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GALOIS COVERINGS OF THE PLANE BY K3 SURFACES

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Dedicated to Professor Iossif V. Ostrovskii on his 71st birthday

Abstract. We study branched Galois coverings of the projective plane by smooth K3 surfaces. Branching data of such a covering determines in a unique way a uniformizable orbifold on the plane. In order to study Galois coverings of the plane by K3 surfaces, it suffices to study orbifolds on the plane uniformized by K3 surfaces. We call these K3 orbifolds and classify K3 orbifolds with an abelian uniformization. We also classify K3 orbifolds with a locus of degree less than 6 and with a non-abelian uniformization. There are no K3 orbifolds with a locus of degree greater than 6. Although we give some examples of K3 orbifolds with a sextic locus, our results are incomplete in this case.

1. Introduction

The aim of this paper is to study the problem: 'classify the branched Galois coverings of $\mathbf{P}^2$ by *smooth* K3 surfaces'. A basic example of such a covering is the following. Let $C \subset \mathbf{P}^2$ be a smooth quartic given by the homogeneous polynomial $f(x, y, z)$. Then the surface $X \subset \mathbf{P}^3$ with equation $w^4 = f(x, y, z)$ is a smooth K3 surface. The projection $\psi : [x : y : z : w] \in X \to [x : y : z] \in \mathbf{P}^2$ is a covering with $\mathbb{Z}/(4)$ as the Galois group, branched along C with branching index 4. Viewing the pair $\mathcal{O} := (\mathbf{P}^2, 4C)$ as an *orbifold*, the projection $\psi : X \to \mathcal{O}$ is a *uniformization* of this orbifold. The curve C is called the *locus* of $\mathcal{O}$. Our approach to the problem consists of studying orbifolds over $\mathbf{P}^2$ uniformized by smooth K3 surfaces. As a result, we obtain a complete list of orbifolds over $\mathbf{P}^2$ uniformized by smooth K3 surfaces and

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with an abelian Galois group (Theorem 1) and a complete list of orbifolds over $\mathbf{P}^2$ with a locus of degree ≤ 5, uniformized by smooth K3 surfaces and with non-abelian Galois groups (Theorem 2). We also present some K3 orbifolds over $\mathbf{P}^2$ with a sextic locus. There are no K3 orbifolds with a locus of degree > 6.

Let $\psi : X \to \mathbf{P}^2$ be a Galois covering of the plane by a smooth surface X with branching divisor $B = \sum_{i=1}^n b_i B_i$ (in other words, ψ is a uniformization of the orbifold $\mathcal{O} := (\mathbf{P}^2, B)$). The smoothness condition on X imposes severe restrictions on B. For example, B may have simple singularities only and the coefficients b_i cannot be arbitrary. Assume, moreover, that X is a K3 surface. It is possible to define the orbifold Chern numbers $c_1^2(\mathcal{O})$ and $c_2(\mathcal{O})$ so that $0 = c_1^2(X) = c_1^2(\mathcal{O}) \deg(\psi)$ and $24 = e(X) = e(\mathcal{O}) \deg(\psi)$. The first equality easily implies that $\deg(B_{\text{red}}) \leq 6$, and the second equality implies that $24/e(\mathcal{O})$ is an integer. These conditions restrict the set of admissible B that may occur as branching loci of Galois coverings of $\mathbf{P}^2$ by K3 surfaces. The classification of plane curves of degree ≤ 5 can be found in [**12**] and [**3**]. Sextic plane curves with simple singularities have been classified by Yang [**19**].

These restrictions gives a finite (up to isotopy) list of admissible orbifolds, and it remains to check if these orbifolds are indeed uniformizable. This is the most difficult part of the project we undertake, since for this verification one must show that local orbifold fundamental groups injects into the global orbifold fundamental group. However, in most cases the uniformizing maps factors through the abelian covering maps $\mathbf{P}^2 \to \mathbf{P}^2$, simplifying the task of verification.

We believe that the list of sextic K3 orbifolds presented in this article is complete, but we are unable to prove this since there are too many sextics with simple singularities (see [**19**]).

2. Orbifolds

2.1. Definitions

We shall mostly follow the terminology used in [**18**]. For details one may consult [**9**]. Let M be a connected smooth complex manifold, $G \subset \text{Aut}(M)$ a properly discontinuous subgroup and put $X := M/G$. Then the projection $\phi : M \to X$ is a branched Galois covering endowing X with a map $\beta_\phi : X \to \mathbb{N}$ defined by $\beta_\phi(p) := |G_q|$, where q is a point in $\phi^{-1}(p)$ and G_q is the isotropy subgroup of G at q. In this setting, the pair (X, β_ϕ) is said to be uniformized by $\phi : M \to (X, \beta_\phi)$. An *orbifold* is a pair (X, β) of an irreducible normal analytic space X with a function $\beta : X \to \mathbb{N}$ such that the pair (X, β) is locally finitely uniformizable. In the case $\gamma|\beta$, the orbifold (X, γ) is said to be a *suborbifold* of (X, β). Let $\phi : (Y, 1) \to (X, \gamma)$ be a uniformization of (X, γ), e.g. $\beta_\phi = \gamma$. Then $\phi : (Y, \beta') \to (X, \beta)$ is called an *orbifold covering*, where $\beta' := \beta \circ \phi / \gamma \circ \phi$. The orbifold (Y, β') is called the *lifting of* (X, β) *to the uniformization* Y *of* (X, γ) and will be denoted by $(X, \beta)/(X, \gamma)$.

Let (X, β) be an orbifold, $B_\beta := \mathrm{supp}(\beta - 1)$ be its *locus* and let $B_1, \ldots, B_n$ be the irreducible components of B_β. Then β is constant on $B_i \setminus \mathrm{sing}(B_\beta)$; so let b_i be this number. The *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(X, \beta)$ of (X, β) is the group defined by $\pi_1^{\mathrm{orb}}(X, \beta) := \pi_1(X \setminus B_\beta)/\langle\langle \mu_1^{b_1}, \ldots, \mu_n^{b_n} \rangle\rangle$, where μ_i is a meridian of B_i and $\langle\langle\ \rangle\rangle$ denotes the normal closure. The *local orbifold fundamental group* $\pi_1^{\mathrm{orb}}(X, \beta)_p$ *at a point* $p \in X$ is the orbifold fundamental group $\pi_1^{\mathrm{orb}}(B, p)$ of the restriction of (X, β) to a sufficiently small neighborhood of B of p.

2.2. *Orbifold singularities in dimension 2*

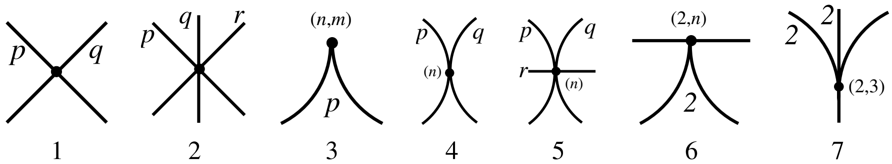
Let (X, β) be an orbifold with X being a smooth surface. Then the map β is determined by the numbers b_i; in fact $\beta(p)$ is the order of the local orbifold fundamental group at p. In the case $\dim X = 2$ the orbifold condition (i.e. locally finitely uniformizability) is equivalent to the finiteness of the local fundamental groups. This implies that the locus B_β of (X, β) may have ADE singularities only, and the admissible values for b_i are very restricted. These singularities and the branching numbers can either be characterized as the branching loci of the quotients of $\mathbb{C}^2$ by finite reflection groups or as the germs with a finite orbifold fundamental group (see [**20**]). A list of these singularities is given in Table 1.

In the last column of Table 1, the order of the local orbifold fundamental group of the corresponding singularity is given. For the node (1), this order is evidently pq. For case (2), i.e. triple point, the orders of the corresponding reflection groups are well known; see, for example,[**20**]. For the remaining cases, the orders can be computed by using Yoshida's result that any of the remaining singular loci is an orbifold covering of one of the triple points in (2) [**20**, pp. 167–169].

2.3. *Finite uniformizations*

Let X be a smooth manifold, (X, β) an orbifold, and let $f : \pi_1^{\mathrm{orb}}(X, \beta) \to G$ be a surjection onto a finite group G. Then there is a corresponding branched covering $\phi : M \to (X, \beta)$ with G as the Galois group. This is a finite smooth uniformization of X if and only if ϕ coincides locally with the local finite uniformizations of (X, β). In terms of G, this condition is expressed as follows (see [**9**, Theorem 1]).

TABLE 1. Orbifold germs in dimension two.



No	Equation	Condition	Order
1	xy	—	pq
2	$xy(x+y)$	$0 < \rho := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$	$4\rho^{-2}$
3	$x^n - y^m (\gcd(n,m) = 1)$	$0 < \rho := \frac{1}{n} + \frac{1}{m} + \frac{1}{p} - 1$	$\frac{4}{nm}\rho^{-2}$
4	$x^2 - y^{2n} (n \geq 2)$	$0 < \rho := \frac{1}{p} + \frac{1}{q} + \frac{1}{n} - 1$	$\frac{4}{n}\rho^{-2}$
5	$y(x^2 - y^{2n})$	$0 < \rho := \frac{1}{p} + \frac{1}{q} + \frac{1}{nr} - 1$	$\frac{4}{n}\rho^{-2}$
6	$y(x^2 - y^n)$ (n odd)	—	$2nq^2$
7	$x(x^2 - y^3)$	—	96

LEMMA 1. *Let $\iota_p : \pi_1^{\text{orb}}(X, \beta)_p \to \pi_1^{\text{orb}}(X, \beta)$ be the homomorphism induced by the inclusion. Then M is a finite smooth uniformization of (X, β) if and only if for any $p \in X$, the map $f \circ \iota_p : \pi_1^{\text{orb}}(X, \beta)_p \to G$ is an injection.*

Let (X, β) be an orbifold, (X, γ) a suborbifold, $(Y, 1) \to (X, \gamma)$ a universal uniformization, and $(Y, \beta') \simeq (X, \beta)/(X, \gamma)$. Then one has the exact sequence of fundamental groups

$$0 \to \pi_1^{\text{orb}}(Y, \beta') \to \pi_1^{\text{orb}}(X, \beta) \to \pi_1^{\text{orb}}(X, \gamma) \to 0$$

so that if $\pi_1^{\text{orb}}(X, \beta)$ is finite, then $|\pi_1^{\text{orb}}(X, \beta)| = |\pi_1^{\text{orb}}(X, \gamma)||\pi_1^{\text{orb}}(Y, \beta')|$.

The following lemma will be useful.

LEMMA 2. *Let $\mathcal{O} = (\mathbf{P}^2, B)$ be an orbifold where $B = \sum_{i=1}^n b_i B_i$ with B_i being an irreducible plane curve of degree d_i. If B is a nodal curve then $\pi_1(\mathbf{P}^2 \backslash B)$ is abelian* *and*

$$\pi_1^{\mathrm{orb}}(\mathcal{O}) \simeq \left\langle \mu_1, \mu_2, \ldots, \mu_n \;\middle|\; b_1\mu_1 = b_2\mu_2 = \cdots = b_n\mu_n = \sum_{i=1}^{n} d_i\mu_1 = 0 \right\rangle.$$

Proof. By the Zariski conjecture proved by Deligne and Fulton (see [**4**]) the fundamental group of a nodal curve is abelian. Therefore

$$\pi_1(\mathbf{P}^2\backslash B) \simeq H_1(\mathbf{P}^2\backslash B, \mathbb{Z}) \simeq \left\langle \mu_1, \mu_2, \ldots, \mu_n \;\middle|\; \sum_{i=1}^{n} d_i\mu_i = 0 \right\rangle,$$

the classes μ_i being representatives of meridians around B_i for $i \in [1, n]$. The lemma follows by adding the orbifold relations to the above presentation. □

2.4. *Orbifold Chern numbers*

Hereafter, we shall exclusively be concerned with orbifolds (X, β) with X being a smooth algebraic surface. Since in this case β is determined by its values b_i on $B_i \backslash \operatorname{sing}(B_\beta)$, an orbifold can alternatively be defined as a pair (X, B), where $B := b_1B_1 + \cdots + b_nB_n$ is a divisor on X with $b_i \geq 2$. For an orbifold (X, B) the corresponding map $X \to \mathbb{N}$ will be denoted by β_B.

Let $\mathcal{O} := (\mathbf{P}^2, B)$ be an orbifold where $B = b_1B_1 + \cdots + b_nB_n$ with B_i being an irreducible curve of degree d_i.

Definition. The *orbifold Chern numbers of* $\mathcal{O}$ are defined as

$$c_1^2(\mathcal{O}) := \left[-3 + \sum_{1\leq i\leq n} d_i \left(1 - \frac{1}{b_i}\right) \right]^2$$

$$e(\mathcal{O}) := 3 - \sum_{1\leq i\leq n} \left(1 - \frac{1}{b_i}\right) e(B_i \backslash \operatorname{sing}(B)) - \sum_{p\in\operatorname{sing}(B)} \left(1 - \frac{1}{\beta_B(p)}\right).$$

Let $\mathcal{O} := (\mathbf{P}^2, B)$ be an orbifold where $B = b_1B_1 + \cdots + b_nB_n$ with B_i being an irreducible curve of degree d_i. If $(X, 1) \to (\mathbf{P}^2, \beta)$ is a finite uniformization of degree d then the Chern numbers of X are given by $e(X) = de(\mathcal{O})$ and $c_1^2(X) = dc_1^2(\mathcal{O})$ (see [**10**]).

2.5. *K3 orbifolds*

We call $\mathcal{O} := (\mathbf{P}^2, B)$ a *K3 orbifold* if there is a finite uniformization $\phi : M \to \mathcal{O}$ by a smooth K3 surface M. Since M is simply connected, ϕ must be the universal

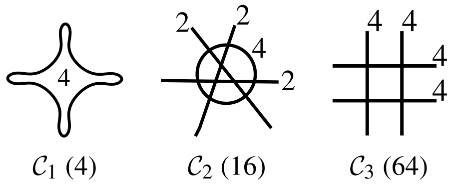


FIGURE 1. The K3 orbifolds $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$.

uniformization of $\mathcal{O}$, which means that the corresponding Galois group is the full group $\pi_1^{\text{orb}}(\mathcal{O})$, and the conditions of Lemma 1 must hold for $G = \pi_1^{\text{orb}}(\mathcal{O})$. Moreover, since $c_1^2(M) = 0$ and $e(M) = 24$, one has the following lemma.

LEMMA 3. *Let $\mathcal{O} := (\mathbf{P}^2, B)$ be a K3 orbifold where $B = b_1 B_1 + \cdots + b_n B_n$ with B_i being an irreducible curve of degree d_i. Then:*

(i) *one has $c_1^2(\mathcal{O}) = 0$; in particular, $4 \leq \deg(B_{\text{red}}) \leq 6$, and if $\deg(B_{\text{red}}) = 6$, then $b_1 = b_2 = \cdots = b_n = 2$;*

(ii) *$\pi_1^{\text{orb}}(\mathcal{O})$ is of order $24/e(\mathcal{O})$; in particular, $24/e(\mathcal{O})$ is an integer.*

3. Abelian K3 orbifolds

In Sections 3.1–3.5 we shall give a complete list of K3 orbifolds with abelian uniformizing Galois group. The completeness of the list will be proved in Section 3.6. The locus of any orbifold considered in this section is assumed to be a nodal curve.

3.1. The K3 orbifolds $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$

These orbifolds are defined as follows. $\mathcal{C}_1$ is the orbifold $(\mathbf{P}^2, 4K)$, where K is a smooth quartic; $\mathcal{C}_2$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_1 + 2L_3 + 4Q)$, where L_1, L_2 and L_3 are three distinct lines and Q is a smooth quadric; $\mathcal{C}_3$ is the orbifold $(\mathbf{P}^2, 4L_1 + 4L_1 + 4L_3 + 4L_4)$ where L_1, L_2, L_3 and L_4 are four distinct lines (see Figure 1).

3.1.1. The orbifold $\mathcal{C}_1$. This is the orbifold mentioned at the beginning of the introduction. Obviously, one has $|\pi_1^{\text{orb}}(\mathcal{C}_1)| = 4$.

3.1.2. The orbifold $\mathcal{C}_2$. Suppose that L_1, L_2, L_3 and Q are given by the equations $x = 0$, $y = 0$, $z = 0$, respectively, and $P[x : y : z] = 0$. The suborbifold

$\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$ of $\mathcal{C}_2$ is uniformized by $\mathbf{P}^2$ via the covering ϕ_2 : $[x : y : z] \to [x^2 : y^2 : z^2]$. Lifting the orbifold $\mathcal{C}_2$ to this uniformization gives the orbifold $\mathcal{C}_1 = (\mathbf{P}^2, 4K)$, where K is the smooth quartic $P(x^2, y^2, z^2) = 0$. In other words, one has $\mathcal{C}_1 \simeq \mathcal{C}_2/\mathcal{A}_2$, and there is a covering $\phi_2 : \mathcal{C}_1 \to \mathcal{C}_2$ of degree 4. The orbifold $\mathcal{C}_1$ in turn is uniformized by the K3 surface X_2 defined by the equation $w^4 = P(x^2, y^2, z^2)$. One has

$$|\pi_1^{\text{orb}}(\mathcal{C}_2)| = |\pi_1^{\text{orb}}(\mathcal{A}_2)||\pi_1^{\text{orb}}(\mathcal{C}_1)| = 4 \times 4 = 16.$$

3.1.3. The orbifold $\mathcal{C}_3$. Since any two arrangement of four lines in general position in $\mathbf{P}^2$ are projectively equivalent, one can assume that the lines L_1, L_2, L_3 and L_4 are given by the equations $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 0$ respectively. Consider the suborbifold $\mathcal{A}_4 := (\mathbf{P}^2, 4L_1 + 4L_2 + 4L_3)$ of $\mathcal{C}_3$, which is uniformized by $\mathbf{P}^2$ via the covering $\phi_4 : [x : y : z] \to [x^4 : y^4 : z^4]$. Lifting the orbifold $\mathcal{C}_3$ to this uniformization gives the orbifold $\mathcal{C}_1 = (\mathbf{P}^2, 4C)$, where C is the Fermat quartic given by the equation $x^4 + y^4 + z^4 = 0$. In other words, one has $\mathcal{C}_1 \simeq \mathcal{C}_3/\mathcal{A}_4$, and there is a covering $\phi_4 : \mathcal{C}_1 \to \mathcal{C}_3$ of degree 16. The orbifold $\mathcal{C}_1$ in turn is uniformized by the Fermat K3 surface X_3 defined by the equation $w^4 = x^4 + y^4 + z^4$ (see [**13**]). One has

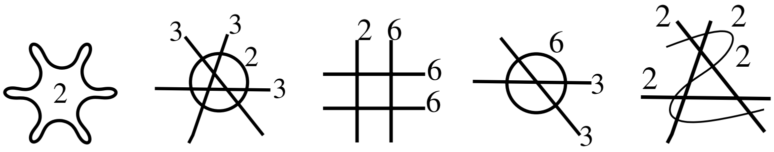
$$|\pi_1^{\text{orb}}(\mathcal{C}_3)| = |\pi_1^{\text{orb}}(\mathcal{A}_4)||\pi_1^{\text{orb}}(\mathcal{C}_1)| = 16 \times 4 = 64.$$

Note that $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$ is also a suborbifold of $\mathcal{C}_3$, which is uniformized by $\mathbf{P}^2$ via the covering $\phi_2' : [x : y : z] \to [x^2 : y^2 : z^2]$. Lifting the orbifold $\mathcal{C}_3$ to this uniformization gives the orbifold $\mathcal{C}_2 = (2L_1 + 2L_2 + 2L_3 + 4Q)$, where L_1, L_2 and L_3 are given by the equations $x = 0$, $y = 0$, $z = 0$, respectively, and Q is the smooth quadric given by the equation $x^2 + y^2 + z^2 = 0$. In other words, one has $\mathcal{C}_2 \simeq \mathcal{C}_3/\mathcal{A}_2'$, and there is a covering $\phi_2' : \mathcal{C}_2 \to \mathcal{C}_3$ of degree 4.

3.2. *The K3 orbifolds $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$ and $\mathcal{D}_5$*

Let L_1, L_2, L_3, L_4 be four distinct lines. $\mathcal{D}_1$ is the orbifold $(\mathbf{P}^2, 2S)$ where S is a smooth sextic; $\mathcal{D}_2$ is the orbifold $(\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3 + 2Q)$, where Q is a smooth quadric; $\mathcal{D}_3$ is the orbifold $(\mathbf{P}^2, 6L_1 + 6L_2 + 6L_3 + 2L_4)$; $\mathcal{D}_4$ is the orbifold $(\mathbf{P}^2, 3L_1 + 3L_2 + 6Q)$, where Q is a smooth quadric. $\mathcal{D}_5$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3 + 2C)$, where C is a smooth cubic (see Figure 2).

3.2.1. The orbifold $\mathcal{D}_1$. The double covering Y_1 of the plane along the sextic is the uniformization in this case. One has $|\pi_1^{\text{orb}}(\mathcal{D}_1)| = 2$.



$\mathcal{D}_1$ (2) $\mathcal{D}_2$ (18) $\mathcal{D}_3$ (72) $\mathcal{D}_4$ (18) $\mathcal{D}_5$ (8)

FIGURE 2. The K3 orbifolds $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$ and $\mathcal{D}_5$.

3.2.2. The orbifold $\mathcal{D}_2$. Consider the suborbifold $\mathcal{A}_3 := (\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3)$ of $\mathcal{D}_3$, which is uniformized by $\mathbf{P}^2$ via ϕ_3. Assume that L_1, L_2, L_3 and Q are defined by $x = 0$, $y = 0$, $z = 0$, respectively, and $P[x : y : z] = 0$. Lifting $\mathcal{D}_2$ by ϕ_3 gives the orbifold $\mathcal{D}_1 = (\mathbf{P}^2, 2S)$, where S is the smooth sextic $P(x^3, y^3, z^3) = 0$. In other words, one has $\mathcal{D}_1 \simeq \mathcal{D}_2/\mathcal{A}_3$, and there is a covering $\phi_3 : \mathcal{D}_1 \to \mathcal{D}_2$ of degree 9. The orbifold $\mathcal{D}_1$ in turn is uniformized by the K3 surface Y_2 which is the double cover of the plane along S. One has

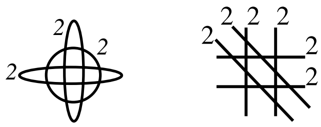
$$|\pi_1^{\text{orb}}(\mathcal{D}_2)| = |\pi_1^{\text{orb}}(\mathcal{A}_3)||\pi_1^{\text{orb}}(\mathcal{D}_1)| = 9 \times 2 = 18.$$

3.2.3. The orbifold $\mathcal{D}_3$. Assume that L_1, L_2, L_3 and L_4 are defined by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 0$, respectively. Consider the suborbifold $\mathcal{A}_6 := (\mathbf{P}^2, 6L_1 + 6L_2 + 6L_3)$ of $\mathcal{D}_3$, which is uniformized by $\mathbf{P}^2$ via ϕ_6. Lifting $\mathcal{D}_3$ by ϕ_6 gives $\mathcal{D}_1 = (\mathbf{P}^2, 2S)$, where S is the Fermat sextic $x^6 + y^6 + z^6 = 0$. In other words, one has $\mathcal{D}_1 \simeq \mathcal{D}_3/\mathcal{A}_6$, and there is a covering $\phi_6 : \mathcal{D}_1 \to \mathcal{D}_3$ of degree 36. The orbifold $\mathcal{D}_1$ in turn is uniformized by the K3 surface Y_3 which is the double cover of the plane along S. One has

$$|\pi_1^{\text{orb}}(\mathcal{D}_3)| = |\pi_1^{\text{orb}}(\mathcal{A}_6)||\pi_1^{\text{orb}}(\mathcal{D}_1)| = 36 \times 2 = 72.$$

3.2.4. The orbifold $\mathcal{D}_4$. One has $\mathcal{D}_4 \simeq \mathcal{D}_3/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$. Hence $|\pi_1^{\text{orb}}(\mathcal{D}_4)| = |\pi_1^{\text{orb}}(\mathcal{D}_3)|/|\pi_1^{\text{orb}}(\mathcal{A}_2)| = 72/4 = 18$.

3.2.5. The orbifold $\mathcal{D}_5$. One has $\mathcal{D}_5 \simeq \mathcal{D}_3/\mathcal{A}_3$, where $\mathcal{A}_3 := (\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3)$. Hence $|\pi_1^{\text{orb}}(\mathcal{D}_5)| = |\pi_1^{\text{orb}}(\mathcal{D}_3)|/|\pi_1^{\text{orb}}(\mathcal{A}_3)| = 72/9 = 8$.



$\mathcal{E}_1$ (8) $\mathcal{E}_2$ (32)

FIGURE 3. The K3 orbifolds $\mathcal{E}_1$ and $\mathcal{E}_2$.

3.3. *The K3 orbifolds $\mathcal{E}_1$ and $\mathcal{E}_2$*

These are defined as follows. $\mathcal{E}_1$ is the orbifold $(\mathbf{P}^2, 2Q_1 + 2Q_2 + 2Q_3)$, where Q_1, Q_2, Q_3 are three smooth quadrics; $\mathcal{E}_2$ is the orbifold $(\mathbf{P}^2, \sum_{i=1}^{6} 2L_i)$, where, L_i $(i \in [1, 6])$ are six distinct lines (see Figure 3).

3.3.1. The orbifold $\mathcal{E}_2$. By Lemma 2 the orbifold fundamental group is the abelian group with the presentation

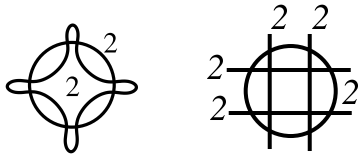
$$\pi_1^{\mathrm{orb}}(\mathcal{E}_2) \simeq \left\langle \lambda_1, \ldots, \lambda_6 \;\middle|\; 2\lambda_1 = \cdots = 2\lambda_6 = \sum_{i=1}^{6} \lambda_i = 0 \right\rangle,$$

which is isomorphic to $(\mathbb{Z}/(2))^5$ and is of order 32. In order to show that $\mathcal{E}_1$ is uniformizable, one must verify that the subgroup $\langle \lambda_i \rangle$ is the group $\mathbb{Z}/(2)$ and the subgroup $\langle \lambda_i, \lambda_j \rangle$ is the group $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ for any $i \neq j \in [1, 6]$. This is easily verified. Moreover, the euler number of this orbifold is easily seen to be $3/4$, so that the euler number of the uniformization Z is $3|\pi_1^{\mathrm{orb}}(\mathcal{E}_2)|/4 = 24$. The surface Z is simply connected since it is the universal uniformization. Since the first Chern number of $\mathcal{E}_2$ is 0, it follows that this is a K3 orbifold.

3.3.2. The orbifold $\mathcal{E}_1$. One has $\mathcal{E}_1 \simeq \mathcal{E}_2/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1+2L_2+2L_3)$. Hence $|\pi_1^{\mathrm{orb}}(\mathcal{E}_1)| = |\pi_1^{\mathrm{orb}}(\mathcal{E}_2)|/|\pi_1^{\mathrm{orb}}(\mathcal{A}_2)| = 32/4 = 8$.

3.4. *The K3 orbifolds $\mathcal{F}_1$ and $\mathcal{F}_2$*

These are defined as follows. $\mathcal{F}_1$ is the orbifold $(\mathbf{P}^2, 2Q + 2K)$, where Q is a smooth quadric and K is a smooth quartic; $\mathcal{F}_2$ is the orbifold $(\mathbf{P}^2, 2Q + \sum_{i=1}^{4} 2L_i)$, where L_i $(i \in [1, 4])$ are four distinct lines and Q is a smooth quadric (see Figure 4).



$\mathcal{F}_1$ (4) $\mathcal{F}_2$ (16)

FIGURE 4. The K3 orbifolds $\mathcal{F}_1$ and $\mathcal{F}_2$.

3.4.1. The orbifold $\mathcal{F}_2$. By Lemma 2 the orbifold fundamental group is the abelian group with the presentation

$$\pi_1^{\text{orb}}(\mathcal{F}_2) \simeq \left\langle \kappa, \lambda_1, \dots, \lambda_4 \;\middle|\; 2\kappa = 2\lambda_1 = \dots = 2\lambda_4 = \sum_{i=1}^{4} \lambda_i = 0 \right\rangle,$$

which is isomorphic to $(\mathbb{Z}/(2))^4$ and is of order 16. In order to show that $\mathcal{F}_1$ is uniformizable, one must verify that the subgroups $\langle\kappa\rangle$ and $\langle\lambda_i\rangle$ are isomorphic to $\mathbb{Z}/(2)$ and the subgroups $\langle\kappa, \lambda_j\rangle$ and $\langle\lambda_i, \lambda_j\rangle$ are isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ for any $i \neq j \in [1, 4]$. These are easily verified. Moreover, the euler number of this orbifold is easily seen to be $3/4$, so that the euler number of the uniformization is $3|\pi_1^{\text{orb}}(\mathcal{F}_2)|/4 = 24$. The universal uniformization is simply connected. Since, moreover, the first Chern number of $\mathcal{F}_2$ is 0, it follows that this is a K3 orbifold.

3.4.2. The orbifold $\mathcal{F}_1$. One has $\mathcal{F}_1 \simeq \mathcal{F}_2/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$. Hence $|\pi_1^{\text{orb}}(\mathcal{F}_1)| = |\pi_1^{\text{orb}}(\mathcal{F}_2)|/|\pi_1^{\text{orb}}(\mathcal{A}_2)| = 16/4 = 4$.

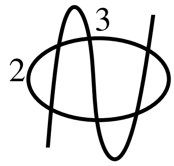
3.5. The K3 orbifold $\mathcal{G}$

This is the orbifold $(\mathbf{P}^2, 2Q + 3C)$, where Q is a smooth quadric and C is a smooth cubic (see Figure 5).

By Lemma 2 one has

$$\pi_1^{\text{orb}}(\mathcal{O}) \simeq \langle \lambda, \kappa \mid 2\lambda = 3\kappa = 0 \rangle \simeq \mathbb{Z}/(6).$$

In order to show that $\mathcal{G}$ is uniformizable, one must verify that the subgroups $\langle\kappa\rangle$ and $\langle\lambda\rangle$ are isomorphic to $\mathbb{Z}/(3)$ and $\mathbb{Z}/(2)$, respectively, and the subgroup $\langle\kappa, \lambda\rangle$ is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$ since the local orbifold fundamental group at any node of $Q \cup C$ is generated by λ and κ. These are easily verified. Moreover, the euler number of $\mathcal{G}$ is



$\mathcal{G}$ (6)

FIGURE 5. The K3 orbifold $\mathcal{G}$.

easily seen to be 4, so that the euler number of the uniformization is $4|\pi_1^{\text{orb}}(\mathcal{F}_2)| = 24$. Since the first Chern number of $\mathcal{G}$ is 0, it follows that this is a K3 orbifold.

3.6. *Completeness of the list*

THEOREM 1. *The list $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{G}$ is the complete list of K3 orbifolds with an abelian fundamental group.*

Suppose that $\mathcal{O} = (\mathbf{P}^2, B)$ is a K3 orbifold over $\mathbf{P}^2$, where $B = \sum_{i=1}^n b_i B_i$ and B_{red} is a plane curve of degree d. By Lemma 3 one has $4 \leq d \leq 6$ and $\sum_{i=1}^n \deg(B_i)/b_i = 1$ if $d = 4$. In the case when $d = 5$ one has $\sum_{i=1}^n \deg(B_i)/b_i = 2$. If $d = 6$, one has $b_1 = \cdots = b_n = 2$. Hereafter it will be assumed that these conditions hold.

On the other hand, the local fundamental group of an orbifold singularity is abelian if and only if the singularity is nodal. By the condition of smoothness on the uniformization, the local orbifold fundamental group of each singular point should inject into the global group $\pi_1^{\text{orb}}(\mathcal{O})$. Since this latter group is assumed to be abelian, and since the only orbifold singularities with abelian local fundamental group are nodes, B_{red} must be a nodal curve. We claim, moreover, that irreducible components of B_{red} must be smooth. Assume the contrary; e.g. suppose that an irreducible component B_i of B has a node p. The local orbifold fundamental group of this node has the presentation $\langle \mu_1, \mu_2 \mid b_i\mu_1 = b_i\mu_2 = 0\rangle \simeq \mathbb{Z}/(b_i) \oplus \mathbb{Z}/(b_i)$, where μ_1 and μ_2 are meridians of the branches of B_i meeting at p. However, since B_i is irreducible, the meridians μ_1 and μ_2 of B_i are conjugate elements of the group $\pi_1^{\text{orb}}(\mathcal{O})$. Since $\pi_1^{\text{orb}}(\mathcal{O})$ is abelian, one actually has $\mu_1 = \mu_2$. Hence, the subgroup of $\pi_1^{\text{orb}}(\mathcal{O})$ generated by μ_1 and μ_2 is at most $\mathbb{Z}/(b_i)$ and cannot be isomorphic to the local orbifold fundamental group $\mathbb{Z}/(b_i) \oplus \mathbb{Z}/(b_i)$ at p.

The proof of Theorem 1 will be given in Sections 3.6.1–3.6.3.

3.6.1. Quartics. Suppose that $B = b_1L_1 + b_2L_2 + b_3L_3 + b_4L_4$ with L_1, L_2, L_3, L_4 being four lines in general position. There are 14 four-tuples (b_1, b_2, b_3, b_4) with $1/b_1 + 1/b_2 + 1/b_3 + 1/b_4 = 1$. The euler number of $\mathcal{O}$ reads

$$e(\mathcal{O}) = 1 - \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4}\right) + \sum_{1\le i\ne j\le 4} \frac{1}{b_i b_j}.$$

By Lemma 3, the number $24/e(\mathcal{O})$ should be an integer. The only four-tuples (b_1, b_2, b_3, b_4) satisfying this are $(4, 4, 4, 4)$ and $(2, 6, 6, 6)$, corresponding to the orbifolds $\mathcal{C}_3$ and $\mathcal{D}_3$, respectively.

Suppose that $B = b_1L_1 + b_2L_2 + bQ$, where L_1, L_2 are two lines and Q is a smooth quadric. The only triples (b_1, b_2, b) satisfying $1/b_1 + 1/b_2 + 2/b = 1$ are $(12, 4, 3)$, $(6, 6, 3)$, $(4, 4, 4)$ and $(3, 3, 6)$. One has

$$e(\mathcal{O}) = 2 - \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{2}{b}\right) + \frac{1}{b_1b_2} + \frac{2}{b_1b} + \frac{2}{b_2b}.$$

One has $24/e(\mathcal{O}) \in \mathbb{N}$ only for the triple $(3, 3, 6)$, corresponding to the orbifold $\mathcal{D}_4$.

Suppose that $B = b_1Q_1 + b_2Q_2$, where Q_1, Q_2 are two smooth quadrics. The only pairs (b_1, b_2) satisfying $2/b_1 + 2/b_2 = 1$ are $(4, 4)$ and $(6, 3)$. One has $e(\mathcal{O}) = 3 - (2/b_1 + 2/b_2) + 4/(b_1b_2)$. The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1L + b_2C$, where L is a line and C is a smooth cubic. The only pairs (b_1, b_2) satisfying $1/b_1 + 3/b_2 = 1$ are $(4, 4)$ and $(2, 6)$. One has $e(\mathcal{O}) = 3 - (1/b_1 + 3/b_2) + 3/(b_1b_2)$. The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1K$, where K is a smooth quartic. Then $b_1 = 4$ and $e(\mathcal{O}) = 6$, corresponding to the orbifold $\mathcal{C}_1$.

3.6.2. Quintics. Suppose that $B = \sum_{i=1}^{5} b_iL_i$, where $L_1, \ldots, L_5$ are five lines in general position. There are only three five-tuples $(2, 2, 2, 3, 6)$, $(2, 2, 2, 4, 4)$, $(2, 2, 3, 3, 3)$ with reciprocals summing up to 2. The euler number of $\mathcal{O}$ reads

$$e(\mathcal{O}) = 3 - \sum_{1\le i\le 5} \frac{2}{b_i} + \sum_{1\le i\ne j\le 5} \frac{1}{b_ib_j}.$$

None of the above five-tuples satisfies $24/e(\mathcal{O}) \in \mathbb{N}$.

Suppose that $B = b_1L_1 + b_2L_2 + b_3L_3 + bQ$ with L_1, L_2 and L_3 being three lines and Q a smooth quadric. The only four-tuples (b_1, b_2, b_3, b) satisfying $1/b_1 + 1/b_2 + 1/b_3 + 2/b = 2$ are $(2, 3, 6, 2)$, $(2, 2, 2, 4)$, $(2, 4, 4, 2)$, $(3, 3, 3, 2)$ and $(2, 2, 3, 3)$. One has

$$e(\mathcal{O}) = 4 - \left(\frac{2}{b_1} + \frac{2}{b_2} + \frac{2}{b_3} + \frac{4}{b}\right) + \frac{1}{b_1b_2} + \frac{1}{b_2b_3} + \frac{1}{b_3b_1} + \frac{2}{b_1b} + \frac{2}{b_2b} + \frac{2}{b_3b}.$$

The number $24/e(\mathcal{O})$ is an integer only for the four-tuples (2, 2, 2, 4) and (3, 3, 3, 2) which corresponds to the orbifolds $\mathcal{C}_2$ and $\mathcal{D}_2$, respectively.

Suppose that $B = bL_1 + b_1Q_1 + b_2Q_2$ with Q_1, Q_2 being two smooth quadrics, and L a line. The only triples (b, b_1, b_2) satisfying $1/b+2/b_1+2/b_2 = 2$ are (2, 2, 4) and (3, 3, 2). One has

$$e(\mathcal{O}) = 5 - \left(\frac{4}{b_1} + \frac{4}{b_2} + \frac{2}{b}\right) + \frac{2}{bb_1} + \frac{2}{b_2b} + \frac{4}{b_1b_2}.$$

The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1L_1+b_2L_2+bC$, where C is a smooth cubic. The only triples (b_1, b_2, b) satisfying $1/b_1 + 1/b_2 + 3/b = 2$ are (4, 4, 2), (3, 6, 2) and (2, 2, 3). One has

$$e(\mathcal{O}) = 6 - \left(\frac{2}{b_1} + \frac{2}{b_2} + \frac{6}{b}\right) + \frac{1}{b_1b_2} + \frac{3}{bb_1} + \frac{3}{bb_2}.$$

The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1Q_+b_2C$, where Q is a smooth quadric and C is a smooth cubic. The only pairs (b_1, b_2) satisfying $2/b_1+3/b_2 = 2$ are (4, 2) and (2, 3). One has $e(\mathcal{O}) = 7 - (4/b_1 + 6/b_2) + 6/(b_1b_2)$. One has $24/e(\mathcal{O}) \in \mathbb{N}$ only for the pair (2, 3), corresponding to the orbifold $\mathcal{G}$.

In the case $B = b_1L_1 + b_2K$ with K being a smooth quartic, there are no pairs (b_1, b_2) satisfying $1/b_1 + 4/b_2 = 2$. Similarly, in the case $B = bP$ with P being a smooth quintic, one never has $5/b = 2$.

3.6.3. Sextics. It suffices to show that if $\deg(B_{\rm red}) = 6$ and $\mathcal{O}$ is not one of the orbifolds $\mathcal{D}_1, \mathcal{D}_5, \mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$ then $\mathcal{O}$ is not a K3 orbifold.

If $B = 2(L + P)$, where P is a smooth quintic, then by Lemma 2 one has $\pi_1^{\rm orb}(\mathcal{O}) \simeq \langle \lambda, \kappa \mid 2\lambda = 2\kappa = \lambda + 5\kappa = 0\rangle \simeq \mathbb{Z}/(2)$, whereas the local orbifold fundamental group of a node of B is $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, which cannot be a subgroup of $\pi_1^{\rm orb}(\mathcal{O})$. Hence $\mathcal{O}$ is not uniformizable.

If $B = 2(C_1 + C_2)$, where C_1 and C_2 are smooth cubics, then one has $\pi_1^{\rm orb}(\mathcal{O}) \simeq \langle \lambda, \kappa \mid 2\lambda = 2\kappa = 3\lambda + 3\kappa = 0\rangle \simeq \mathbb{Z}/(2)$, so that $\mathcal{O}$ is not uniformizable.

If $B = 2(L_1 + L_2 + K)$ with K being a smooth quartic then one has $\pi_1^{\rm orb}(\mathcal{O}) \simeq \langle \lambda_1, \lambda_2, \kappa \mid 2\lambda_1 = 2\lambda_2 = 2\kappa = \lambda_1 + \lambda_2 = 0\rangle$. Consider the intersection point p of L_1 and L_2. The local orbifold fundamental group at the node p is generated by λ_1 and λ_2 and is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, whereas the subgroup of $\pi_1^{\rm orb}(\mathcal{O})$ generated by λ_1 and λ_2 is isomorphic to $\mathbb{Z}/(2)$. Hence $\mathcal{O}$ is not uniformizable.

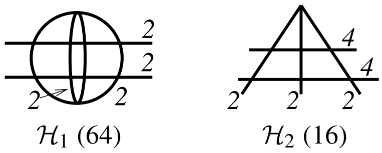


FIGURE 6. The K3 orbifolds $\mathcal{H}_1$ and $\mathcal{H}_2$.

If $B = 2(L + Q + C)$ with Q being a quadric and C a cubic, then one has $\pi_1^{\text{orb}}(\mathcal{O}) \simeq \langle \lambda, \kappa, \mu \mid 2\lambda = 2\kappa = 2\mu = \lambda + \mu = 0 \rangle$. Consider the intersection point p of L and C. The local orbifold fundamental group at the node p is generated by λ and μ and is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, whereas the subgroup of $\pi_1^{\text{orb}}(\mathcal{O})$ generated by λ and μ is isomorphic to $\mathbb{Z}/(2)$. Hence $\mathcal{O}$ is not uniformizable.

If $B = 2(L_1 + L_2 + Q_1 + Q_2)$ with Q_1 and Q_2 being smooth quadrics then $\pi_1^{\text{orb}}(\mathcal{O}) \simeq \langle \lambda_1, \lambda_2, \kappa_1, \kappa_2 \mid 2\lambda_1 = 2\lambda_2 = 2\kappa_1 = 2\kappa_2 = \lambda_1 + \lambda_2 = 0 \rangle$. Consider the intersection point p of L_1 and L_2. The local orbifold fundamental group at the node p is generated by λ_1 and λ_2 and is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, whereas the subgroup of $\pi_1^{\text{orb}}(\mathcal{O})$ generated by λ_1 and λ_2 is isomorphic to $\mathbb{Z}/(2)$. Hence $\mathcal{O}$ is not uniformizable.

4. Non-abelian K3 orbifolds

4.1. *The K3 orbifolds $\mathcal{H}_1$ and $\mathcal{H}_2$*

These are defined as follows. $\mathcal{H}_1$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2Q_1 + 2Q_2)$, where Q_1, Q_2 are two smooth quadrics tangent to each other at two distinct points, and L_1, L_2 are two lines in general position with respect to $Q_1 \cup Q_2$; $\mathcal{H}_2$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3 + 4L_4 + 4L_5)$, where L_1, L_2 and L_3 are three lines with a common point and L_4, L_5 are two lines in general position with respect to $L_1 \cup L_2 \cup L_3$ (see Figure 6).

4.1.1. The orbifold $\mathcal{H}_2$. One has $\mathcal{H}_2/\mathcal{A}_2 \simeq \mathcal{C}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$. Since $\mathcal{C}_2$ is a K3 orbifold, so is $\mathcal{H}_2$. One has

$$|\pi_1^{\text{orb}}(\mathcal{H}_2)| = |\pi_1^{\text{orb}}(\mathcal{A}_2)||\pi_1^{\text{orb}}(\mathcal{C}_2)| = 4 \times 16 = 64.$$

4.1.2. The orbifold $\mathcal{H}_1$. One has $\mathcal{H}_1 \simeq \mathcal{H}_2/\mathcal{A}_2'$, where $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_4 + 2L_5)$. Hence $|\pi_1^{\text{orb}}(\mathcal{H}_1)| = |\pi_1^{\text{orb}}(\mathcal{H}_2)|/|\pi_1^{\text{orb}}(\mathcal{A}_2)| = 64/4 = 16$.

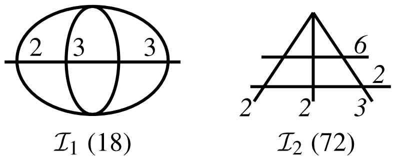


FIGURE 7. The K3 orbifolds $\mathcal{I}_1$ and $\mathcal{I}_2$.

4.2. The K3 orbifolds $\mathcal{I}_1$ and $\mathcal{I}_2$

These are defined as follows. $\mathcal{I}_1$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2Q_1 + 2Q_2)$, where Q_1, Q_2 are two smooth quadrics tangent to each other at two distinct points, and L_1 is a line in general position with respect to $Q_1 \cup Q_2$; $\mathcal{I}_2$ is the orbifold $(\mathbf{P}^2, 2L_1+2L_2+3L_3+2L_4+6L_5)$, where L_1, L_2, L_3 are three lines with a common point and $L_4 \cup L_5$ is in general position with respect to $L_1 \cup L_2 \cup L_3$ (see Figure 7).

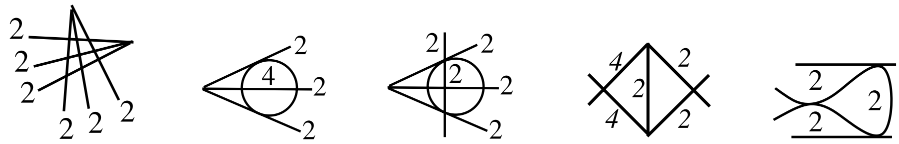
4.2.1. The orbifold $\mathcal{I}_2$. One has $\mathcal{I}_2/\mathcal{A}_2 \simeq \mathcal{D}_4$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1+2L_2+2L_4)$. Hence $|\pi_1^{\text{orb}}(\mathcal{I}_2)| = |\pi_1^{\text{orb}}(\mathcal{A}_2)||\pi_1^{\text{orb}}(\mathcal{D}_4)| = 4 \times 18 = 72$. Moreover, one has $\mathcal{I}_2/\mathcal{A}_2' \simeq \mathcal{D}_2$, where $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_5)$.

4.2.2. The orbifold $\mathcal{I}_1$. One has $\mathcal{I}_1 \simeq \mathcal{I}_2/\mathcal{A}_2''$, where $\mathcal{A}_2'' := (\mathbf{P}^2, 2L_1+2L_4+2L_5)$. Hence $|\pi_1^{\text{orb}}(\mathcal{I}_1)| = |\pi_1^{\text{orb}}(\mathcal{I}_2)|/|\pi_1^{\text{orb}}(\mathcal{A}_2'')| = 72/4 = 18$.

4.3. The K3 orbifolds $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_5$

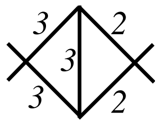
These are defined as follows. $\mathcal{J}_1$ is the orbifold $(\mathbf{P}^2, \sum_{i=1}^6 2L_i)$, where the lines L_1, L_2, L_3 meet at a point, L_3, L_4, L_5 meet at another point, and $L_1 \cup L_2 \cup L_2$ is in general position with respect to $L_4 \cup L_5 \cup L_6$; $\mathcal{J}_2$ is the orbifold $(\mathbf{P}^2, 2L_1+2L_2+2L_3+4Q)$, where L_1, L_2, L_3 meet at a point not lying on the smooth quadric Q and L_1, L_3 are tangent to Q; $\mathcal{J}_3$ is the orbifold $(\mathbf{P}^2, 2L_1+2L_2+2L_3+2L_4+2Q)$, where L_1, L_2, L_3 meet at a point not lying on Q, the lines L_1, L_3 are tangent to Q and L_4 pass through these tangent points; $\mathcal{J}_4$ is the orbifold $(\mathbf{P}^2, 4L_1+2L_2+2L_3+2L_4+4L_5)$, where L_3 pass through the points $L_1 \cap L_2$ and $L_4 \cap L_5$; $\mathcal{J}_5$ is the orbifold $(\mathbf{P}^2, 2L_1+2L_2+2K)$, where K is a quartic with a tacnode and L_1, L_2 are two flexes (with multiplicity 4) of K (see Figure 8).

One has $\mathcal{J}_4/\mathcal{A}_2 \simeq \mathcal{C}_3$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$ is a suborbifold of $\mathcal{J}_4$. Hence $|\pi_1^{\text{orb}}(\mathcal{J}_4)| = |\pi_1^{\text{orb}}(\mathcal{A}_2)||\pi_1^{\text{orb}}(\mathcal{C}_3)| = 4 \times 64 = 256$. Moreover, let $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_3 + 2L_5)$, $\mathcal{A}_2'' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$ and



$\mathcal{J}_1$ (64) $\mathcal{J}_2$ (64) $\mathcal{J}_3$ (64) $\mathcal{J}_4$ (256) $\mathcal{J}_5$ (16)

FIGURE 8. The K3 orbifolds $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_5$.



$\mathcal{K}$ (576)

FIGURE 9. The K3 orbifold $\mathcal{K}$.

$\mathcal{A}_2''' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_5)$ be suborbifolds of $\mathcal{J}_4$. Then one has $\mathcal{J}_4/\mathcal{A}_2' \simeq \mathcal{J}_1$, $\mathcal{J}_4/\mathcal{A}_2'' \simeq \mathcal{J}_2$ and $\mathcal{J}_4/\mathcal{A}_2''' \simeq \mathcal{J}_3$, showing that $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ are K3 orbifolds.

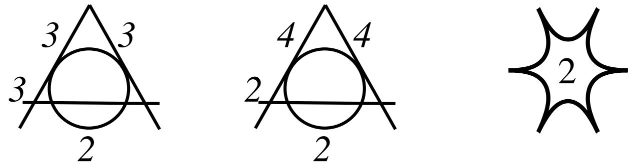
On the other hand, one has $\mathcal{J}_1/\mathcal{A}_2 \simeq \mathcal{H}_1$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1+2L_2+2L_4)$ is a suborbifold of $\mathcal{J}_1$, $\mathcal{J}_3/\mathcal{A}_2 \simeq \mathcal{H}_1$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1+2L_3+2L_4)$ is a suborbifold of $\mathcal{J}_3$, and $\mathcal{J}_3/\mathcal{A}_2' \simeq \mathcal{J}_5$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$ is a suborbifold of $\mathcal{J}_3$.

4.4. The K3 orbifold $\mathcal{K}$

This is the orbifold $(\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3 + 2L_4 + 4L_5)$ where L_3 pass through the points $L_1 \cap L_2$ and $L_4 \cap L_5$. One has $\mathcal{K}/\mathcal{A}_3 \simeq \mathcal{J}_1$, showing that $\mathcal{K}$ is a K3 orbifold (see Figure 9).

4.5. The K3 orbifolds $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$

The orbifolds $\mathcal{L}_1$, $\mathcal{L}_2$ are defined as $(\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3 + 2Q)$ and $(\mathbf{P}^2, 4L_1 + 4L_2 + 2L_3 + 2Q)$, respectively, where L_1, L_2 are tangent to Q and L_3 is in general position with respect to $L_1 \cup L_2 \cup Q$. By definition, $\mathcal{L}_3$ is the orbifold $\mathcal{L}_1/\mathcal{A}_3$, where $\mathcal{A}_3 := (\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3)$ is a suborbifold of $\mathcal{L}_1$ (see Figure 10). The locus of $\mathcal{L}_3$ is an irreducible sextic with six cusps and no other singularities. For other examples of non-abelian branched coverings along a sextic see [**14**].



$\mathcal{L}_1$ (54) $\mathcal{L}_2$ (64) $\mathcal{L}_3$ (6)

FIGURE 10. The K3 orbifolds $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

4.5.1. The orbifold $\mathcal{L}_2$. One has $\mathcal{L}_2/\mathcal{A}_2 \simeq \mathcal{P}_2$, which is a K3 orbifold (see Section 5.2).

4.5.2. The orbifold $\mathcal{L}_1$. An application of Zariski–Van Kampen (see [**5**]) gives the (redundant) presentation

$$\pi_1(\mathbf{P}^2\backslash(L_1 \cup L_2 \cup Q)) \simeq \langle \kappa, \tau_1, \tau_2 \mid (\kappa\tau_1)^2 = (\tau_1\kappa)^2,\ \kappa^2\tau_1\tau_2 = 1\rangle,$$

where κ is a meridian of Q and τ_i is a meridian of L_i. Since L_3 is in general position with respect to $L_1 \cup L_2 \cup Q$, the fundamental group $\pi_1(\mathbf{P}^2\backslash(L_1 \cup L_2 \cup L_3 \cup Q))$ admits the presentation

$$\langle \kappa, \tau_1, \tau_2, \tau_3 \mid (\kappa\tau_1)^2 = (\tau_1\kappa)^2,\ [\tau_3, \kappa] = [\tau_3, \tau_1] = [\tau_3, \tau_2] = \kappa^2\tau_3\tau_2\tau_1 = 1\rangle,$$

so that the group $\pi_1^{\text{orb}}(\mathcal{L}_1)$ admits the presentation (see [**7**])

$$\pi_1^{\text{orb}}(\mathcal{L}_1) \simeq \left\langle \kappa, \tau_1, \tau_2, \tau_3 \;\middle|\; \begin{array}{l} (\kappa\tau_1)^2 = (\tau_1\kappa)^2 \\ [\tau_3, \kappa] = [\tau_3, \tau_1] = [\tau_3, \tau_2] = 1 \\ \tau_3\tau_2\tau_1 = 1 \\ \tau_1^3 = \tau_2^3 = \tau_3^3 = \kappa^2 = 1 \end{array} \right\rangle. \tag{1}$$

The group theory software GAP [**6**] gives $|\pi_1^{\text{orb}}(\mathcal{L}_1)| = 54$. In order to show that $\mathcal{L}_1$ is a K3 orbifold, one must verify:
(i) $|\langle \tau_i \rangle| = 3$;
(ii) $|\langle \tau_i, \tau_j \rangle| = 9$ $(i \neq j)$;
(iii) $|\langle \kappa \rangle| = 2$;
(iv) $|\langle \kappa, \tau_3 \rangle| = 6$ and
(v) $|\langle \kappa, \tau_1 \rangle| = |\langle \kappa, \tau_2 \rangle| = 18$ in the group $\pi_1^{\text{orb}}(\mathcal{L}_1)$.
The conditions (i)–(iv) already hold in the abelianized orbifold fundamental group and are easily verified. By symmetry, it suffices to verify only $|\langle \kappa, \tau_1 \rangle| = 18$ in (v).

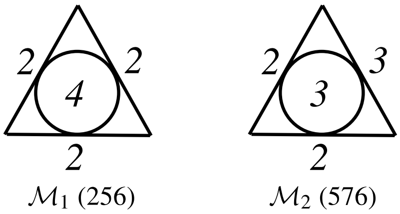


FIGURE 11. The orbifolds $\mathcal{M}_1$ and $\mathcal{M}_2$.

This latter is a routine computer check by GAP. Hence, $\mathcal{L}_1$ is uniformized by a smooth surface Y with $c_1^2(Y) = 0$. Since $e(\mathcal{L}_1) = 3/2$, one has $e(Y) = |\pi_1^{\text{orb}}(\mathcal{L}_1)|e(\mathcal{L}_1) = 24$, and Y is a K3 surface.

4.6. *The orbifolds $\mathcal{M}_1$ and $\mathcal{M}_2$*

The orbifolds $\mathcal{M}_1$, $\mathcal{M}_2$ are defined as $(\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3 + 4Q)$ and $(\mathbf{P}^2, 2L_1 + 2L_2 + 3L_3 + 3Q)$, respectively, where Q is a smooth quadric and L_1, L_2, L_3 are three distinct tangents (see Figure 11). Note that any two such arrangements are projectively equivalent. This arrangement is named the *Apollonius cycle* by Holzapfel and Vladov [**8**] and shown to support several ball and bidisc-quotient orbifolds (see also [**15**]).

4.6.1. The orbifold $\mathcal{M}_1$. One has $\mathcal{M}_1/\mathcal{A}_2 \simeq \mathcal{C}_3$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$. Note that $\mathcal{M}_1$ is invariant under the S_3-action on $\mathbf{P}^2$.

4.6.2. The orbifold $\mathcal{M}_2$. The group $\pi_1(\mathbf{P}^2 \backslash (Q \cup L_1 \cup L_2 \cup L_3))$ is the braid group on two strands of the thrice punctured sphere, and its presentation is known; see [**1**] and see [**2, 11**] for generalizations. Adding the orbifold relations to this presentation gives

$$\pi_1^{\text{orb}}(\mathcal{M}_2) \simeq \left\langle \begin{array}{c} \tau_1, \tau_2, \tau_3, \\ \kappa_1, \kappa_2, \kappa_3 \end{array} \;\middle|\; \begin{array}{l} \kappa_i = \tau_{i-1}\kappa_{i-1}\tau_{i-1}^{-1}, \ 2 \le i \le 3 \\ (\kappa_i\tau_i)^2 = (\tau_i\kappa_i)^2, \ 1 \le i \le 3 \\ [\kappa_i^{-1}\tau_i\kappa_i, \tau_j] = 1, \ 1 \le i < j \le 3 \\ \tau_3\tau_2\tau_1\kappa_1^2 = \tau_1^2 = \tau_2^2 = \tau_3^3 = \kappa^3 = 1 \end{array} \right\rangle, \quad (2)$$

where κ_i are meridians of Q and τ_i is a meridian of T_i for $1 \le i \le 3$. GAP gives $|\pi_1^{\text{orb}}(\mathcal{M}_2)| = 576$.

As a by-product of the method used in the computation of the group $\pi_1(\mathbf{P}^2\backslash(Q \cup L_1 \cup L_2 \cup L_3))$ in [**1**], the generators of the local fundamental groups around the singular points of $Q \cup L_1 \cup L_2 \cup L_3$ can be read easily from the above presentation. For the nodes the local groups are $\langle \kappa_i^{-1}\tau_i\kappa_i, \tau_j\rangle$, and for the tangency points they are $\langle \kappa_i, \tau_i\rangle$. In order to show that $\mathcal{M}_2$ is uniformizable, one needs to verify the following:

(i) $|\langle \kappa_1\rangle| = |\langle \tau_3\rangle| = 3$ and $|\langle \tau_1\rangle| = |\langle \tau_2\rangle| = 2$;
(ii) $|\langle \kappa_1^{-1}\tau_1\kappa_1, \tau_2 j\rangle| = 4$ and $|\langle \kappa_1^{-1}\tau_1\kappa_1, \tau_3\rangle| = |\langle \kappa_2^{-1}\tau_2\kappa_2, \tau_3\rangle| = 6$;
(iii) $|\langle \kappa_1, \tau_1\rangle| = |\langle \kappa_2, \tau_2\rangle| = 18$ and $|\langle \kappa_3, \tau_3\rangle| = 72$.

This is a routine check with GAP. Hence, the orbifold $\mathcal{M}_2$ is uniformizable by a smooth surface Z. Since $e(\mathcal{M}_2) = 1/24$, one has $e(Z) = e(\mathcal{M}_2)|\pi_1^{\mathrm{orb}}(\mathcal{M}_2)| = 24$, so that Z is a K3 surface, showing that $\mathcal{M}_2$ is a K3 orbifold.

4.7. Completeness of the list

THEOREM 2. *The list $\mathcal{H}_2$, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{J}_2$, $\mathcal{J}_4$, $\mathcal{K}$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$ is the complete list of K3 orbifolds over $\mathbf{P}^2$, with a locus of degree ≤ 5 and with a non-abelian fundamental group.*

It is easily shown that there are no quartic K3 orbifolds with a non-abelian fundamental group; see Section 4.7.1. Suppose that $\mathcal{O} = (\mathbf{P}^2, B)$ is a quintic K3 orbifold with a non-abelian orbifold fundamental group. In particular, the group $\pi_1(\mathbf{P}^2\backslash B)$ should be non-abelian. All quintic curves with a non-abelian fundamental group have been listed and computed by Degtyarev [**3**], so it suffices to consider the orbifolds with loci in Degtyarev's list. Evidently, those quintics with non-simple singularities are immediately eliminated. The proof of the theorem for quintic orbifolds will be given in Sections 4.7.2–4.7.6.

4.7.1. Quartics. Suppose that $\mathcal{O} = (\mathbf{P}^2, B)$ is a quartic K3 orbifold with a non-abelian orbifold fundamental group.

If $B = \sum_{i=1}^4 b_i L_i$ is linear, then since $\pi_1(\mathbf{P}^2\backslash B)$ is non-abelian, there should be a triple point. Suppose L_1, L_2 and L_3 meet at a triple point. By the orbifold condition, one has $1/b_1 + 1/b_2 + 1/b_3 > 1$, whereas the condition $c_1^2(\mathcal{O}) = 0$ forces $1/b_1 + \cdots + 1/b_4 = 1$. Hence B cannot be linear. Suppose now that $B = b_1L_1 + b_2L_2 + bQ$, where Q is a smooth quadric and L_1, L_2 are two lines. The group $\pi_1(\mathbf{P}^2\backslash B)$ is non-abelian only if L_i are both tangents of Q. But then the orbifold condition forces $1/b_i + 1/b > 1/2$ $(i = 1, 2)$, which implies $c_1^2(\mathcal{O}) = (1/b_1 + 1/b_2 + 2/b)^2 > 1$. Suppose now that $B = b_1Q_1 + b_2Q_2$.

Then $c_1^2(\mathcal{O}) = 0$ forces $2/b_1 + 2/b_2 = 1$, which implies $(b_1, b_2) \in \{(3, 6), (4, 4)\}$. The group $\pi_1(\mathbf{P}^2 \backslash B)$ is non-abelian only if Q_1 and Q_2 has a non-nodal intersection point, and it is readily seen that the orbifold condition cannot be satisfied for this point and for the above values of (b_1, b_2). Suppose now that $B = b_1 L + b_2 C$, where L is a line and C is a cubic. The group $\pi_1(\mathbf{P}^2 \backslash B)$ is non-abelian only if C is cuspidal and either L passes through the cusp of C or L is the flex of C. The condition $c_1^2(\mathcal{O}) = 0$ forces $1/b_1 + 3/b_2 = 1$. Therefore $(b_1, b_2) \in \{(2, 6), (4, 4)\}$. In either case, it is easy to see that the orbifold condition cannot be satisfied for the point $L \cdot C$. Finally, suppose that $B = bK$, where K is an irreducible quartic. The condition $c_1^2(\mathcal{O}) = 0$ forces $b = 4$. The only irreducible quartic with a non-abelian fundamental group is the 3-cuspidal quartic K (see [**21, 5**]). The group $\pi_1(\mathbf{P}^2 \backslash K)$ is non-abelian of order 12. The euler number of the orbifold $\mathcal{O} = (\mathbf{P}^2, 4K)$ is easily computed. Since K is rational, one has $e(K) = 2$ and $e(\mathbf{P}^2 \backslash K) = 1$. Since $e(K \backslash \mathrm{sing}(K)) = 2 - 3 = -1$, one has $e(\mathcal{O}) = 1 - 1/4 + 3/96 = 25/32$, so that $24/e(\mathcal{O}) \notin \mathbb{N}$.

4.7.2. Linear quintics. For Sections 4.7.2–4.7.6 set $\beta_i := 1/b_i$. Let $\mathcal{O} = (\mathbf{P}^2, B)$, where $B = \sum_{i=1}^5 b_i L_i$. One has $c_1^2(\mathcal{O}) = 0$ if and only if $\sum_{i=1}^5 \beta_i = 2$. There are only three five-tuples $(2, 2, 2, 3, 6)$, $(2, 2, 2, 4, 4)$, $(2, 2, 3, 3, 3)$ satisfying this condition.

Suppose that L_1, L_2, L_3 meet at a point, and that $L_4 \cup L_5$ is in general position with respect to $L_1 \cup L_2 \cup L_3$. Then

$$e(\mathcal{O}) = 2 - (\beta_1 + \beta_2 + \beta_3 + 2\beta_4 + 2\beta_5) + (\beta_4 + \beta_5)(\beta_1 + \beta_2 + \beta_3) + \tfrac{1}{4}[\beta_1 + \beta_2 + \beta_3 - 1]^2.$$

For the triple point to be an orbifold point one must have $\beta_1 + \beta_2 + \beta_3 > 1$. The only five-tuples satisfying this are $(2, 2, 2, 4, 4)$ and $(2, 2, 3, 6, 2)$, corresponding to the orbifolds $\mathcal{H}_2$ and $\mathcal{I}_2$, respectively.

Suppose that L_1, L_2, L_3 meet at a triple point, and L_1, L_4, L_5 meet at another point. Then

$$\begin{aligned} e(\mathcal{O}) = 1 &- (\beta_2 + \beta_3 + \beta_4 + \beta_5) + (\beta_4 + \beta_5)(\beta_2 + \beta_3) \\ &+ \tfrac{1}{4}[\beta_1 + \beta_2 + \beta_3 - 1]^2 + \tfrac{1}{4}[\beta_1 + \beta_5 + \beta_4 - 1]^2. \end{aligned}$$

For the triple points to be orbifold points one must have $\beta_1 + \beta_2 + \beta_3 > 1$ and $\beta_1 + \beta_4 + \beta_5 > 1$. The only five-tuples satisfying these conditions are $(2, 4, 2, 2, 4)$ and $(3, 3, 2, 3, 2)$, corresponding to the orbifolds $\mathcal{J}_4$ and $\mathcal{K}$ respectively.

4.7.3. A quadric with three lines. Let $B = b_1 L_1 + b_2 L_2 + b_3 L_3 + bQ$. One has $c_1^2(\mathcal{O}) = 0$ if and only if $\beta_1 + \beta_2 + \beta_3 + 2\beta = 2$. The possible values of (b_1, b_2, b_3, b) are (i) $(2, 3, 6, 2)$, (ii) $(2, 4, 4, 2)$, (iii) $(2, 2, 2, 4)$, (iv) $(3, 3, 3, 2)$ and (v) $(2, 2, 3, 3)$.

Suppose that L_1, L_2, L_3 have a common point and $L_1 \cup L_2 \cup L_3$ is in general position with respect to Q. Then

$$e(\mathcal{O}) = 3 - (\beta_1 + \beta_2 + \beta_3 + 4\beta) + 2\beta_4(\beta_1 + \beta_2 + \beta_3) + \tfrac{1}{4}[\beta_1 + \beta_2 + \beta_3 - 1]^2.$$

For the triple point to be an orbifold point one must have $\beta_1+\beta_2+\beta_3 > 1$. This occurs only in (iii), which forces $b_1 = b_2 = b_3 = 2$ and $b = 4$. But then $e(\mathcal{O}) = 21/16$, so that $24/e(\mathcal{O}) \notin \mathbb{N}$.

Suppose that L_1, L_2, L_3 have a common point, L_1 is tangent to Q, and $L_2 \cup L_4$ is in general position with respect to Q. Then

$$\begin{aligned} e(\mathcal{O}) = 2 - (\beta_2 + \beta_3 + 3\beta) + 2\beta(\beta_2 + \beta_3) + \tfrac{1}{4}[\beta_1 + \beta_2 + \beta_3 - 1]^2 \\ + \tfrac{1}{2}[\beta_1 + \beta - \tfrac{1}{2}]^2. \end{aligned}$$

For the triple point to be an orbifold point one must have $\beta_1+\beta_2+\beta_3 > 1$. This occurs only in (iii), which forces $b_1 = b_2 = b_3 = 2$ and $b = 4$. But then $e(\mathcal{O}) = 31/16$, so that $24/e(\mathcal{O}) \notin \mathbb{N}$.

Suppose that L_1, L_2, L_3 have a common point and L_1, L_2 are tangent to Q. For the triple point to be an orbifold point one must have $\beta_1+\beta_2+\beta_3 > 1$. This occurs only in (iii), which forces $b_1 = b_2 = b_3 = 2$ and $b = 4$. But then $\mathcal{O}$ is the orbifold $\mathcal{J}_2$.

Suppose that L_1, L_2, L_3 have no point in common, L_1, L_2 are tangent to Q and L_3 is in general position with respect to Q. Then

$$\begin{aligned} e(\mathcal{O}) = 2 - (\beta_1 + \beta_2 + 2\beta_3 + 2\beta) + \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 + 2\beta\beta_3 \\ + \tfrac{1}{2}[\beta + \beta_1 - \tfrac{1}{2}]^2 + \tfrac{1}{2}[\beta + \beta_2 - \tfrac{1}{2}]^2. \end{aligned}$$

For the tangent points to be orbifold singularities one must have $\beta + \beta_1 > 1/2$ and $\beta + \beta_2 > 1/2$. The number $24/e(\mathcal{O})$ is an integer only for four four-tuples (b_1, b_2, b_3, b). The four-tuples $(3, 3, 3, 2)$ and $(4, 4, 2, 2)$ yield the K3 orbifolds $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The remaining two orbifolds are not uniformizable; indeed, for $(3, 6, 2, 2)$, one should have $|\pi_1^{\mathrm{orb}}(\mathcal{O})| = 24/e(\mathcal{O}) = 64$, whereas a meridian of L_1 in this group should be of order 3. For $(2, 6, 3, 2)$ one has the presentation

$$\pi_1^{\mathrm{orb}}(\mathcal{L}_1) \simeq \left\langle \kappa, \tau_1, \tau_2, \tau_3 \;\middle|\; \begin{array}{l} (\kappa\tau_1)^2 = (\tau_1\kappa)^2 \\ [\tau_3, \kappa] = [\tau_3, \tau_1] = [\tau_3, \tau_2] = 1 \\ \tau_3\tau_2\tau_1\kappa^2 = 1 \\ \tau_1^2 = \tau_2^6 = \tau_3^3 = \kappa^2 = 1 \end{array} \right\rangle. \qquad (3)$$

This group is of order 24. But then the corresponding orbifold is not uniformizable since $24/e(\mathcal{O}) = 54 \neq |\pi_1^{\mathrm{orb}}(\mathcal{O})|$.

Suppose that L_1, L_2, L_3 have no point in common, L_1, L_2 are tangent to Q, the line L_3 pass through $L_1 \cap Q$ and in general position with respect to $L_2 \cup Q$. Then

$$e(\mathcal{O}) = 1-(\beta_2+\beta_3+\beta)+\beta_1\beta_2+\beta_2\beta_3+\beta\beta_3+\tfrac{1}{2}[\beta+\beta_1+\tfrac{\beta_3}{2}-1]^2+\tfrac{1}{2}[\beta+\beta_2-\tfrac{1}{2}]^2.$$

For the tangent points to be orbifold points one must have $\beta + \beta_1 + \beta_3/2 > 1$ and $\beta + \beta_2 > 1/2$. The number $24/e(\mathcal{O})$ is never an integer.

Let $B = b_1L_1 + b_2L_2 + b_3L_3 + bQ$, where L_1, L_2, L_3 have no point in common, L_1, L_2 are tangent to Q and L_3 pass through the tangent points. Then

$$e(\mathcal{O}) = \beta_1\beta_2 + \tfrac{1}{2}[\beta + \beta_1 + \beta_3/2 - 1]^2 + \tfrac{1}{2}[\beta + \beta_2 + \beta_3/2 - 1]^2.$$

For the tangent points to be orbifold singularities one must have $\beta + \beta_1 + \beta_3/2 > 1$ and $\beta + \beta_2 + \beta_3/2 > 1$. The number $24/e(\mathcal{O})$ is never an integer.

Let $B = b_1L_1+b_2L_2+b_3L_3+bQ$, where L_1, L_2, L_3 are tangent to Q. This case was settled in [**15**]. It yields the orbifolds $\mathcal{M}_1$ and $\mathcal{M}_2$.

4.7.4. Two quadrics with a line. Let $B = bL+b_1Q_1+b_2Q_2$. Since $\beta+2\beta_1+2\beta_2 = 2$, one has $(b, b_1, b_2) \in \{(2, 2, 4), (3, 2, 3)\}$.

Suppose that Q_1, Q_2 have an intersection point of multiplicity 4. The point $Q_1 \cap Q_2$ is an orbifold point only when $Q_1 \cap Q_2 \notin L$ and (b_1, b_2) is one of $(2, 2)$, $(3, 2)$ or $(2, 3)$. Since $\beta + 2\beta_1 + 2\beta_2 = 2$, the case $(b_1, b_2) = (2, 2)$ cannot occur. This leaves the possibilities $(3, 2, 3)$ and $(3, 3, 2)$ for (b, b_1, b_2). If L is in general position with respect to $Q_1 \cup Q_2$, then for $(b, b_1, b_2) = (3, 3, 2)$ one has $24/e(\mathcal{O}) = 384/17 \notin \mathbb{N}$. On the other hand, if L is tangent to Q_1, then for the triple $(3, 2, 3)$ one has $e(\mathcal{O}) = -7/144$, whereas for the triple $(3, 3, 2)$ one has $e(\mathcal{O}) = 25/48$.

Suppose that Q_1 and Q_2 are tangent at two distinct points. If L is in general position with respect to $Q_1 \cup Q_2$, then

$$e(\mathcal{O}) = 3 - 2(\beta + \beta_1 + \beta_2) + 2\beta(\beta_1 + \beta_2) + [\beta_1 + \beta_2 - \tfrac{1}{2}]^2.$$

Since $\beta + 2(\beta_1 + \beta_2) = 2$, this becomes $e(\mathcal{O}) = (5 + 2\beta - 3\beta^2)/4$. Recall that b can only take the values 2 and 3. For $b = 2$, one has $24/e(\mathcal{O}) = 384/21 \notin \mathbb{N}$. For $b = 3$ one has $24/e(\mathcal{O}) = 18$. The corresponding triple is $(b, b_1, b_2) = (3, 3, 2)$, which gives the orbifold $\mathcal{I}_1$. If L is tangent to Q_1 and in general position with respect to Q_2, then

$$e(\mathcal{O}) = 2 - (\beta + \beta_1 + 2\beta_2) + 2\beta\beta_2 + \tfrac{1}{2}[\beta + \beta_1 - \tfrac{1}{2}]^2 + [\beta_1 + \beta_2 - \tfrac{1}{2}]^2,$$

and $24/e(\mathcal{O})$ is never an integer. If L intersects $Q_1 \cup Q_2$ transversally at one or both of the points of $Q_1 \cap Q_2$, then this point is an orbifold point only if $\beta/2+\beta_1+\beta_2 > 1$, which contradicts the condition $\beta + 2\beta_1 + 2\beta_2 = 2$. If L is tangent to $Q_1 \cup Q_2$ at one of the points $Q_1 \cap Q_2$, then this point is an orbifold point only if $b = b_1 = b_2 = 2$. But then one cannot have $\beta + 2\beta_1 + 2\beta_2 = 2$.

4.7.5. A cubic with two lines. Let $B := bC + b_1L_1 + b_2L_2$, where C is a cubic curve. The condition $3\beta + \beta_1 + \beta_2 = 2$ implies that (b, b_1, b_2) is one of $(4, 4, 2)$, $(2, 2, 3)$ or $(3, 6, 2)$. In this subsection, we shall adopt Degtyarev's notation for quintic curves [**3**] and give only the euler numbers of the orbifolds. Non of these orbifolds are uniformizable.

The quintic $C_3(A_2) \sqcup \{\times 3\} \sqcup \{\times 2, \times 1\}$. The euler number $e(\mathcal{O})$ is

$$1 - (2\beta + \beta_2) + \beta_2(\beta + \beta_1) + \tfrac{3}{2}[\beta - \tfrac{1}{6}]^2 + \tfrac{1}{2}[\beta_1 + \beta - \tfrac{1}{2}]^2 + \tfrac{3}{4}[\beta_2 + \beta - \tfrac{2}{3}]^2.$$

The quintic $C_3(A_2) \sqcup \{A_2^*\} \sqcup \{\times 3\}$. The line L_1 is tangent to $C_3(A_2)$ at the cusp A_2. This is an orbifold point only if $b = b_1 = 2$. But then one cannot have $3\beta + \beta_1 + \beta_2 = 2$.

The quintics $C_3(A_2) \sqcup \{\underline{\times 3}\} \sqcup \{A_2, \underline{\times 1}\}$ *and* $C_3(A_2) \sqcup \{\underline{\times 3}\} \sqcup \{\underline{\times 1}, \times 1, \times 1\}$. *The flex point of* $C_3(A_2)$ *is an orbifold point only if* $\beta + \beta_1 + \beta_2/3 > 1$. But then one cannot have $3\beta + \beta_1 + \beta_2 = 2$.

The quintic $C_3(A_2) \sqcup \{\times 3\} \sqcup \{A_2, \times 1\}$. The euler number is

$$e(\mathcal{O}) = 1 - (\beta_2 + \beta) + \beta_2(\beta_1 + \beta) + \tfrac{3}{4}[\beta + \beta_1 - \tfrac{2}{3}]^2 + \tfrac{3}{2}[\beta + \beta_2/3 - \tfrac{1}{2}]^2.$$

The quintic $C_3(A_2) \sqcup \{\times 3\} \sqcup \{\times 1, \times 1, \times 1\}$. The euler number is

$$e(\mathcal{O}) = 2 - (2\beta_2 + 3\beta) + \beta_2(3\beta + \beta_1) + \tfrac{3}{2}[\beta - \tfrac{1}{6}]^2 + \tfrac{3}{4}[\beta + \beta_1 - \tfrac{2}{3}]^2.$$

The quintic $C_3(A_2) \sqcup \{\underline{\times 2}, \times 1\} \sqcup \{\times 2, \underline{\times 1}\}$. The euler number is

$$e(\mathcal{O}) = 1 - 2\beta + \beta_1\beta + \tfrac{3}{2}[\beta - \tfrac{1}{6}]^2 + \tfrac{1}{2}[\beta + \beta_2 - \tfrac{1}{2}]^2 + \tfrac{1}{2}[\beta + \beta_2/2 + \beta_1 - 1]^2.$$

The quintic $C_3(A_2) \sqcup \{A_2, \underline{\times 1}\} \sqcup \{\times 2, \underline{\times 1}\}$. The euler number is

$$e(\mathcal{O}) = 1 - \beta + \tfrac{1}{2}[\beta + \beta_2 - \tfrac{1}{2}]^2 + \tfrac{1}{4}[\beta + \beta_2 + \beta_1 - 1]^2 + \tfrac{3}{2}[\beta + \beta_1/3 - \tfrac{1}{2}]^2.$$

The cusp of C is an orbifold point only if $\beta + \beta_1/3 > 1/2$, which forces $b = 2$. But then $\beta_1 + \beta_2 + 3\beta = 2$ implies that $\beta + \beta_1 + \beta_2 = 1$. On the other hand, the triple point is an orbifold point only if $\beta + \beta_1 + \beta_2 > 1$.

The quintic $C_3(A_1) \sqcup \{\times 3\} \sqcup \{\times 3\}$. The euler number is

$$e(\mathcal{O}) = 1 - 2\beta + \beta^2 + \beta_1\beta_2 + \tfrac{3}{4}[\beta + \beta_1 - \tfrac{2}{3}]^2 + \tfrac{3}{4}[\beta + \beta_2 - \tfrac{2}{3}]^2.$$

The quintic $C_3(A_1) \sqcup \{\underline{\times 3}\} \sqcup \{\times 2, \underline{\times 1}\}$. The point of intersection of the lines is an orbifold point only if $\beta = \beta_1 = 2$. But then $\beta_1 + 3\beta = 2$, so that one cannot have $\beta_1 + \beta_2 + 3\beta = 2$.

The quintic $C_3(A_1) \sqcup \{\times 2, \underline{\times 1}\} \sqcup \{\times 2, \underline{\times 1}\}$. The euler number is

$$2 - 3\beta + \beta^2 + \tfrac{1}{2}[\beta + \beta_2 - \tfrac{1}{2}]^2 + \tfrac{1}{2}[\beta + \beta_1 - \tfrac{1}{2}]^2 + \tfrac{1}{4}[\beta + \beta_2 + \beta_1 - 1]^2.$$

4.7.6. A cubic with a quadric. Let $B = b_1 Q + b_2 C$. Then (b_1, b_2) is one of $(4, 2)$ or $(2, 3)$. Suppose that Q intersects C at two points of multiplicity 3 at each. Then

$$e(\mathcal{O}) = 1 - \beta_2 + \tfrac{3}{2}[\beta_2 - \tfrac{1}{6}]^2 + \tfrac{3}{2}[\beta_1 + \beta_2 - \tfrac{2}{3}]^2.$$

For $(\beta_1, \beta_2) = (4, 2)$ one has $e(\mathcal{O}) = 65/96$. For $(b_1, b_2) = (2, 3)$ one has $24/e(\mathcal{O}) = 32$. But if $|\pi_1^{\text{orb}}(\mathcal{O})| = 32$ then this group cannot contain elements of order 3. Hence, this orbifold is not uniformizable neither.

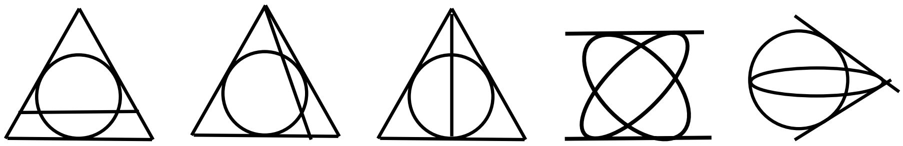
5. Some sextic K3 orbifolds

5.1. Sextic K3 orbifolds $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$ and $\mathcal{N}_5$

The orbifolds $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ are defined as follows. Let $B = 2L_1 + 2L_2 + 2L_3 + 2L_4 + 2Q$, where L_1, L_2, L_3 are three tangents of Q. The case where L_4 is in general position with respect to $Q \cup L_1 \cup L_2 \cup L_2$ gives $\mathcal{N}_1$. The case where L_4 passes through the point $L_1 \cap L_2$ and not through the point $L_3 \cap Q$ gives $\mathcal{N}_2$. Finally, the case where L_4 is the line through the points $L_1 \cap L_2$ and $L_3 \cap Q$ gives $\mathcal{N}_3$. The orbifolds $\mathcal{N}_4$ and $\mathcal{N}_5$ are defined as follows. Let $B = 2L_1 + 2L_2 + 2Q_1 + 2Q_2$, where L_1, L_2 are two common tangents of the quadrics Q_1 and Q_2. Assume that the lines do not pass through any point in $Q_1 \cap Q_2$. The case where $Q_1 \cap Q_2$ consists of four distinct points gives $\mathcal{N}_4$, and the case where $Q_1 \cap Q_2$ consists of three distinct points gives $\mathcal{N}_5$. (See Figure 12.)

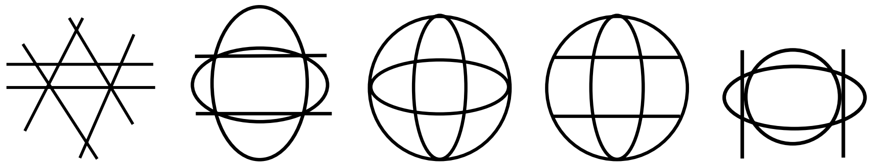
Let $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$ and $\mathcal{A}'_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$ be suborbifolds of $\mathcal{N}_1$. One has $\mathcal{N}_1/\mathcal{A}_2 \simeq \mathcal{F}_2$. Since $\mathcal{F}_2$ is a K3 orbifold, so is $\mathcal{N}_1$ and $|\pi_1^{\text{orb}}(\mathcal{N}_1)| = |\pi_1^{\text{orb}}(\mathcal{A}_2)||\pi_1^{\text{orb}}(\mathcal{F}_2)| = 4 \times 16 = 64$. On the other hand, one has $\mathcal{N}_1/\mathcal{A}'_2 = \mathcal{N}_3$, $\mathcal{N}_3/\mathcal{A}_2 = \mathcal{J}_2$ and $\mathcal{N}_3/\mathcal{A}'_2 = \mathcal{N}_5$.

One has $\mathcal{N}_2/\mathcal{A}_2 \simeq \mathcal{E}_2$. Since $\mathcal{E}_2$ is a K3 orbifold, so is $\mathcal{N}_2$ and $|\pi_1^{\text{orb}}(\mathcal{N}_1)| = |\pi_1^{\text{orb}}(\mathcal{A}_2)||\pi_1^{\text{orb}}(\mathcal{F}_2)| = 4 \times 16 = 64$. On the other hand, one has $\mathcal{N}_2/\mathcal{A}'_2 \simeq \mathcal{N}_4$.



$\mathcal{N}_1$ (64) $\mathcal{N}_2$ (128) $\mathcal{N}_3$ (256) $\mathcal{N}_4$ (32) $\mathcal{N}_5$ (64)

FIGURE 12. Sextic K3 orbifolds $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ and $\mathcal{N}_5$.



$\mathcal{P}_1$ (64) $\mathcal{P}_2$ (16) $\mathcal{P}_3$ (16) $\mathcal{P}_4$ (16) $\mathcal{P}_5$ (16)

FIGURE 13. The sextic K3 orbifolds $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and $\mathcal{P}_5$.

5.2. *The sextic K3 orbifolds $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$*

Let $\mathcal{P}_1$ be the orbifold $(\mathbf{P}^2, B)$, where $B = \sum_{i=1}^{\infty} 2L_i$ such that L_1, L_2 and L_3 meet at a point, L_3, L_4 and L_5 meet at another point, and L_6 is in general position with respect to $\cup_{i=1}^{5} L_i$. One has $\mathcal{P}_1/\mathcal{A}_2 \simeq \mathcal{F}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$. Since $\mathcal{F}_2$ is a K3 orbifold, so is $\mathcal{P}_1$. On the other hand, one has (see Figure 13):

- $\mathcal{P}_1/\mathcal{A}'_2 \simeq \mathcal{P}_2$ where $\mathcal{A}'_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_6)$;
- $\mathcal{P}_1/\mathcal{A}''_2 \simeq \mathcal{P}_3$ where $\mathcal{A}''_2 := (\mathbf{P}^2, 2L_2 + 2L_4 + 2L_6)$;
- $\mathcal{P}_1/\mathcal{A}'''_2 \simeq \mathcal{P}_4$ where $\mathcal{A}'''_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_6)$;
- $\mathcal{P}_1/\mathcal{A}''''_2 \simeq \mathcal{P}_5$ where $\mathcal{A}''''_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$.

5.3. *The sextic K3 orbifolds $\mathcal{Q}_1$ and $\mathcal{Q}_2$*

Let $\mathcal{Q}_1$ be the orbifold $(\mathbf{P}^2, B)$, where $B = \sum_{i=1}^{\infty} 2L_i$ such that L_1, L_2, L_3 meet at a point, L_3, L_4, L_5 meet at another point, and L_5, L_6, L_1 meet at another point. The orbifold $\mathcal{Q}_2$ is defined to be the lifting $\mathcal{Q}_1/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_4 + 2L_6)$; its locus consists of three quadrics Q_1, Q_2, Q_3 with six tacnodes.

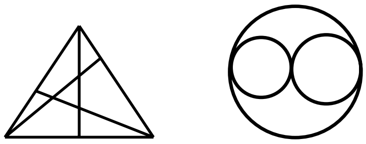


FIGURE 14. The sextic K3 orbifolds $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

(See Figure 14.) On the other hand, one has $\mathcal{Q}_1/\mathcal{A}_2' \simeq \mathcal{E}_2$, where $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_3 + 2L_5)$, showing that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are both K3 orbifolds.

5.4. *Completeness of the list*

As remarked in the introduction, we believe that our list of sextic K3 orbifolds is complete. However we leave this question open since we have no simple way to check it; the list of sextics with simple singularities found by Yang [**19**] is very long.

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