JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **227,** 335–358 (1998)
ARTICLE NO. AY986091

On Possible Deterioration of Smoothness under the Operation of Convolution

A. Muhammed Uludağ*

Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey

Submitted by N. H. Bingham

Received March 8, 1996

We give some sufficient conditions of deterioration of smoothness under the operation of convolution. We show that the convolution of two probability densities which are restrictions to $\mathbb{R}$ of entire functions can possess infinite essential supremum on each interval.

1. INTRODUCTION

It is known that, as a rule, the operation of convolution improves smoothness. This rule was mentioned by Paul Lévy in his book [1, p. 91]. In order to elaborate the domain of applicability of this rule, D. Raikov [2] constructed two probability densities p_1, p_2 on $\mathbb{R}$ which are restrictions to $\mathbb{R}$ of entire functions, but their convolution

$$p(x) = (p_1 * p_2)(x) = \int_{-\infty}^{\infty} p_1(x-s) p_2(s)\, ds, \qquad x \in \mathbb{R},$$

although infinitely differentiable, is not analytic everywhere on $\mathbb{R}$. We show that the deterioration of smoothness under convolution can be much greater than in Raikov's example. We prove this by a method different from Raikov's. Nevertheless, Raikov's method permits us to obtain some conditions of deterioration presented in this article.

* Current address for correspondence: A. Muhammed Uludağ, Institut Fourier, B.P. 74, 38402 Saint Martin-d'Hères Cedex, France. E-mail: uludag@ujf-grenoble.fr.

0022-247X/98 $25.00

2. NOTATION

We shall adopt the following notation for some subsets of $L_1(\mathbb{R})$:

- L_1^+ is the set of all nonnegative functions on $\mathbb{R}$ belonging to $L_1(\mathbb{R})$ and not equivalent to 0;
- EL_1^+ is the set of all functions of L_1^+ which are restrictions to $\mathbb{R}$ of entire functions;
- $E^\infty L_1^+$ is the subset of EL_1^+ consisting of the restrictions to $\mathbb{R}$ of entire functions bounded in each strip $\{z: |\mathrm{Im}\, z| \le r\}$, $r > 0$;
- $E_\rho L_1^+$ is the subset of EL_1^+ consisting of the functions which are restrictions to $\mathbb{R}$ of entire functions of order not exceeding ρ;
- UL_1^+ is the set of all functions $f \in L_1^+$ possessing the following property: for any nonempty interval $]\alpha, \beta[$, the equality

$$\operatorname*{ess\,sup}_{x \in]\alpha, \beta[} f(x) = \infty \tag{1}$$

is valid.

The set L_1^+ consists of functions equal to a probability density up to a positive constant factor. The sets $EL_1^+, E^\infty L_1^+, E_\rho L_1^+$ can be viewed as subsets of L_1^+ consisting of functions with "extremely good smoothness." The set UL_1^+ can be viewed as a subset of L_1^+ consisting of functions with "extremely bad smoothness."

We define the operators $S: L_1^+ \to L_1^+$ by the equality

$$(Sf)(x) = \int_{-\infty}^{\infty} f(x+t) f(t)\, dt, \qquad x \in \mathbb{R}. \tag{2}$$

We accept the agreement that S is defined by (2) *everywhere* on $\mathbb{R}$. Note that Sf is an even function, and S is the operator of convolution of $f(x)$ and $f(-x)$. We call Sf the *symmetrization* of f.

A standard characterization of growth of a function analytic in the disc $\{z: |z| < R\}$, $R \le \infty$, is

$$M(r, f) := \max_{|z| \le r} |f(z)|, \qquad 0 \le r < R.$$

If f is analytic in the strip $\{z: |\mathrm{Im}\, z| < R\}$, we shall use, besides $M(r, f)$, the characteristic

$$H(r, f) := \sup_{|\mathrm{Im}\, z| \le r} |f(z)|, \qquad 0 \le r < R.$$

Evidently, $M(r, f) \le H(r, f)$, for $0 \le r < R$. If $R = \infty$, i.e., f is an entire function, then, besides its order, defined by

$$\rho[f] := \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

we shall consider another characteristic $\kappa[f]$, defined by

$$\kappa[f] := \limsup_{r\to\infty} \frac{\log\log H(r, f)}{\log r}.$$

Evidently, $0 \le \rho[f] \le \kappa[f] \le \infty$.

If $f \in L_1^+$, we define the quantity $h[f]$ as

$$h[f] := \sup\{r > 0\colon f \text{ is the restriction to } \mathbb{R} \text{ of a function analytic and bounded in the strip } \{z\colon |\mathrm{Im}\, z| \le r\}\}.$$

If $f \in E^\infty L_1^+$, we define $h[f] = \infty$. If there is no function whose restriction to $\mathbb{R}$ is f and analytic and bounded in some strip $\{z\colon |\mathrm{Im}\, z| \le r\}$, we define $h[f] = 0$.

3. STATEMENT OF RESULTS

As we have mentioned, the set UL_1^+ consists of functions with extremely bad smoothness. For example, if $f \in UL_1^+$, then f cannot coincide almost everywhere with a continuous function in any interval.

THEOREM 1. *There exists $f \in EL_1^+$ such that $Sf \in UL_1^+$, i.e., $S(EL_1^+) \cap UL_1^+ \ne \emptyset$.*

In the proof of this theorem, we use a theorem of T. Carleman on "touching" approximation by entire functions on $\mathbb{R}$. By the help of the generalization of this theorem due to Keldysh, it is possible to prove the following refinement of Theorem 1:

THEOREM 2. *There is a function $f \in E_3L_1^+$ such that $Sf \in UL_1^+$, i.e., $S(E_3L_1^+) \cap UL_1^+ \ne \emptyset$.*

Now we give some conditions of deterioration of smoothness obtained by use of Raikov's method. The basic result in this direction is the next theorem.

THEOREM 3. *Let $f \in L_1^+$. Then*

(i) *$h[Sf] \geq h[f]$, and $M(r, Sf) = H(r, Sf) \leq \|f\|_1 H(r, f)$ for $r < h[f]$.*

(ii) *If Sf is analytic in the disc $\{z: |z| < R\}$, then $h[Sf] \geq R$, $h[f] \geq R/2$, and the following inequality is valid:*

$$M(r, Sf) = H(r, Sf) \leq \|f\|_1 H(r, f) \leq \|f\|_1 \left\{ \frac{1}{\pi h} M(2(r+h), Sf) \right\}^{1/2},$$
$$r > 0,\ h > 0,\ 2(r+h) \leq R. \quad (3)$$

Since $f \in E^\infty L_1^+ \Leftrightarrow h[f] = \infty$, the following corollary is immediate:

COROLLARY 1. *In order that $f \in E^\infty L_1^+$ it is necessary and sufficient that $Sf \in EL_1^+$. Moreover, $Sf \in EL_1^+$ implies $Sf \in E^\infty L_1^+$. If $f \in E^\infty L_1^+$ then the relation $\rho[f] \leq \kappa[f] = \rho[Sf] = \kappa[Sf]$ is valid.*

Now we describe the possible pairs $(\rho[f], \kappa[f])$ for $f \in E^\infty L_1^+$:

THEOREM 4. *Let (ρ, κ) be a pair of numbers such that $1 \leq \rho \leq \kappa \leq \infty$. There exists a function $f \in E^\infty L_1^+$ such that $\rho[f] = \rho$, $\kappa[f] = \kappa$.*

Therefore, if $f \in E^\infty L_1^+$ is of fixed order $\rho[f] = \rho$, then the order $\rho[Sf]$ of Sf can be arbitrarily large. Now let $f, g \in E^\infty L_1^+$. If $\rho[f] < \rho[g]$, then it is natural to consider f as “smoother” than g. Since $\rho[Sf] = \kappa[f]$, the functions constructed in Theorem 4 can be interpreted as examples of deterioration of smoothness under convolution.

From Theorem 3(ii), it also follows that if $h[f] = 0$, then Sf cannot be analytic at the origin. This is Raikov’s result [2]. To show that convolution can deteriorate smoothness, he then considered the function $f(x) = d/dx \exp\{1 - \exp[e^{-x}]\}$, which belongs to EL_1^+, but $h[f] = 0$. However, although not analytic at the origin, Sf is infinitely differentiable on $\mathbb{R}$, and $\rho[f] = \infty$. We construct the following examples:

THEOREM 5. *There exists $f \in E_1 L_1^+$ with $h[f] = 0$, i.e., Sf is not analytic at the origin.*

THEOREM 6. *For each n, there exists an $f \in E_{1+1/n} L_1^+$ such that Sf is not $(2n+2)$-times differentiable at the origin.*

Theorem 6 is proved by the help of the following theorem, which is obtained by a refinement of Raikov’s method.

THEOREM 7. *If $f \in L_1^+$ is not n-times differentiable, or if it is but not all of the n derivatives are bounded, then Sf is not $(2n+2)$-times differentiable at the origin.*

Note that, by Theorem 7, for any function $f \in EL_1^+$ unbounded on $\mathbb{R}$, Sf is not twice differentiable at the origin.

4. PROOF OF THEOREM 1

We begin by the construction of a continuous function $g \in L_1^+$ such that $(Sg)(x) = \infty$ for any $x \in \mathbb{Q}$. Note that $(Sg)(0) = \|g\|_2$. However, there are continuous functions $g \in L_1^+$ such that $g \notin L_2(\mathbb{R})$. This is the basic fact in this construction.

For $2 \le n \in \mathbb{N}$ denote by s_n the function continuous on $\mathbb{R}$, equal to zero for $x \notin]n - n^{-3}, n + 2n^{-3}[$, equal to n for $x \in [n, n + n^{-3}]$, and linear for $x \in [n - n^{-3}, n]$, and for $x \in [n + n^{-3}, n + 2n^{-3}]$. Define

$$q = \sum_{n=2}^{\infty} s_n. \tag{4}$$

Since the supports of s_ns do not overlap, and $\|s_n\|_1 = 2n^{-2}$, it is easy to verify that q is continuous on $\mathbb{R}$ and belongs to L_1^+. For any nonnegative integer a, we have

$$\begin{aligned}(Sq)(a) &= \int_{-\infty}^{\infty} \left\{ \sum_{n=2}^{\infty} s_n(t) \right\} \left\{ \sum_{n=2}^{\infty} s_n(t+a) \right\} dt \\ &= \int_{-\infty}^{\infty} \sum_{n=2}^{\infty} s_n(t) s_{n+a}(a+t)\, dt \\ &\ge \sum_{n=2}^{\infty} \int_{n}^{n+(n+a)^{-3}} n(n+a)\, dt \ge \sum_{n=2}^{\infty} \frac{n}{(n+2)^2} = \infty.\end{aligned}$$

Set

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{k} q(kx - k^2). \tag{5}$$

Since each summand of (5) is continuous on $\mathbb{R}$ and, moreover, the support of the kth summand is contained in $[k, \infty[$, the series converges everywhere and g is continuous on $\mathbb{R}$. Since

$$\|g\|_1 \le \sum_{k=1}^{\infty} \frac{1}{k} \left\| q(kx - k^2) \right\|_1 = \|q\|_1 \sum_{k=1}^{\infty} \frac{1}{k^2},$$

we have $g \in L_1^+$. Let x be a non-negative rational number; set $x = a/b$, where $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$. We have

$$\begin{aligned}(Sg)(x) &= \int_{-\infty}^{\infty} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} q(kt - k^2) \right\} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} q(kt + kx - k^2) \right\} dt \\ &\geq \int_{-\infty}^{\infty} \left\{ \frac{1}{b} q(bt - b^2) \right\} \left\{ \frac{1}{b} q(bt + a - b^2) \right\} dt \\ &= \frac{1}{b^3} \int_{-\infty}^{\infty} q(s) q(s+a)\, ds \\ &= \frac{1}{b^3} (Sq)(a) = \infty.\end{aligned}$$

Since Sg is an even function, we conclude that $(Sg)(x) = \infty$ for any $x \in \mathbb{Q}$. Thus, the function g with the properties mentioned at the beginning of the proof has been constructed. In order to construct the desired function $f \in EL_1^+$, we need the following theorem by Carleman [4].

THEOREM (Carleman). *Let g be a (complex-valued) continuous function on $\mathbb{R}$. Let $\epsilon = \epsilon(r)$ be a positive decreasing continuous function on $[0, \infty[$. There exists an entire function f such that*

$$|g(x) - f(x)| < \epsilon(|x|), \qquad x \in \mathbb{R}. \tag{6}$$

We shall use the following corollary to this theorem.

COROLLARY 2. *If g is assumed to be real valued on $\mathbb{R}$, then f can be chosen real valued and such that $f(x) > g(x)$ on $\mathbb{R}$.*

To derive the corollary, note that, by Carleman's theorem, there exists an entire function f_1 such that

$$\left| g(x) + \tfrac{1}{2}\epsilon(|x|) - f_1(x) \right| < \tfrac{1}{4}\epsilon(|x|), \qquad x \in \mathbb{R}.$$

It is easy to see that the function $f(z) = \frac{1}{2}\{f_1(z) + \overline{f_1(\bar{z})}\}$ is entire, satisfies $f(x) > g(x)$ on $\mathbb{R}$, and (6) is valid. Now we can construct the function $f \in EL_1^+$ such that $Sf \in UL_1^+$. Let g be the function defined by (5). By the corollary to Carleman's theorem, there exists an entire function f positive on $\mathbb{R}$ and satisfying the condition

$$|g(x) - f(x)| < e^{-|x|}, \qquad x \in \mathbb{R}.$$

Hence, $f \in EL_1^+$. It remains to show that $Sf \in UL_1^+$. From $f(x) > g(x) \geq 0$, it follows that $(Sf)(x) \geq (Sg)(x)$. Since $(Sg)(x) = \infty$ for $x \in \mathbb{Q}$, we conclude that

$$(Sf)(x) = \infty, \qquad x \in \mathbb{Q}. \tag{7}$$

In order to derive from (7) that $Sf \in UL_1^+$, we shall use the following two lemmas.

LEMMA 1. *If f, g are continuous nonnegative functions, then the convolution $f * g$ is lower semicontinuous.*

Proof. The function $f * g$ can be represented as the pointwise limit of the nondecreasing sequence of continuous functions

$$\left\{ \int_{-n}^{n} f(x-t) g(t)\, dt \right\}_{n=1}^{\infty}.$$

Since the limit of a nondecreasing sequence of continuous functions is lower semicontinuous, so is $f * g$. ∎

LEMMA 2. *If f is a lower semicontinuous function such that $f(x) = +\infty$ for x in a dense subset M of $\mathbb{R}$, then f possesses infinite essential supremum in any interval.*

Proof. By the lower semicontinuity of f, the set $\{x\colon f(x) > C\} \cap]\alpha, \beta[\}$ is open for any $C > 0$ and for any interval $]\alpha, \beta[$. By the condition of the lemma, this set is nonempty. Since any nonempty open set has a positive Lebesgue measure, we obtain, for any set E with $\operatorname{meas} E = 0$, $\sup_{x \in]\alpha, \beta[\setminus E} f(x) > C$. Hence $\operatorname{ess\,sup}_{x \in]\alpha, \beta[} f(x) > C$. Using the arbitrariness of C, we get the desired result. ∎

We are now ready to complete the proof of Theorem 1. By Lemma 1, Sf is a lower semicontinuous function. Since $(Sf)(x) = \infty$ for $x \in \mathbb{Q}$, $Sf \in UL_1^+$ according to Lemma 2. ∎

5. PROOF OF THEOREM 2

Now we proceed to show that there exists an entire function f of order ≤ 3 such that $Sf \in UL_1^+$. In order to construct the function f, we shall use a refinement of the Carleman theorem due to Keldysh [5]. For a detailed exposition of this theorem see [6].

THEOREM (Keldysh). *Let g be a (complex-valued) differentiable function on $\mathbb{R}$. Put*

$$\mu(r) := \max_{|x| \leq r} |g'(x)| \quad \textit{and} \quad \nu[g] := \limsup_{r \to \infty} \frac{\log \mu(r)}{\log r}.$$

Then for each $\epsilon \geq 0$ there exists an entire function f whose order does not exceed $\nu[g] + 1$ and satisfying $|f(x) - g(x)| < \epsilon$ for all $x \in \mathbb{R}$.

The following corollary is easily derived by imitating the proof of Corollary 2.

COROLLARY 3. *Let g be a real-valued differentiable function on $\mathbb{R}$. Then there exists an entire function f whose order does not exceed $\nu[g]+1$ which is real valued on $\mathbb{R}$ and satisfies* $0 < f(x) - g(x) < 1$.

Now we start with the construction of the function f whose existence is asserted by Theorem 2.

Step 1. For $3 \le n \in \mathbb{N}$ denote by s_n the function continuous on $\mathbb{R}$, equal to zero for $x \notin]n - n^{-1}\log^{-3} n, n + 2n^{-1}\log^{-3} n[$, equal to $n\log^3 n$ in the interval $[n, n + n^{-1}\log^{-3} n]$, and linear for $[n - n^{-1}\log^{-3} n, n]$ and for $[n + n^{-1}\log^{-3} n, n + 2n^{-1}\log^{-3} n]$. One can make the edges of s_n smoother, so that it becomes a differentiable function. Define

$$q(x) = \sum_{n=3}^{\infty} s_n(x) \tag{8}$$

and set

$$g(x) = \sum_{k=3}^{\infty} \frac{q(kx - k!)}{k^2}. \tag{9}$$

Since the supports of s_ns do not overlap, q is differentiable on $\mathbb{R}$. Likewise, the support of the function $q(kx - k!)$ is contained in $[k-1, \infty[$, so that the series defining g converges everywhere and g is also differentiable. We will approximate this function by an entire function according to the corollary to Keldysh's theorem. Let us first calculate $\mu(r)$ for the function g. If $x < 1$, then $g'(x)$ is identically 0; so it suffices to consider $x > 1$ only. So assume that $1 < x < r$. Then, since the function $q(kx - k!)$ vanishes for $kx - k! < 1$, only the finite number $n(r) := \#\{k\colon (k-1)! < r\}$ of terms contributes to g. Note that by Stirling's formula $n(r) = O(\log r)$ as $r \to \infty$. Hence

$$g(x) = \sum_{k=3}^{n(r)} \frac{q(kx - k!)}{k^2}, \qquad 1 \le x \le r,$$

$$|g'(x)| \le \sum_{k=3}^{n(r)} \left| \frac{q'(kx - k!)}{k} \right| \le \sum_{k=3}^{n(r)} |q'(kx - k!)|, \qquad 1 \le x \le r.$$

Now clearly we have $|q'(x)| \le (x+1)^2 \log^6 (x+1)$. Inserting this in the above inequality, we get, as $r \to \infty$,

$$|g'(x)| \le \sum_{k=3}^{n(r)} (kx - k! + 1)^2 \log^6 (kx - k! + 1)$$
$$\le n(r) x^2 \log^6 x = O\big(r^2 \log^7 r\big).$$

Therefore $\mu(r) = O(r^2 \log^7 r)$ as $r \to \infty$, and hence

$$\nu[g] = \limsup_{r \to \infty} \frac{\log \mu(r)}{\log r} \le 2.$$

By Corollary 3, we conclude that there is an entire function f_0, real valued and nonnegative on $\mathbb{R}$ with $\rho[f_0] \le 3$, satisfying $f_0(x) = g(x) + \delta(x)$, where $0 < \delta(x) < 1$. The desired function will be obtained by "shrinking" f_0 by multiplying with the function h we describe in the lemma below, whose proof is rather technical and will be given at the end of this section.

LEMMA 3. *There exists a function* $h \in EL_1^+$ *such that* $\rho[h] = 1$ *and* $h(x) = 1/(x \log^2 x) + O(x^{-3/2})$ *as* $x \to \infty$ *in* $\mathbb{R}$.

Put $f(z) := f_0(z) h(z)$. We claim that f is a function with desired properties.

Step 2. Now we shall prove that $f \in E_3 L_1^+$. Since $\rho[h] = 1$, $\rho[f_0] \le 3$, f is entire and $\rho[f] \le 3$. Clearly f is nonnegative on $\mathbb{R}$, and it remains only to show that it is integrable. Put $\delta := f_0 - g$. Then we have

$$\|f\|_1 = \|(g+\delta)h\|_1 \le \|\delta h\|_1 + \|gh\|_1.$$

Since δ is bounded and h is integrable, $\|\delta h\|_1 < \infty$. On the other hand, by (9) we have

$$\|gh\|_1 \le \sum_{k=3}^{\infty} \frac{1}{k^2} \|q(kx - k!) h(x)\|_1.$$

By the change of variable $kx - k! = y$ we obtain

$$\|q(kx - k!) h(x)\|_1 = \frac{1}{k} \left\| q(y) h\left(\frac{y + k!}{k} \right) \right\|_1.$$

Hence, in order to show that $\|gh\|_1 < \infty$, it suffices to show that $\|q(y) h((y + k!)/k)\|_1 = O(k)$. Indeed, by (9) we have

$$\left\| q(y) h\left(\frac{y+k!}{k} \right) \right\|_1 \le \sum_{n=3}^{\infty} \left\| s_n(y) h\left(\frac{y+k!}{k} \right) \right\|_1.$$

Since the support of s_n is contained in $]n - 2n^{-1}\log^{-3} n, n + 2n^{-1}\log^{-3} n[$, and $s_n(x) \leq n \log^3 n$, we obtain

$$\left\| s_n(y) h\left(\frac{y+k!}{k}\right) \right\|_1 \leq \int_{n-2/(n\log^3 n)}^{n+2/(n\log^3 n)} h\left(\frac{y+k!}{k}\right) dy \leq 4 \max_{y\in[n-1,n+1]} h\left(\frac{y+k!}{k}\right).$$

Now define $r(x) := 1/(\log^2 x)$. From $h(x) = r(x) + O(x^{-3/2})$ as $x \to \infty$ it follows that $h(x) \leq Cr(x)$, $x \geq 2$ with some positive constant C. The function r is decreasing, so we have, for $k \geq 3$

$$\max_{y\in[n-1,n+1]} h\left(\frac{y+k!}{k}\right) \leq r\left(\frac{n-1+k!}{k}\right).$$

On the other hand, recall the formula

$$\sum_{k=k_0}^{\infty} f(k) \leq f(k_0) + \int_{k_0}^{\infty} f(y)\, dy,$$

which is valid if f is a decreasing function. Using this formula, we get

$$\begin{aligned}\left\| q(y) h\left(\frac{y+k!}{k}\right) \right\|_1 &\leq \sum_{n=3}^{\infty} 4Cr\left(\frac{n-1+k!}{k}\right) \\ &\leq 4Cr\left(\frac{2+k!}{k}\right) + 4C\int_3^{\infty} r\left(\frac{y-1+k!}{k}\right) dy.\end{aligned}$$

Substitute $x = (y-1+k!)/k$ in the integral to get, for some C,

$$\begin{aligned}\left\| q(y) h\left(\frac{y+k!}{k}\right) \right\|_1 &\leq Cr\left(\frac{2+k!}{k}\right) + Ck\int_{(2+k!)/k}^{\infty} r(x)\, dx \\ &= Cr\left(\frac{2+k!}{k}\right) - \frac{Ck}{\log x}\Bigg|_{(2+k!)/k}^{\infty} = O(k) \qquad \text{as } k \to \infty.\end{aligned}$$

We conclude that f is integrable.

Step 3. We shall prove $Sf \in UL_1^+$. Let us first show $(Sf)(x) = \infty$ for $x \in \mathbb{Q}$. We shall, for $b \in \mathbb{N}$,

$$Sf = S(hg + h\delta) \geq S(hg) = S\left(h \sum_{k=3}^{\infty} \frac{q(kt-k!)}{k^2}\right) \geq S\left(h \frac{q(bt-b!)}{b^2}\right).$$

Therefore,

$$(Sf)(x) \geq \frac{1}{b^4} \int_{-\infty}^{\infty} h(t)h(x+t)q(bt-b!)q(bx+bt-b!)\,dt.$$

Now let x be a nonnegative rational number; set $x = a/b$, where $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$. Upon the change of variable $bt - b! = y$ and recalling that $bx = a$, the above inequality becomes

$$(Sf)(x) \geq \frac{1}{b^5} \int_{-\infty}^{\infty} h\left(\frac{y+b!}{b}\right) h\left(\frac{y+a+b!}{b}\right) q(y)q(a+y)\,dy.$$

Recall that $h(x) = r(x) + O(x^{-3/2})$ as $x \to +\infty$, so that $h(y) \geq r(y)/2$ for $y > y_0$. If we increase the lower limit of the above integral to by_0, the inequality will be preserved:

$$(Sf)(x) \geq \frac{1}{4b^5} \int_{by_0}^{\infty} r\left(\frac{y+b!}{b}\right) r\left(\frac{y+a+b!}{b}\right) q(y)q(a+y)\,dy.$$

Since r is a decreasing function we have

$$(Sf)(x) \geq \frac{1}{4b^5} \int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right) q(y)q(a+y)\,dy.$$

Now we insert the series defining q into the last inequality:

$$\begin{aligned}(Sf)(x) &\geq \frac{1}{4b^5} \int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right) \left\{\sum_{n=3}^{\infty} s_n(y)\right\} \left\{\sum_{n=3}^{\infty} s_n(a+y)\right\} dy \\ &\geq \frac{1}{4b^5} \int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right) \sum_{n=3}^{\infty} s_n(y)s_{a+n}(a+y)\,dy.\end{aligned}$$

Put $n_0 := a + [by_0] + 1$. Simply by eliminating the terms for which $n < n_0$ above, we obtain

$$\begin{aligned}(Sf)(x) &\geq \frac{1}{4b^5} \int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right) \sum_{n=n_0}^{\infty} s_n(y)s_{a+n}(a+y)\,dy \\ &\geq \sum_{n=n_0}^{\infty} \frac{1}{4b^5} \int_{n}^{n+1/(n+a)\log^3(n+a)} r^2\left(\frac{y+a+b!}{b}\right) \\ &\quad \times \sum_{n=n_0}^{\infty} s_n(y)s_{a+n}(a+y)\,dy.\end{aligned}$$

The last inequality follows from the fact that $\bigcup_{n=n_0}^{\infty}[n, n+1/\{(n+a)\log^3(n+a)\}] \subset [by_0, \infty]$. On the other hand, note that $s_n(y) = n\log^3 n$, $s_{a+n}(a+y) = (a+n)\log^3(a+n)$, and $r^2(\{y+a+b!\}/b) \geq r^2(\{2n+a+b!\}/b)$ for $y \in [n, n+1/\{(n+a)\log^3(n+a)\}]$. Using these, we obtain

$$\begin{aligned}(Sf)(x) &\geq \frac{1}{b^5}\sum_{n=n_0}^{\infty} n\log^3 n\left(\frac{2n+b+b!}{b}\right)^{-2}\log^{-4}\left(\frac{2n+b+b!}{b}\right)\\ &= \infty.\end{aligned}$$

Hence, the result $(Sf)(x) = \infty$ for all $x \in \mathbb{Q}^+$ is proved. Since Sf is an even function, we conclude that $(Sf)(x) = \infty$ for all $x \in \mathbb{Q}$. Since Sf is lower semicontinuous to Lemma 1, $Sf \in UL_1^+$ by Lemma 2. ∎

We believe that $S(E_1L_1^+) \cap UL_1^+ \neq \varnothing$, but we have failed to prove it.

The latter condition cannot be improved since $E_\rho L_1^+ = \varnothing$ for $\rho < 1$ by the Phragmén–Lindelöf theorem.

Proof of Lemma 3. For the construction of a function with properties described in Lemma 3, we will use the following theorem based on an idea first used by Mittag–Leffler in 1903 [7]. This theorem will be used extensively throughout this paper.

THEOREM 8 (Mittag-Leffler). *Denote by G_θ the angle $\{z\colon |\arg z| < \theta\}$, $0 < \theta < \pi$, and let g be a function analytic in G_γ for some γ, satisfying*

$$g(z) = O(|z|^{-2}) \qquad \text{as } |z| \to \infty \text{ in } G_\gamma \setminus G_\alpha, \tag{10}$$

*where $0 < \alpha < \gamma$. For $z \in$ **int**$(\mathbb{C} \setminus G_\delta)$, define*

$$f(z) := -\frac{1}{2\pi i}\int_{\partial G_\delta}\frac{g(\zeta)\,d\zeta}{\zeta - z} \tag{11}$$

for some δ such that $\alpha < \delta < \gamma$. Then

(i) *The function f does not depend on $\delta \in (\alpha, \gamma)$ and can be continued to $\mathbb{C}$ as an entire function.*

(ii) *The following asymptotic formulas are valid for any $n = 0, 1, 2, \ldots$:*

$$f^{(n)} = \begin{cases} O(|z|^{-n-1}), & \text{as } |z| \to \infty \text{ in } \mathbb{C} \setminus G_\delta,\\ g^{(n)}(z) + O(|z|^{-n-1}), & \text{as } |z| \to \infty \text{ in } G_\delta. \end{cases} \tag{12}$$

Proof. (i) By virtue of (10), integral (11) converges uniformly on every compact subset of **int**$(\mathbb{C} \setminus G_\delta)$. Hence f is analytic in **int**$(\mathbb{C} - G_\delta)$. Now put $G_{\delta,R} := \{z \in G_\delta\colon\ |z| \geq R\}$. If $z \in \mathbf{int}(\mathbb{C} \setminus G_\delta)$, then, by the Cauchy theorem, the integral (11) does not change if we replace ∂G_δ by $\partial G_{\delta,R}$ for any $R > 0$. Since the integral along $\partial G_{\delta,R}$ is analytic in $z \in \mathbf{int}(\mathbb{C} - G_{\delta,R})$, and, moreover, R is arbitrary, we conclude that f can be analytically continued into $\mathbb{C}$. According to this rule of continuation, for $z \in G_\delta$ we have the representation

$$f(z) = -\frac{1}{2\pi i}\int_{\partial G_{\delta,R}} \frac{g(\zeta)\,d\zeta}{\zeta - z}, \tag{13}$$

where $R > |z|$. The function f does not depend on $\delta \in (\alpha, \gamma)$ since, for $z \in \mathbf{int}(\mathbb{C} \setminus G_\gamma)$, the integral (11) does not change if we replace ∂G_δ by $\partial G_{\delta'}$ for any $\delta' \in (\alpha, \gamma)$: This follows from the Cauchy theorem and condition (10).

(ii) For $z \in \mathbf{int}(\mathbb{C} \setminus G_\delta)$, we choose $\delta' \in (\alpha, \delta)$ and represent $f^{(n)}$ in the form

$$f^{(n)} = -\frac{n!}{2\pi i}\int_{\partial G_{\delta'}} \frac{g(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}. \tag{14}$$

Since

$$|\zeta - z| \geq |z| \sin(\delta - \delta') \qquad \text{for } \zeta \in \partial G_{\delta'} \text{ and } z \in \mathbf{int}(\mathbb{C} \setminus G_\delta), \tag{15}$$

by using (14) and (10), we obtain the first part of (12). On the other hand, for $z \in \partial G_\delta$, by the Cauchy theorem, the representation (13) gives

$$-\int_{\partial G_{\delta,R}} \frac{g(\zeta)\,d\zeta}{\zeta - z} + \int_{\partial G_\delta} \frac{g(\zeta)\,d\zeta}{\zeta - z} = -\int_{\partial (G_\delta \setminus G_{\delta,R})} \frac{g(\zeta)\,d\zeta}{\zeta - z} = 2\pi i g(z).$$

Hence, for $z \in \mathbf{int}(G_\delta)$ we have the representation

$$f(z) = g(z) - \frac{1}{2\pi i}\int_{\partial G_{\delta'}} \frac{g(\zeta)\,d\zeta}{\zeta - z}. \tag{16}$$

Now choose $\delta' \in (\delta, \gamma)$. By (16), for $z \in G_\delta$ we have

$$f^{(n)}(z) = g^{(n)}(z) - \frac{n!}{2\pi i}\int_{\partial G_{\delta'}} \frac{g(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}. \tag{17}$$

Using (15) and (10), we obtain the second part of (12). ∎

Now we are ready to prove Lemma 3. We apply the Mittag-Leffler theorem to the function $g(z) := e^{-iz}/(\sqrt{z}\log z)$ which is analytic in the region $G_0 := \{z = re^{i\theta}\colon\ -\pi/2 < \theta < 3\pi/2;\ z \neq 1, 0\}$, with the branch of the logarithm real on $\mathbb{R}^+$.

1. Let us first look at the asymptotic behaviour of g in the subset of G_0 lying in the lower half-plane. Let δ be such that $0 < \delta < \pi/2$. Then if $\arg z < -\delta$ or $\arg z > \pi + \delta$, we have

$$|g(z)| = \frac{e^{-|z|\sin\delta}}{|\sqrt{z}\log z|} \leq e^{-|z|\sin\delta} \qquad \text{for } |z| > 3.$$

Denote $G := \{z\colon\ -\pi/4 < \arg z < 5\pi/4\}$. By Theorem 8, the function given in $\mathbb{C} \setminus G$ by

$$f(z) := -\frac{1}{2\pi i}\int_{\partial G} \frac{g(\zeta)\,d\zeta}{\zeta - z}$$

can be continued analytically to $\mathbb{C}$ and satisfies

$$f(z) = \begin{cases} O(|z|^{-1}), & \text{as } |z| \to \infty \text{ in } \mathbb{C} \setminus G, \\ g(z) + O(|z|^{-1}), & \text{as } |z| \to \infty \text{ in } G. \end{cases}$$

Clearly, $\rho[f] = 1$ and $f(x) = g(x) + O(1/|x|)$ as $|x| \to \infty$ in $\mathbb{R}$. Hence we have

$$f(x) = \begin{cases} \dfrac{e^{-ix}}{\sqrt{x}\log x} + O(x^{-1}), & \text{as } x \to +\infty, \\[2ex] \dfrac{e^{-ix}}{i\sqrt{|x|}\,(\log x + i\pi)} + O(|x|^{-1}), & \text{as } x \to -\infty. \end{cases} \tag{18}$$

2. Consider the function $t(z) := \{f(z) + \overline{f(\bar{z})}\}^2$, which is also entire, and nonnegative on $\mathbb{R}$. From the estimate for f we obtain $|t(x)| \leq 2/(|x|\log^2|x|)$. Therefore t is integrable. On the other hand, as $x \to \infty$, by (18) we have

$$t(x) = \left\{\frac{e^{-ix}}{\sqrt{x}\log x} + \frac{e^{ix}}{\sqrt{x}\log x} + O(x^{-1})\right\}^2 = \frac{4\cos^2 x}{x\log^2 x} + O(x^{-3/2}).$$

3. Finally, define the entire function h of order 1 as $h(z) := \{t(z) + t(z + \pi/2)\}/4$. Then h is the desired function. Clearly, $h \in EL_1^+$. As $x \to +\infty$ we have

$$\begin{aligned} h(x) &= \frac{\cos^2 x}{x\log^2 x} + \frac{\sin^2 x}{(x + \pi/2)\log^2(x + \pi/2)} + O(x^{-3/2}) \\ &= \frac{1}{x\log^2 x} + O(x^{-3/2}). \end{aligned}$$

∎

6. PROOF OF THEOREM 3

Before proving Theorem 3, we recall some of Raikov's results cited in [3].

THEOREM 9 (Raikov). *Let $\hat{g}$ be the Fourier transform of the function $g \in L_1^+$. If $\hat{g}$ is analytic in the disc $\{z\colon |z| < R\}$ then*

$$\int_{-\infty}^{\infty} e^{rx} g(x)\, dx \leq \infty, \qquad -R < r < R.$$

Moreover, $h[\hat{g}] \geq R$ and the following representation is valid in the strip $\{z\colon |\mathrm{Im}\, z| < R\}$:

$$\hat{g}(z) = \int_{-\infty}^{\infty} e^{izx} g(x)\, dx.$$

Now note that $\widehat{f(-x)}(t) = \overline{\hat{f}(t)}$. Thus, we have

$$\widehat{(Sf)}(t) = \hat{f}(t)\overline{\hat{f}(t)} = |\hat{f}(t)|^2 \geq 0.$$

Hence, the transform of Sf is always nonnegative. For such functions, the following fact is valid:

THEOREM 10 (Raikov). *Let $f \in L_1^+$ be continuous at 0. If $\hat{f}(t) \geq 0$, then $\hat{f} \in L_1^+$ and*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \hat{f}(t)\, dt.$$

COROLLARY 4. *If Sf is continuous at 0, then Theorem 9 is applicable for $\hat{g} = Sf$ and $g = \widehat{Sf}/(2\pi)$.*

Now we pass to the proof of Theorem 3.

(i) First, we shall prove the following lemma:

LEMMA 4. *Let f, g be functions such that $g \in L_1$. Then $h[f * g] \geq h[f]$, and the inequality $H(r, f * g) \leq \|g\|_1 H(r, f)$ is satisfied for $r < h[f]$.*

Proof. Clearly, for $|\mathrm{Im}\, z| < h[f]$ we have $|f(z-t)| \leq H(|\mathrm{Im}\, z|, f)$. Hence the convolution integral converges uniformly in the strip $\{z\colon |\mathrm{Im}\, z| \leq r < h[f]\}$, and $f * g$ is analytic in this strip. Moreover,

$$H(r, f * g) = \sup_{|\mathrm{Im}\, z| < r} \left| \int_{-\infty}^{\infty} f(z-t) g(t)\, dt \right| \leq \|g\|_1 H(r, f),$$

which also shows that $h[f * g] \geq r$. ∎

Now, by Lemma 4, $h[Sf] \geq h[f]$ and $H(r, Sf) \leq \|f\|_1 H(r, f)$. On the other hand, by Corollary 4 we have, for $|\mathrm{Im}\, z| < h[f]$,

$$\begin{aligned} |(Sf)(z)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izt} |\hat{f}(t)|^2 \, dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{Im}(zt)} |\hat{f}(t)|^2 \, dt = (Sf)(i \,\mathrm{Im}\, z), \end{aligned}$$

which shows that $H(r, Sf) = M(r, Sf)$.

(ii) We want to show that the integral

$$\int_{-\infty}^{\infty} e^{r|t|} |\hat{f}(t)| \, dt$$

is finite for $0 \leq r < R/2$. Let $r < r' < R/2$. Then

$$\int_{-\infty}^{\infty} e^{rt} |\hat{f}(t)| \, dt \leq \int_{-\infty}^{\infty} e^{r|t|} |\hat{f}(t)| \, dt = \int_{-\infty}^{\infty} e^{(r-r')|t|} e^{r'|t|} |\hat{f}(t)| \, dt.$$

By Schwarz's inequality, it follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{(r-r')|t|} e^{r'|t|} |\hat{f}(t)| \, dt \\ &\qquad \leq \left\{ \int_{-\infty}^{\infty} e^{2(r-r')|t|} \, dt \int_{-\infty}^{\infty} e^{2r'|t|} |\hat{f}(t)|^2 \, dt \right\}^{1/2}. \end{aligned} \tag{19}$$

For the first integral in the right-hand side of (19) we have

$$\int_{-\infty}^{\infty} e^{2(r-r')|t|} \, dt = \frac{1}{r' - r}.$$

For the second integral,

$$\int_{-\infty}^{\infty} e^{2r'|t|} |\hat{f}(t)|^2 \, dt \leq \int_{-\infty}^{\infty} e^{2r't} |\hat{f}(t)|^2 \, dt + \int_{-\infty}^{\infty} e^{-2r't} |\hat{f}(t)|^2 \, dt.$$

Now assume that Sf is analytic in the disc $\{z\colon |z| < R\}$. Then, both of the last two integrals are finite by Corollary 4, and

$$\int_{-\infty}^{\infty} e^{2r't} |\hat{f}(t)|^2 \, dt = 2\pi (Sf)(2ir') \leq 2\pi M(2r', Sf),$$

$$\int_{-\infty}^{\infty} e^{-2r't} |\hat{f}(t)|^2 \, dt = 2\pi (Sf)(-2ir') \leq 2\pi M(2r', Sf).$$

Hence

$$\int_{-\infty}^{\infty} e^{2r'|t|}|\hat{f}(t)|^2\, dt \le 4\pi M(2r', Sf),$$

and we finally have

$$\int_{-\infty}^{\infty} e^{r|t|}|\hat{f}(t)|\, dt \le \left\{\frac{4\pi M(2r', Sf)}{r'-r}\right\}^{1/2} \le \infty.$$

It follows that the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izt}\hat{f}(t)\, dt$$

converges uniformly in the strips $\{z\colon |\mathrm{Im}\, z| \le r\}$ for $r < R/2$, and f is analytic in the strip $\{z\colon |\mathrm{Im}\, z| < R/2\}$. On the other hand

$$\begin{aligned} H(r, f) = \sup_{|\mathrm{Im}\, z|<r} |f(z)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{r|t|}|\hat{f}(t)|\, dt \\ &\le \frac{1}{2\pi}\left\{\frac{4\pi M(2r', Sf)}{r'-r}\right\}^{1/2}, \end{aligned} \tag{20}$$

which means that $H(r, f) < \infty$ for $r < R/2$; i.e., $h[f] \ge r < R/2$. Now put $r' - r = h$ and substitute in (20) to get

$$H(r, f) \le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{r|t|}|\hat{f}(t)|\, dt \le \left\{\frac{M(2(r+h), Sf)}{\pi h}\right\}^{1/2}.$$

By part (i), $M(r, Sf) = H(r, Sf) \le \|f\|_1 H(r, f)$. Joining this with the above inequality, we obtain the desired result.

Proof of Corollary 1. If $f \in E^{\infty}L_1^+$, then $h[f] = \infty$ and by Theorem 3(i) it follows that $h[Sf] = \infty$, i.e., $Sf \in E^{\infty}L_1^+ \subset EL_1^+$. Similarly if $Sf \in EL_1^+$, then by Theorem 3(ii) it follows that both $h[Sf] = \infty$ and $h[f] = \infty$, i.e., $Sf \in E^{\infty}L_1^+$, $f \in E^{\infty}L_1^+$. Substituting $h = 1$ in inequality (3) one has

$$\begin{aligned} \rho[Sf] = \kappa[Sf] &\le \kappa[f] \\ &\le \limsup_{r\to\infty} \frac{\log\log\{M(2(1+r), Sf)\}^{1/2}}{\log r} = \rho[Sf]. \end{aligned}$$

Hence $\rho[Sf] = \kappa[Sf] = \kappa[f]$. Since the inequality $\rho[f] \le \kappa[f]$ is always valid, we get the desired result.

7. PROOF OF THEOREM 4

To calculate the quantity $\kappa[f]$ for the functions we shall construct, the following lemma will be helpful.

LEMMA 5. *For* $0 < \beta < \pi/2$, *define the set* $A_{r,\beta} := \{z\colon |\operatorname{Im} z| < r, |\arg z| < \beta$ *or* $|\arg z - \pi| < \beta\}$, *and for the entire function* f *put* $H_\beta(r, f) := \sup_{z \in A_{r,\beta}} |f(z)|$. *Then the inequality*

$$\kappa[f] = \max(\rho[f], \kappa_\beta[f])$$

is valid, where

$$\kappa_\beta[f] := \limsup_{r \to \infty} \frac{\log\log H_\beta(r, f)}{\log r}.$$

Proof. Evidently, $\kappa[f] \geq \max(\rho[f], \kappa_\beta[f])$. On the other hand, put $B_{r,\beta} := \{z\colon |\operatorname{Im} z| < r\} \setminus A_{r,\beta}$, and let $B_\beta(r, f) := \sup_{z \in B_{r,\beta}} |f(z)|$. Define

$$b_\beta[f] := \limsup_{r \to \infty} \frac{\log\log B_\beta(r, f)}{\log r}.$$

Since $H(r, f) = \max(B_\beta(r, f), H_\beta(r, f))$, we have $\kappa[f] \leq \max(b_\beta[f], \kappa_\beta[f])$. Finally, $b_\beta[f] \leq \rho[f]$ since

$$B_\beta(r, f) \leq \sup_{|z| \leq r/\sin\beta} |f(z)| = M(r/\sin\beta, f),$$

so that $\kappa[f] \leq \max(\rho[f], \kappa_\beta[f])$. ∎

Now we pass to the proof of the theorem.

Case 1. $1 < \rho < \kappa < \infty$. For $0 < \rho < \sigma - 1 < \rho$, consider the function

$$g(z) := \exp(-iz^\rho - z^\sigma),$$

where the branches of the power functions are taken to be positive on $\mathbb{R}^+$. It is analytic in the region $\{z = re^{i\theta}\colon -\pi < \theta < \pi\}$. For $-\pi < \theta_1 < \theta_2 < \pi$ denote by $G(\theta_1, \theta_2)$ the angle $\{z = re^{i\theta}\colon \theta_1 < \theta < \theta_2\}$; and put $\gamma := \min(2\pi/\rho, \pi)$.

1. We have

$$\log|g(re^{i\theta})| = r^\rho \sin\rho\theta - r^\sigma \cos\sigma\theta, \tag{21}$$

which is majorized by the term $r^\rho \sin\rho\theta$ since $\rho > \sigma$. Hence, $g(z) = O(\exp(-K_\delta r^\rho))$ as $r \to \infty$, and $z = re^{i\theta} \in G(\pi/\rho + \delta, \gamma - \delta) \cup G(-\pi/$

$\rho + \delta, -\delta)$, δ being sufficiently small. Note that, for $x \in \mathbb{R}^+$, one has $g(x) = O(1/x)$.

Fix an α satisfying $\pi/\rho < \alpha < \gamma$. By Theorem 8, the function f given in **int**$(\mathbb{C} \setminus G(-\pi/2\rho, \alpha))$ by

$$f(z) := -\int_{\partial G(-\pi/2\rho,\alpha)} \frac{g(\zeta)\, d\zeta}{\zeta - z}$$

can be continued analytically to $\mathbb{C}$ and satisfies

$$f^{(n)} = \begin{cases} O(|z|^{-1}), & \text{as } |z| \to \infty \text{ in } \mathbb{C} \setminus G(-\pi/2\rho, \alpha), \\ g^{(n)}(z) + O(|z|^{-n-1}), & \text{as } |z| \to \infty \text{ in } G(-\pi/2\rho, \alpha). \end{cases} \tag{22}$$

2. Clearly, $\rho[f] = \rho$. Now let us show that $\kappa[f] = \sigma/(\sigma - \rho + 1)$. There exists an angle $\beta < \pi/(3\rho)$ such that f is bounded in the angle $\{z: \pi - \arg z| < \beta\}$. Moreover, f is bounded on the lower half-plane. By Lemma 5 we want to estimate $H_\beta(r, f)$. It suffices to consider the angle $\{z: 0 < \arg z < \beta\}$ only. In order to estimate $H_\beta(r, f)$, we shall find the supremum of $|f|$ on the lines $l_y^+ := \{z = x + iy: 0 < \arg z < \beta\}$. By the construction of f we have

$$f(z) = g(z) + O(|z|^{-1}) \qquad \text{as } |z| \to \infty,\ z \in l_y^+,$$

so that by (21) it follows that

$$\log|f(z)| = r^\rho \sin \rho\theta - r^\sigma \cos \sigma\theta + O(r^{-1}) \qquad \text{as } r \to \infty,\ z = re^{i\theta} \in l_y^+.$$

Substitute $\sin\theta = y/r$, and use the estimates

$$\frac{2\rho\theta}{\pi} \le \sin \rho\theta \le \frac{\pi\rho\theta}{2} \qquad \text{for } 0 \le \theta \le \frac{\pi}{2\rho},$$

$$\frac{1}{2} \le \cos \rho\theta \le 1, \qquad \text{for } 0 \le \theta \le \frac{\pi}{3\rho},$$

to get, for $z = x + iy = re^{i\theta} \in l_y^+$, as $r \to \infty$,

$$\frac{2\rho y}{\pi} r^{\rho-1} - r^\sigma + O(r^{-1}) \le \log|f(z)| \le \frac{\pi\rho y}{2} r^{\rho-1} - \frac{1}{2} r^\sigma + O(r^{-1}).$$

Hence, for r large enough, we have

$$\frac{2\rho y}{\pi} r^{\rho-1} - r^{\sigma} - 1 \le \log|f(z)| \le \frac{\pi\rho y}{2} r^{\rho-1} - \frac{1}{2} r^{\sigma} + 1. \qquad (23)$$

For any fixed $y_0 > 0$, the largeness of r can be taken uniformly in y with $0 < y \le y_0$. In the estimation of f from above, we see that the dominant term as $r \to \infty$ for fixed y is $-r^{\sigma}/2$ (since $\sigma \ge \rho - 1$), so that the function is bounded on l_y^+ and $H_\beta(y, f) < \infty$ for all $y > 0$. Calculation of the maximum of both sides of (23) by the usual method of differentiation gives

$$K_1 y^{\sigma/(\sigma-\rho+1)} \le \sup_{z \in l_y^+} \log|f(z)| \le \log H_\beta(y, f) \le K_2 y^{\sigma/(\sigma-\rho+1)} \quad (24)$$

for y large enough with some positive constants K_1, K_2. On the other hand, (23) tells us that $\sup_{z \in l_y^+} \log|f(z)|$ is bounded in $0 < y < y_0$ for any y_0. Hence $\log H_\beta(y, f) \le K_2 y^{\sigma/(\sigma-\rho+1)}$ for large y. We conclude that $\kappa[f] = \sigma/(\sigma - \rho + 1)$. Hence, for given κ, if we substitute $\sigma = \kappa(\rho - 1)/(\kappa - 1)$, then $\kappa[f] = \kappa$.

3. Put $h(z) := \{f(z) + \overline{f(\bar{z})}\}^2$. Then $h \in E^\infty L_1^+$, $\rho[h] = \rho$, and $\kappa[h] = \kappa$. Indeed, $f(x) = O(1/|x|)$ as $|x| \to \infty$ in $\mathbb{R}$ by construction, so $h(x) = O(1/x^2)$ as $|x| \to \infty$ on $\mathbb{R}$, and h is an integrable function. Being nonnegative on $\mathbb{R}$, we conclude that $h \in EL_1^+$. On the other hand, f is bounded on the lower half-plane, say by the constant C, therefore $(|f(z)| - C)^2 \le |h(z)| \le (|f(z)| + C)^2$ if $\mathrm{Im}\, z \ge 0$. Applying the same argument for $\mathrm{Im}\, z \le 0$ we obtain

$$(M(r, f) - C)^2 \le M(r, h) \le (M(r, f) + C)^2,$$

$$(H(r, f) - C)^2 \le H(r, h) \le (H(r, f) + C)^2.$$

Hence, $\rho[h] = \rho[f] = \rho$ and $\kappa[h] = \kappa[f] = \kappa$.

For the remaining cases, we shall give a sketch of proof.

Case 2. $1 < \rho < \kappa = \infty$. For $\rho > 1$, apply the procedure in Case 1 to the function $g(z) := \exp(-iz^{\rho}\log^2 z - z^{\rho-1}\log^3 z)$.

Case 3. $1 = \rho < \kappa < \infty$. Put $\sigma = (2\kappa)/(\kappa - 1)$, and consider the function $g(z) := \exp(-iz \log^2 z - \log^{\sigma} z)$, $\pi/2 < \arg z < 3\pi/2$; $|z| > 1$. Define the entire function by the Cauchy-type integral along the contour $L := \{z\colon |z| = 2, -\pi/4 < \arg z < 5\pi/4\} \cup \{z\colon |z| \ge 3, \arg z = -\pi/4 \text{ or } 5\pi/4\}$. Then apply the same procedure in Case 1.

Case 4. $1 = \rho < \kappa = \infty$. Consider the function $g(z) := \exp(-iz \log^2 z - \log^2 z \log\log z)$, which is analytic in the region $G = \{z = re^{i\theta}\colon r > 1; -\pi/2 < \theta < \pi/2\}$. Then apply the procedure in Case 3.

Case 5. $1 \le \rho = \kappa \le \infty$. The desired functions are Sf, where f is one of the functions constructed in previous cases. Indeed, we have exhibited functions f with given $\kappa[f] \ge 1$. By Corollary 1 for each of these functions we have $Sf \in E^\infty L_1^+$ and $\kappa[f] = \rho[Sf] = \kappa[Sf]$. ∎

Proof of Theorem 5. Theorem 3 can be considered as a test for analyticity of Sf. For example, it immediately implies that if $h[f] < \infty$, then $Sf \neq EL_1^+$. For each $h > 0$, there exists a function $f \in EL_1^+$ such that $h[f] = h$; for example, consider the function $f(x) = \exp[-\cosh(\pi x/(2h))]$. Hence, symmetrizations Sf of these functions cannot be entire. As we have already mentioned, another immediate consequence of Theorem 3 is that if $h[f] = 0$, then Sf cannot be analytic at 0. Now we shall show the existence of a function $E_1 L_1^+$ with $h[f] = 0$. From the function

$$g(z) := \frac{e^{-iz\log^2 z}}{z}, \qquad \text{where } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2},$$

construct the entire function f as in the proof of Theorem 4, Case 3, and put $\tilde{f} := \{f(z) + \overline{f(\bar{z})}\}^2$. Then $\tilde{f}$ is the desired function. Clearly, $\tilde{f} \in EL_1^+$ and $\rho[\tilde{f}] = 1$. Now let us show $h[\tilde{f}] = 0$. We have

$$\log|g(re^{i\theta})| = r\log^2 r \sin\theta + 2r\theta\log r\cos\theta - r\theta^2\sin\theta - \log r,$$

so that for $0 \le \theta \le \pi/2$ and for sufficiently large r one has $\log|g(re^{i\theta})| \ge r\log^2 r\sin\theta - r\theta^2\sin\theta - \log r$. Now assume $y > 0$. For $z = re^{i\theta} \in l_y := \{z = x + iy\colon x > 0\}$, one has $\sin\theta = y/r$. Hence, for $z = re^{i\theta} \in l_y$ we have

$$\log|g(re^{i\theta})| \ge y\log^2 y - \frac{\pi y}{2} - \log r \to \infty \qquad \text{as } r \to \infty.$$

Since $f(z) = g(z) + O(1/|z|)$ in the upper half-plane, we get $H(y, f) = \infty$. On the other hand, note that $H_1(\overline{f(\bar{z})}) < \infty$ since $\overline{f(\bar{z})}$ is bounded in the upper half-plane. Hence

$$H(y, \tilde{f}) \ge \left\{H(y, f) - H(y, \overline{f(\bar{z})})\right\}^2 \ge \left\{H(y, f) - H(1, \overline{f(\bar{z})})\right\}^2 = \infty.$$

Since $y > 0$ is arbitrary, we conclude that $h[\tilde{f}] = 0$. ∎

However, symmetrization of the function constructed in Theorem 5, and the symmetrization of Raikov's example, $f(x) = d/dx\,\exp\{1 - \exp[e^{-x}]\}$, are infinitely differentiable by the following theorem.

THEOREM 11. *Assume $f \in L_1^+$ is a bounded function with continuous, bounded derivatives up to the order n. Then Sf has continuous, bounded derivatives up to the order n.*

Proof. For $k \leq n$ we have

$$|(Sf)^{(k)}(x)| = |f^{(k)} * \bar{f}| \leq \|f^{(k)}\|_\infty \|\bar{f}\|_1 < \infty.$$

It follows that the convolution integrals

$$\int_{-\infty}^{\infty} f^{(k)}(x+t) f(t)\, dt$$

are uniformly convergent, and Sf has n bounded derivatives. ∎

By the refinement of Raikov's method, a statement of the converse type can also be proved. This is the content of Theorem 7.

8. PROOF OF THEOREM 7

We begin with a theorem of Raikov (cited in [3]), which is actually the first step of the proof of Theorem 9.

THEOREM 12 (Raikov). *Let $\hat{g}$ be the transform of $g \in L_1^+$. If $\hat{g}$ is $2n$-times differentiable on $\mathbb{R}$, then*

$$\int_{-\infty}^{\infty} |x|^m g(x)\, dx < \infty, \qquad m = 0, 1, \ldots, 2n.$$

Moreover, $\hat{g}$ is $2n$-times differentiable on $\mathbb{R}$, and these derivatives can be represented by the integrals

$$\hat{g}^{(m)}(t) = i^m \int_{-\infty}^{\infty} x^m e^{itx} g(x)\, dx, \qquad m = 0, 1, \ldots, 2n.$$

Since the transform of Sf is nonnegative, by Theorem 10 we have the following corollary:

COROLLARY 5. *If Sf is continuous at 0, then Theorem 11 is applicable to $\hat{g} = Sf$ and $g = (\widehat{Sf})/(2\pi)$.*

Now we are ready to prove Theorem 7. We have

$$\int_{-\infty}^{\infty} |\hat{f}(t)|\, |t|^\alpha\, dt = \int_{-\infty}^{\infty} \frac{|\hat{f}(t)|}{1+|t|^\beta} (1+|t|^\beta)|t|^\alpha\, dt.$$

By Schwarz's inequality, this is bounded above by

$$\left\{ \int_{-\infty}^{\infty} |\hat{f}(t)|^2 \left(1 + |t|^{\beta}\right)^2 |t|^{2\alpha} \, dt \int_{-\infty}^{\infty} \frac{1}{\left(1 + |t|^{\beta}\right)^2} \, dt \right\}^{1/2}.$$

The second integral above is finite whenever $\beta > 1/2$. For the first integral, we can consider $|\hat{f}|^2$ as the transform of Sf. Suppose that Sf is $2n$-times differentiable at the origin. By Corollary 5 the first integral above is finite whenever $2\alpha + 2\beta \leq 2n$, that is, when $\alpha < n - 1/2$. Hence the integral $\int_{-\infty}^{\infty} \hat{f}(t) e^{-itx} (it)^k \, dt$ converges uniformly for $k \leq n - 1$. Therefore f is $(n-1)$-times differentiable on $\mathbb{R}$, and since these derivatives tends to 0 at infinity by the Riemann–Lebesgue theorem, they are bounded. ∎

Proof of Theorem 6. From the function

$$g(z) = \frac{e^{-iz^{\rho} \log^3 z}}{\sqrt{z} \log z},$$

construct the entire function f as in the proof of Theorem 4, Case 1, and put $h(z) := \{f(z) + \overline{f(\bar{z})}\}$ again. Clearly, h is entire, $\rho[h] = \rho$, and it is nonnegative on $\mathbb{R}$. Since $h(x) = O(1/|x|\log^2 |x|)$ as $|x| \to \infty$ in $\mathbb{R}$, h is integrable and $h \in E_{\rho} L_1^{+}$. Now let us show that h' is unbounded on $\mathbb{R}$. Indeed, by Theorem 8(ii), we have $f'(x) = g'(x) + O(1/x^2)$ as $x \to +\infty$ on $\mathbb{R}$. Note that, as $x \to +\infty$, we have

$$f(x) + \overline{f(x)} = \frac{2\cos\left(x^{\rho} \log^3 x\right)}{\sqrt{x} \log x} + O(x^{-1})$$

and

$$f'(x) + \overline{f'(x)} = -2\rho x^{\rho - 3/2} \log^2 x \sin\left(x^{\rho} \log^3 x\right) + O\left(x^{\rho - 3/2} \log x\right).$$

Hence

$$h'(x) = -4\rho x^{\rho - 2} \log x \sin\left(2x^{\rho} \log^3 x\right) + O(x^{\rho - 2}),$$

so that h' is unbounded if $\rho \geq 2$.

Note that, if $\rho \geq 1 + 1/n$ in the above construction, then $h^{(n)}$ is unbounded and Sh is not $2(n+1)$-times differentiable at 0. ∎

ACKNOWLEDGMENTS

I express my gratitude to Professor I. V. Ostrovskii for supervising this work and to Prof. A. Kerimov for his valuable critiquing.

REFERENCES

1. P. Lévy, ‘‘Théorie de l’Addition des Variables Aléatoires,’’ Gauthier-Villars, Paris, 1954.
2. D. A. Raikov, On composition of analytic distribution functions, *Dokl. Akad. Nauk SSSR* **23** (1939), 511–514.
3. Ju. V. Linnik and I. V. Ostrovskii, ‘‘Decomposition of Random Variables and Vectors,’’ Am. Math. Soc., Providence, 1977.
4. T. Carleman, Sur un théorème de Weierstrass, *Ark. Mat. Astronom. Fyz.* **20B**(4) (1927), 275–279.
5. M. V. Keldysh, On approximation of holomorphic functions by entire functions, *Dokl. Akad. Nauk SSSR* **47** (1945), 239–241.
6. S. N. Mergelyan, Uniform approximations to functions of a complex variable, *in* ‘‘Series and Approximation,’’ Vol. 3, pp. 357–363, Am. Math. Soc., Providence, 1962.
7. G. Mittag-Leffler, Sur la nouvelle fonction, *C. R. Acad. Sci. Paris* (2) **137** (1903), 554–558.