. The group $\mathbf{B}_n(a; b_0, \dots, b_m)$ is a subgroup of $\mathbf{B}_{n+k}(a; b_k, \dots, b_m)$ in case the equality $a = b_0 = \dots = b_{k-1}$ holds. The group $\mathbf{B}_n(2a; b_0, \dots, b_m)$ is a normal subgroup of index $n!$ in the group $\mathbf{P}_n(a; b_0, \dots, b_m)$. The group $\mathbf{B}_n(a; b_0, \dots, b_m)$ admits the presentation (see [2] for the case $n = 2$ and [4], [5], [14] for the general case)

(1) *generators:* $\sigma_1, \dots, \sigma_{n-1}, \tau_0, \dots, \tau_m$
(2) *braid relations:* $[\sigma_i, \sigma_j] = 1$, $|i - j| > 1$,
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \le i \le n-1$
(3) *mixed relations* $(\sigma_1\tau_i)^2 = (\tau_i\sigma_1)^2$, $1 \le i \le m$,
$[\tau_i, \sigma_j] = 1$, $j \ne 1$, $1 \le i \le m$
$[\sigma_1\tau_i\sigma_1^{-1}, \tau_j] = 1$, $1 \le i < j \le m$
(4) *projective relation*[1] $\sigma_1\sigma_2 \dots \sigma_{n-1}\tau_0 \cdots \tau_m\sigma_{n-1} \dots \sigma_2\sigma_1 = 1$
(5) *orbifold relations* $\tau_0^{b_0} = \dots = \tau_m^{b_m} = \sigma_1^a = 1$

In particular, the group $\mathbf{B}_n(\infty; \infty)$ is the usual braid group of $\mathbb{C}$ introduced by Artin [3]. The group $\mathbf{B}_n(\infty) \simeq \mathbf{B}_n(\infty; 1)$ is the *braid group of the sphere*, see [18]. On the other hand, one has

$$\mathbf{B}_1(b_0, \dots, b_m) \simeq \langle \tau_0, \dots, \tau_m \,|\, \tau_0^{b_0} = \dots = \tau_m^{b_m} = \tau_0 \cdots \tau_m = 1\rangle$$

In case $n = 2$, the discriminant hypersurface $D_2^{(1)}$ is a smooth quadric, and the lines H_{q_i} are tangent to $D_2^{(1)}$ (see [16]). In particular, the groups $\mathbf{B}_2(a; b)$ are abelian. The group $\mathbf{B}_2(a; b, c)$ admits the presentation

$$\mathbf{B}_2(a; b, c) \simeq \langle \tau, \sigma \,|\, (\tau\sigma)^2 = (\sigma\tau)^2, \quad \tau^b = (\tau\sigma^2)^c = \sigma^a = 1\rangle \tag{5.1}$$

Proposition 5.1. *For* $b, c < \infty$*, the group* $\mathbf{B}_2(a; b, c)$ *is a finite central extension of the triangle group* $T_{2,a,d} : \langle \tau, \sigma \,|\, (\tau\sigma)^2 = \tau^d = \sigma^a = 1\rangle$*, where* $d := \gcd(b, c)$*. Hence,* $\mathbf{B}_2(a; b, c)$ *is finite* $1/a + 1/b > 1/2$*, infinite almost solvable if* $1/d + 1/a = 1/2$*, and big otherwise (i.e. it contains non-abelian free subgroups). The group* $\mathbf{B}_2(a; b, b)$ *is of order* $2b\big[a^{-1} + b^{-1} - 2^{-1}\big]^{-1}$ *if* $1/a + 1/b > 1/2$*.*

Proof. Note that $\delta := (\tau\sigma)^2$ is central in $\mathbf{B}_2(a; b, c)$, so that $(\tau\sigma^2)^c = 1 \Leftrightarrow (\sigma\tau\sigma)^c = 1 \Leftrightarrow (\tau^{-1}\delta)^c = \tau^{-c}\delta^c = 1$. The element δ is of finite order. Adding the relation $\delta = 1$ to the presentation (5.1) yields the triangle group $T_{2,a,d}$, which is finite if $1/a + 1/d > 1/2$, infinite solvable if $1/a + 1/d = 1/2$, and big otherwise. In case $c = b$, one has $d = b$ and the triangle group is of order $2\big[a^{-1} + b^{-1} - 2^{-1}\big]^{-1}$ if $1/a + 1/b > 1/2$, which shows that $\mathbf{B}(a; b, b)$ is of order $2b\big[a^{-1} + b^{-1} - 2^{-1}\big]^{-1}$. □

[1]The projective relation was kindly communicated by Paolo Bellingeri.

Let R^n be a uniformization of the orbifold $\big(\mathcal{F}(b_0,\dots,b_m)\big)^n$. If $k \geq m$ then any orbifold $\mathcal{G}_n(2a; c_0b_0, \dots c_mb_m, c_{m+1}, \dots, c_k)$ can be lifted to R^n. In case $R \simeq \mathbb{P}^1$ or $R \simeq \mathbb{C}$ one obtains some arrangements associated to reflection groups as follows. Suppose that $q_0 = [0:1]$ and $q_1 = [1:0]$. Lifting $\mathcal{G}_n(2a; cb, \infty)$ to the uniformization of $\mathcal{G}_n(2; b, b)$ yields the orbifold $(\mathbb{C}^n, a\Delta_n^{(b)} + cF)$ where $F := \{(X_1,\dots,X_n) \in \mathbb{C}^n : X_1 \cdots X_n = 0\}$ and $\Delta_n^{(b)}$ is the lifting of the superdiagonal

$$\Delta_n^{(b)} := \{(X_1,\dots,X_n) \in \mathbb{C}^n \,:\, \psi_b(p_i) = \psi_b(p_j) \text{ for some } 1 \leq i \neq j \leq n\}$$

with $\psi_b(X) = X^b$ if $b < \infty$ and $\psi_\infty(X) = \exp(2\pi \mathrm{i} X)$. Setting $b = 2$ in this construction identifies the group $\mathbf{B}_n(\infty; \infty, \infty)$ with the Artin group corresponding to the diagrams B_n (see [1]).

The groups $\mathbf{B}_2(a; b, c, d)$ admits the simplified presentation (see [16])

$$\mathbf{B}_2(a;b,c,d) \simeq \left\langle \tau, \rho, \sigma \;\middle|\; \begin{array}{l} (\tau\sigma)^2 = (\sigma\tau)^2,\ (\rho\sigma)^2 = (\sigma\rho)^2,\ [\rho,\tau] = 1, \\ \tau^b = (\sigma\tau\sigma\rho)^d = \rho^c = \sigma^a = 1 \end{array} \right\rangle$$

We summarized the known information about the orbifolds $\mathcal{H}$ and the corresponding braid groups in Table 1 below. Suppose that if $(n,m) = (2,1)$ then $b_0 = b_1$. We believe that the group $\mathbf{B}_n(a; b_0, \dots, b_m)$ is finite if

$$\frac{2(n-1)}{a} + \sum_{i \in [0,m]} \frac{1}{b_i} > n + m - 2,$$

at most infinite solvable if the equality holds, and big otherwise.

6. Remarks

Consider the restriction of $D_{n,k}$ to the $n-k+1$ dimensional linear subspace $M_{n-k+1} := \{[Y_0 : \cdots : Y_n] \in \mathbb{P}^n \,|\, Y_{n-k+2} = \cdots = Y_n = 0\}$ of $\mathbb{P}^n$. Setting $[u:v] = [x_n : y_n]$ and $[u_i : v_i] = [x_{n-i} : y_{n-i}]$ for $i \in [1, k-2]$ in (4.3) we see that $D_{n,k}$ has a 1-dimensional linear component L in $M_{n-k+1} \simeq \mathbb{P}^{n-k+1}$, parametrized as $[Y_0 : \cdots : Y_{n-k+1}] \in M_{n-k+1}$ where

$$Y_l = (u_n y_l - v_n x_l)(x_n y_l - y_n x_l)^{n-k+1} \prod_{i \in [2,k-1]} (x_{n-i} y_l - y_{n-i} x_l)$$

for $l \in [0, n-k+1]$ and $[u_n : v_n] \in \mathbb{P}^1$. It is readily seen that there are $k-1$ such lines. In case $[u_i : v_i] = 0$ for $i \in [1, k-1]$, one has the curve C in $D_{n,k} \cap M_{n-k+1}$ parametrized as $[Y_0 : \cdots : Y_{n-k+1}] \in M_{n-k+1}$ where

$$Y_l = (u y_l - v x_l)^{n-k+1} \prod_{i \in [1,k-1]} (x_{n-i} y_l - y_{n-i} x_l)$$

for $l \in [0, n-k+1]$ and $[u:v] \in \mathbb{P}^1$, which shows that C is the curve $E^{(1/n-k+1)}$ for some line E in $\mathbb{P}^{n-k+1}$. The lines L are tangent to C with multiplicity $n-k+1$. In case $k = n-1$, one has $M_{n-k+1} \simeq \mathbb{P}^2$, and one obtains an arrangement of a quadric C with $n-2$ tangent lines. The lines $Y_0 = 0$, $Y_1 = 0$ and $Y_2 = 0$ are also tangent to this quadric.

From these considerations it is easy to obtain a description of the intersection of $D_{n,k}^{(b)}$ with $\mathbb{P}^{n-k+1} \simeq Z_{n-k+2} = \cdots = Z_n = 0$. For $D_{3,2}^{(2)}$, this is the arrangement of a quadric with four tangent lines.

Let $H \subset \mathbb{P}^n$ be a hyperplane. The intersection $H^{(1/2)} \cap M_2$ is a quadric, tangent to the lines $Y_0 = 0$, $Y_1 = 0$ and $Y_2 = 0$, which is very similar to the intersections

Orbifold	**Uniform.**	**Braid group**	**Ref.**
$\mathcal{H}_n(2)$	$(\mathbb{P}^1)^n$	$n!$	Cor. 3.4
$\mathcal{H}_n(2;b,b)$	$(\mathbb{P}^1)^n$	$n!b^n$	Cor. 3.4
$\mathcal{H}_n(2;b,c,d)$ $\big(1/b+1/c+1/d>1\big)$	$(\mathbb{P}^1)^n$	$n!2^n\big[\frac{1}{b}+\frac{1}{c}+\frac{1}{d}-1\big]^{-n}$	Cor. 3.4
$\mathcal{H}_n(2;b,c,d)$ $\big(1/b+1/c+1/d=1\big)$	$\mathbb{C}^n$	Crystallographic	Cor. 3.4
$\mathcal{H}_n(2;2,2,2,2)$	$\mathbb{C}^n$	Crystallographic	Cor. 3.4
$\mathcal{H}_n(2;b_0,\ldots,b_m)$ (otherwise)	$(\mathbb{B}_1)^n$	Linear	Cor. 3.4
$\mathcal{H}_n(\infty;\infty,\infty)$	-	B_n-Artinian	[6]
$\mathcal{H}_2(\infty;\infty,\infty,\infty)$	-	$\widetilde{C}_2$-Artinian	[6]
$\mathcal{H}_2(a;b,b)$ $(1/a+1/b>1/2)$	-	$2b\big[\frac{1}{a}+\frac{1}{b}-\frac{1}{2}\big]^{-1}$	Prop. 5.1
$\mathcal{H}_2(a;b,b)$ $(1/a+1/b=1/2)$	-	∞ almost solvable	Prop. 5.1
$\mathcal{H}_3(a;\infty)$ $(a=3,4,5)$	-	24, 96, 600	[7]
$\mathcal{H}_n(3;\infty)$ $(n=4,5)$	-	648, 155520	[7]
$\mathcal{H}_3(\infty;2)$	-	192	Maple
$\mathcal{H}_4(a)$ $(a=4,5)$	-	192, 60	Maple
$\mathcal{H}_5(4)$	-	120	Maple
$\mathcal{H}_2(a;2,2,2)$	K3 $(a=4)$	$4a^3$	[16]
$\mathcal{H}_2(3;3,2,2)$	K3	576	[16]
$\mathcal{H}_2(3;3,4,4)$ $\mathcal{H}_2(4;4,4,4)$ $\mathcal{H}_2(3;6,6,2)$ $\mathcal{H}_2(3;3,3,6)$	$\mathbb{B}_2$	Picard Modular	[11],[16]
$\mathcal{H}_2(3;3,4,2)$ $\mathcal{H}_2(6;3,3,2)$	$\mathbb{B}_1\times\mathbb{B}_1$?	Unknown	[16]

TABLE 1

$D_n\cap M_2$. In contrast with this, there is the following fact: In a recent article [13], it was proved that the dual of D_n is one dimensional (we believe that $D_{n,k}$ and $D_{n,n-k}$ are duals), whereas it is easy to show that $H^{(r/s)}$ and $H^{(r/r-s)}$ are duals, so that the dual of $H^{(1/2)}$ is the degree-$(n-1)$ hypersurface $H^{(-1)}$. Note also that D_n is of degree $2(n-1)$, whereas $H^{(1/2)}$ is of degree 2^{n-1}. It is of interest to know more about the varieties $D^{(r/s)}_{n,k}$ and their duals.

7. Appendix: The curves $L^{(r/s)}$

In $\mathbb{P}^2$, many interesting curves appears as $L^{(r/s)}$, where L is a line. For example, $L^{(1/2)}$ is the curve $D_{2,1}$, a quadric tangent to the coordinate lines, $L^{(3/2)}\simeq D^3_{2,1}$ is a nine cuspidal sextic, $L^{(2/3)}$ is a Zariski sextic with 4 nodes and 6 cusps, $L^{(-1/2)}\simeq D^{-1}_{2,1}$ is a three cuspidal quartic, $L^{(-1)}$ is a quadric passing through the intersection points of the coordinate lines.

Proposition 7.1. *If $r,s\geq 0$ are coprime integers , then $L^{(r/s)}$ is an irreducible curve of degree sr and genus $(r-1)(r-2)/2$, with $3r$ points of type $x^r=y^s$ and $r^2(s-1)(s-2)/2$ nodes.*

Proof. We begin by proving that the curves $L^{(1/s)}$ are nodal. For this, it suffices to show that the orbit of L under the action of the group $\mathbb{Z}/(s)\oplus\mathbb{Z}/(s)$ has only double points on $\mathbb{P}^2\backslash\{xyz=0\}$. If $\omega := e^{2\pi i/s}$, then the orbit of L consists of the lines $L_{ij} := a\omega^i x + b\omega^j y + cz = 0$ for $1 \leq i, j \leq s$. Suppose that no pairs of lines among the lines $L_{i,j}$, $L_{k,l}$, $L_{p,q}$ meet on $xyz = 0$. Then they meet at a point $\notin \{xyz = 0\}$ only if the determinant of the matrix

$$\begin{vmatrix} a\omega^i & b\omega^j & c \\ a\omega^k & b\omega^l & c \\ a\omega^p & b\omega^q & c \end{vmatrix}$$

vanish. Since $abc \neq 0$, this is equivalent to the vanishing of

$$\det \begin{vmatrix} \omega^\alpha - 1 & \omega^\beta - 1 \\ \omega^\gamma - 1 & \omega^\theta - 1 \end{vmatrix}$$

where $\alpha := k - i$, $\beta := l - j$, $\gamma := p - i$ and $\theta := q - j$. The integers α, β, γ, θ are not multiples of s by hypothesis. Then vanishing of the determinant implies

$$\frac{(\omega^\alpha - 1)(\omega^\theta - 1)}{(\omega^\beta - 1)(\omega^\gamma - 1)} = 1 \Rightarrow \frac{(\omega^{\alpha/2} - \omega^{-\alpha/2})(\omega^{\theta/2} - \omega^{-\theta/2})}{(\omega^{\beta/2} - \omega^{-\beta/2})(\omega^{\gamma/2} - \omega^{-\gamma/2})} = \omega^{(\beta+\gamma-\alpha-\theta)/2}$$

Since the left-hand side of the latter expression is real, so must be the right-hand side. Therefore

$$\mathrm{Im}(e^{\pi i(\beta+\gamma-\alpha-\theta)/s}) = 0 \Rightarrow s|\beta + \gamma - \alpha - \theta.$$

But this means that there is a pair of lines meeting at $z = 0$, contradiction. This shows that the curves $L^{(1/s)}$ are nodal.

Since $L^{(1/s)}$ is a rational curve of degree s, it must have $(s-1)(s-2)/2$ nodes. Since $L^{(r/s)} = \phi_r^{-1}(L^{(1/s)})$, the number of nodes of $L^{(r/s)}$ is $r^2(s-1)(s-2)/2$. Obviously, three flex points of $L^{(1/s)}$ are lifted as $3r$ cusps of type $x^r = y^s$. The genus of $L^{(r/s)}$ can be calculated by the genus formula, or by noting that the curves $L^{(r/s)}$ are coverings of $L^{(1/s)}$ branched at these three flex points, with the branching index r.

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