Orbifolds and their uniformization

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Abstract. This is an introduction to complex orbifolds with an emphasis on orbifolds in dimension 2 and covering relations between them.

Mathematics Subject Classification (2000). Primary 32Q30; Secondary 14J25.

Keywords. orbifold, orbiface, uniformization, ball-quotient.

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This work was supported by TÜBİTAK grant KARİYER-104T136 and achieved thanks to the support of the European Commission through its 6th Framework Program "Structuring the European Research Area" and the contract Nr. RITA-CT-2004-505493 for the provision of Transnational Access implemented as Specific Support Action.

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Introduction

These notes aims to give an introduction to the theory of orbifolds and their uniformizations, along the lines settled in 1986 by M. Kato [12], with special emphasis on complex 2-dimensional orbifolds (orbifaces).

An orbifold is a space locally modeled on a smooth manifold modulo a finite group action, which is said to be uniformizable if it is a global quotient. They were first studied in the 50's by Satake under the name "V-manifold" and renamed by Thurston in the 70's. Orbifolds appear naturally in various fields of mathematics and physics and they are studied from several points of view. In these notes we focus on the uniformization problem and consider almost exclusively orbifolds with a smooth base space. In most cases this base will be a complex projective space. From this perspective, orbifolds can be viewed as a refinement of the double covering construction of special algebraic varieties. The first steps in this refinement were taken by Hirzebruch [9], culminating in the monograph [1] devoted to Kummer coverings of $\mathbb{P}^2$ branched along line arrangements. Kobayashi [13] studied more general coverings with non-linear branch loci with non-nodal singularities.

Many basic topological invariants such as the fundamental group has an orbifold version, and the usual notion of Galois covering is extended to orbifolds in a straightforward way. It was observed by Yoshida that orbifold germs are related via covering maps. We elaborate on this observation and show that many interesting orbifolds (e.g. the ball-quotient orbifolds) are related via covering maps. Note that a covering relation between ball-quotient orbifolds is nothing but a commensurability among the corresponding lattices acting on the ball.

The plan of the paper is as follows: Section 1 gives some background on branched coverings. Section 2 includes fundamental facts and definitions about orbifolds. Section 3 is devoted to the local structure and singularities of orbifolds, especially in dimension 2. Section 4 sketches the solutions of the global uniformization problem for some special orbifolds. In particular, Section 4.1 includes a complete classification of abelian finite smooth branched coverings of $\mathbb{P}^2$. This amounts to the classification of algebraic surfaces with an abelian group action whose quotient is isomorphic to $\mathbb{P}^2$. There are also many examples of non-abelian coverings.

These notes were typeset during my stay in IHES during August 2005 and are based on talks delivered at the CIMPA Summer School *Arithmetic and Geometry around hypergeometric functions* (2005) held in Istanbul and the EMS Summer School *Braid Groups and Related Topics* (2005) held in Tiberias. I am grateful to Professor Mina Teicher and Dr. Tzachi Ben-Itzhak for their hospitality. Diagrams of the paper were typeset by Paul Taylor's *Commutative Diagrams* package.

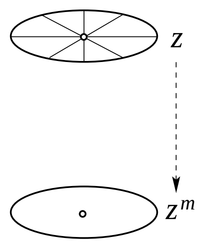


Figure 1.1. A model branched covering

1. Branched Coverings

Here we collect some facts about branched coverings which can mostly be found in Namba's book [16]. In what follows a *variety* is always irreducible, defined over $\mathbb{C}$ and endowed with the strong topology.

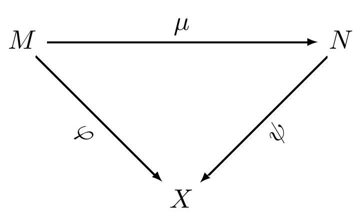
A surjective finite holomorphic map $\varphi : M \to X$ of normal varieties is called a *branched covering*. A topological finite covering map is a very special kind of branched covering. Any non-constant map between compact Riemann surfaces is a finite branched covering. If $M \subset \mathbb{P}^n$ is an irreducible hypersurface, then the restriction onto M of a *generic* projection $\mathbb{P}^n \to \mathbb{P}^{n-1}$ is a finite branched covering. A blow-down is not a branched covering since it is not a finite map. An immersion into a higher dimensional space is not a branched covering since it is not surjective.

Example 1.1. (model branched coverings) The map $\varphi_m : z \in \mathbb{C} \to z^m \in \mathbb{C}$ is a branched covering. More generally, the map

$$\varphi_m : (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \to (z_1^m, z_2, z_3, \ldots, z_n) \in \mathbb{C}^n$$

is a branched covering.

A *morphism* between branched coverings $\varphi : M \to X$ and $\psi : N \to Y$ is a surjective holomorphic map $\mu : M \to N$ such that the following diagram commutes:



An isomorphism of branched covering spaces is a morphism that is a biholomorphism. The group G_φ of all automorphisms of φ is finite and acts on every fiber of φ. A finite branched covering $\varphi : M \to X$ is called a *finite branched Galois covering* if G_φ acts transitively on every fiber of φ. In this case the orbit space M/G_φ is biholomorphic to X (see [4]).

The *ramification locus* of a finite branched covering $\varphi : M \to X$ is defined by

$$R_\varphi := \{p \in M : \varphi \text{ is not biholomorphic around } p\}$$

The image $B_\varphi := \varphi(R_\varphi)$ is called the *branch locus* of φ and the map φ is said to be branched *along* B_φ. In case φ is a topological covering then both R_φ and B_φ are empty, such φ is said to be *unbranched*. The restriction $\varphi : M\backslash R_\varphi \to X\backslash B_\varphi$ is an unbranched covering. Conversely, the Grauert-Remmert theorem states (see [18])

Theorem 1.1. *Let X be a normal variety and B a finite union of proper subvarieties of codimension one. Given a topological unbranched finite covering $\varphi' : M' \to X\backslash B$ with M' being connected, there exists an irreducible normal variety M with a finite branched covering $\varphi : M \to X$ and a homeomorphism $s : M' \to \varphi^{-1}(X\backslash B)$ such that $\varphi(x) = \varphi'(s(x))$ for all $x \in M'$.*

By this theorem, there is a correspondence between subgroups of $\pi_1(X\backslash B)$ of finite index and finite coverings of X branched along B. If φ' is Galois then so is φ ([8], Proposition I.2.8) and therefore the covering φ is Galois if and only if the corresponding subgroup is normal.

Consider the model branched covering φ_m introduced in Example 1.1. Both R_φ and B_φ are hypersurfaces in $\mathbb{C}^n$ defined by the equation $z_1 = 0$. Let $\varphi : M \to X$ be a branched covering. If X is singular, then R_φ and B_φ can be of codimension > 1, even when φ is a non-trivial branched cover. If X is smooth, then by Zariski's "purity of the branch locus" theorem (see [25]), R_φ is a hypersurface in M and B_φ is a hypersurface in X.

The *ramification divisor* of a finite branched covering $\varphi : M \to X$ of smooth spaces is the divisor of its jacobian; for singular spaces it can be defined for the restriction of φ to smooth parts of M and X and then extended. (If φ is ramified only along a singular part then the ramification divisor is empty). The ramification divisor lives on M. If $\varphi : M \to X$ is Galois, it is possible to define the *branch divisor* on X as follows: let $H_1, \dots , H_k$ be the irreducible components of the branch locus B_φ. Let $p \in H_i$ be a smooth point of B_φ. Let U be a small neighborhood of p and V be a connected component of $\varphi^{-1}(U)$. The degree m_i of $\varphi|_V$ does not depend on p and is called the *branching index* of φ along H_i. Then the branch divisor is defined as

$$D_\varphi := \sum_{i=1}^{k} m_i H_i$$

Definition 1.2. Let X be a complex manifold and $D = \Sigma m_i H_i$ be a divisor with coefficients in $\mathbb{Z}_{>0}$. A Galois covering $\varphi : M \to X$ is said to be *branched at* D if $D_\varphi = D$.

Example 1.2. Let $H \subset \mathbb{C}^n$ be a hypersurface given by the reduced polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ and let $M \subset \mathbb{C}^{n+1}$ be the hypersurface defined by the polynomial $z^m - f \in \mathbb{C}[z, x_1, \dots, x_n]$. Let π be the projection

$$\pi : (z, x_1, \dots, x_n) \in \mathbb{C}^{n+1} \to (x_1, \dots, x_n) \in \mathbb{C}^n$$

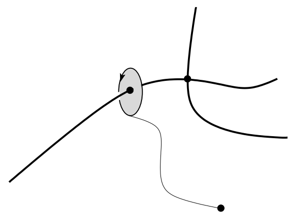


FIGURE 1.2. A meridian

Then the restriction $\pi : M \to \mathbb{C}^n$ is a finite branched Galois covering with $\mathbb{Z}/(m)$ as the Galois group. The branch locus of π is precisely the hypersurface H, and the branch divisor is mH. Note that if the origin is a singular point of H then M also has a singularity at the origin.

Let X be a normal variety and $B = \cup H_i$ be a hypersurface with irreducible components H_i. Take a base point $\star \in X\backslash B$ and let $p \in H_i$ be a smooth point of B. A *meridian* of H_i in $X\backslash B$ is the homotopy class of a loop μ_p in $X\backslash B$ constructed as follows: Take a small disc Δ intersecting B transversally at p. Let ω be a path in $X\backslash B$ connecting $\star$ to a point of $\partial\Delta$. Then $\mu := \omega \cdot \delta \cdot \omega^{-1}$, where δ is the path obtained by following $\partial\Delta$ in the positive sense. It is well known that any two meridians of a fixed irreducible component H_i are conjugate elements in $\pi_1(X\backslash B)$.

Let $D = \sum_1^k m_i H_i$, where H_i are irreducible and take meridians μ_1 of H_1 $\ldots \mu_k$ of H_k in $X\backslash \cup H_i$. The *orbifold fundamental group* of the pair (X, D) is defined as

$$\pi_1^{orb}(X, D) := \pi_1(X\backslash D, \star)/\langle\langle \mu_1^{m_1}, \ldots, \mu_k^{m_k} \rangle\rangle,$$

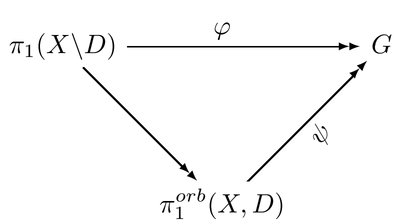
where $\langle\langle\rangle\rangle$ denotes the normal closure. (Note that the definition of an *orbifold* will wait till the next section.)

Let $D = \Sigma_1^k m_i H_i$ and let K be a normal subgroup of finite index in $\pi_1(X\backslash D)$. The Galois covering corresponding to K is branched at D if and only if the following two conditions are satisfied:

Condition (i) K contains the elements $\mu_1^{m_1}, \ldots, \mu_k^{m_k}$

Condition (ii) $\mu_i^m \notin K$ for $m < m_i$

Condition (ii) will be called the *Branching Condition* in the sequel. Let $G := \pi_1(X\backslash D)/K$ be the corresponding Galois group. Then Condition (i) amounts to the existence of the factorization



whereas the branching condition means that $\varphi(\mu_i) \in G$ is strictly of order m_i. We conclude that the coverings of X branched at D are really controlled by the group $\pi_1^{orb}(X, D)$, and there is a Galois correspondence between the Galois coverings of X branched at D and normal subgroups of $\pi_1^{orb}(X, D)$ satisfying the branching condition. In particular a covering of X branched at D is simply connected if and only if it is universal, i.e. the Galois group is the full group $\pi_1^{orb}(X, D)$. Observe that the group $\pi_1^{orb}(X, D)$ may fail to satisfy the branching condition. In this case there are no coverings of X branched at D (see Example 1.3 below).

The following lemma follows from ([7], & 7)

Lemma 1.3. *Let $M \to X$ be a Galois covering branched at D and with the Galois group G. Then there is an exact sequence*

$$0 \to \pi_1(M) \to \pi_1^{orb}(X, D) \to G \to 0$$

Example 1.3. Let $X := \mathbb{P}^n$ where $n > 1$ and let $H_0, \dots, H_k$ be k hyperplanes in general position. Let $m_0, m_1, \dots, m_k$ be $k+1$ distinct prime numbers and put $D := \Sigma_0^k m_i H_i$. By a result of Zariski the group $\pi_1(\mathbb{P}^n \backslash D)$ is abelian and it admits the presentation

$$\pi_1(\mathbb{P}^n \backslash D) \simeq \left\langle \mu_0, \dots, \mu_k \,\middle|\, \textstyle\sum_0^k m_i \mu_i = 0 \right\rangle,$$

where μ_i is a meridian of H_i for $i \in [0, k]$. Consequently, one has

$$\pi_1^{orb}(\mathbb{P}^n, D) \simeq \left\langle \mu_0, \dots, \mu_k \,\middle|\, m_0\mu_0 = \dots = m_k \mu_k = \textstyle\sum_0^k m_i \mu_i = 0 \right\rangle$$

It is easy to see that this latter group is trivial, hence there are no coverings of $\mathbb{P}^n$ branched at D. On the other hand, in case $m_0 = \dots = m_k = m$ there is a covering branched at D, since the group

$$\pi_1^{orb}(\mathbb{P}^n, D) \simeq \mathbb{Z}/(m) \oplus \ldots \oplus \mathbb{Z}/(m) \ (k \text{ copies})$$

satisfy the Branching Condition. As we will see in the next section, the universal covering branched at D is smooth if $k \geq n$. In case $k = n$ we can show this immediately: The power map

$$\varphi_m : [z_0 : \dots : z_n] \in \mathbb{P}^n \to [z_0^m : \dots : z_n^m] \in \mathbb{P}^n$$

is a Galois covering map branched at the divisor $\Sigma_0^n m H_i$ where H_i is the hyperplane $\{z_i = 0\}$ (the arrangement $H_0 \cup \dots \cup H_n$ is unique up to projective transformations). Note that φ_m is the universal covering branched at D.

1.1. Branched coverings of $\mathbb{P}^1$

Example 1.3 concerns projective spaces of dimension > 1. The situation is very different in dimension 1. Let $X = \mathbb{P}^1$, take distinct points $p_0, \dots, p_k$ in $\mathbb{P}^1$ and let $m_0, \dots, m_k$ be integers > 1. Put $D := \Sigma_0^k m_i p_i$. (According to the definition that will be given in the next section, the pair $(\mathbb{P}^1, D)$ will be an orbifold). One has the presentation

$$\pi_1(\mathbb{P}^1 \backslash \{p_0, \dots, p_k\}) \simeq \left\langle \mu_0, \dots, \mu_k \,\middle|\, \mu_0 \dots \mu_k = 1 \right\rangle,$$

which is a free group of rank k. For the orbifold fundamental group one has

$$\pi_1^{orb}(\mathbb{P}^1, D) = \left\langle \mu_1, \dots, \mu_k \,\middle|\, \mu_0^{m_0} = \dots = \mu_k^{m_k} = \mu_0 \dots \mu_k = 1 \right\rangle$$

Let $M \to \mathbb{P}^1$ be a covering branched at D with G as the Galois group. By the Riemann-Hurwitz formula the euler number $e(M)$ of M equals

$$e(M) = |G| \left[e(\mathbb{P}^1 \backslash \{p_0, \dots, p_k\}) + \textstyle\sum_0^k \dfrac{1}{m_i} \right] = |G| \left[1 - k + \textstyle\sum_0^k \dfrac{1}{m_i} \right] \tag{1.1}$$

Recall that by the Koebe-Poincaré theorem, up to biholomorphism there are only three simply connected Riemann surfaces: the Riemann sphere $\mathbb{P}^1$, the affine plane $\mathbb{C}$, and the Poincaré disc Δ. If M is a compact Riemann surface, either $e(M) > 0$ and $M \simeq \mathbb{P}^1$ (and therefore $e(M) = 2$), or $e(M) = 0$ and the universal cover of M is $\mathbb{C}$, or $e(M) < 0$ and the universal cover of M is Δ. Note that in (1.1) the signature of $e(M)$ is already determined by the data $(\mathbb{P}^1, D)$ and no information on G is needed. Accordingly, let us define

$$e^{orb}(\mathbb{P}^1, D) = 1 - k + \sum_0^k \frac{1}{m_i} \quad \Rightarrow \quad e(M) = |G| e^{orb}(\mathbb{P}^1, D) \tag{1.2}$$

In particular, if $M \to \mathbb{P}^1$ is a covering branched at D and with G as the Galois group, then

$$|G| = \frac{e(M)}{e^{orb}(\mathbb{P}^1, D)} \tag{1.3}$$

For $k = 0$ one has $e^{orb}(\mathbb{P}^1, D) = 1 + 1/m_0$, which is positive. Hence if $M \to \mathbb{P}^1$ is a covering branched at D, then $e(M) > 0 \Rightarrow M \simeq \mathbb{P}^1$. Suppose that this covering exists and let G be the Galois group. By (1.3) one has $|G| = 2/(1 + 1/m_0)$, which is not integral unless $m_0 = 1$. Hence for $k = 0$ there are no coverings branched at D, unless $m_0 = 1$. We could also deduce this result by looking at the group $\pi_1^{orb}(\mathbb{P}^1, D)$. Indeed, for $k = 0$ this group is trivial and the Branching Condition can not be satisfied.

In case $k = 1$ one has $e^{orb}(\mathbb{P}^1, D) = 1/m_0 + 1/m_1 > 0$. Hence, if a covering $M \to \mathbb{P}^1$ branched at D exists, then $M \simeq \mathbb{P}^1$. Suppose that it exists and let G be its Galois group. By (1.3) one should have $|G| = 2m_0 m_1/(m_0 + m_1) \in \mathbb{Z}_{>0}$. By the Branching Condition G must contain elements of order m_0 and m_1, in other words $|G|$ must be divisible by m_0 and m_1. This is possible only if $m := m_0 = m_1$. In this case a covering branched at D exists, it is the power map $\varphi_m : [z_0 : z_1] \in \mathbb{P}^1 \to [z_0^m : z_1^m] \in \mathbb{P}^1$. We could also deduce this result by looking at the group $\pi_1^{orb}(\mathbb{P}^1, D)$ as in the example above.

Now let us consider the case $k = 2$. Observe that a three-point set in $\mathbb{P}^1$ is projectively rigid, i.e. any two such sets can be mapped onto each other by a projective transformation. Assume $m_0 \leq m_1 \leq m_2$ and put $\rho := 1/m_0 + 1/m_1 + 1/m_2$. The orbifold euler number is then $\rho - 1$.

If $\rho - 1 > 0$ then the covering must be $\mathbb{P}^1$. Hence, if a covering branched at D exists, the Galois group must be of order $2\rho^{-1}$. In this case, either $m_0 = m_1 = 2$

Group	(m_0, m_1, m_2)	*order*
Cyclic	$(1, m, m)$	m
Dihedral	$(2, 2, m)$	$2m$
Tetrahedral	$(2, 3, 3)$	12
Octahedral	$(2, 3, 4)$	24
Icosahedral	$(2, 3, 5)$	60

TABLE 1. Finite subgroups of $\mathrm{PGL}(2, \mathbb{C})$.

or (m_0, m_1, m_2) is one of $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$, the corresponding Galois groups must be of orders $2n$, 12, 24 and 60 respectively. The group

$$\pi_1^{orb}(\mathbb{P}^1, m_0p_0 + m_1p_1 + m_2p_2) \simeq \left\langle \mu_0, \mu_1, \mu_2 \,|\, \mu_0^{m_0} = \mu_1^{m_1} = \mu_2^{m_2} = \mu_0\mu_1\mu_2 = 1 \right\rangle$$

is called a *triangle group*, it turns out that it is finite of (the right) order $2\rho^{-1}$ if $\rho > 1$ and satisfies the Branching Condition. Hence there exists Galois coverings $\mathbb{P}^1 \to \mathbb{P}^1$ branched at D. Historically this follows from Klein's classification of finite subgroups of $\mathrm{PGL}(2, \mathbb{C}) \simeq \mathrm{Aut}(\mathbb{P}^1)$. Each group is the symmetry group of one of the platonic solids inscribed in a sphere. An independent proof of this result will be given in Section 2.6, except in the icosahedral case.

If $\rho - 1 = 0$ then the orbifold euler number of $(\mathbb{P}^1, m_0p_0 + m_1p_1 + m_2p_2)$ vanish, and (m_0, m_1, m_2) is one of $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$ (one may also add the triple $(2, 2, \infty)$). In these cases, the abelianizations $\mathcal{A}b\left(\pi_1^{orb}(X, D)\right)$ are finite and satisfies the Branching Condition. Hence, they are covered by Riemann surfaces of genus 1 (an elliptic curve), and their universal covering is $\mathbb{C}$. The groups $\pi_1^{orb}(X, D)$ are infinite solvable. Similary, the Galois coverings branched at the divisors $D := 2m_0 + 2m_1 + 2m_2 + 2m_3$ are also elliptic curves. Each one of these coverings corresponds to a regular tessellation of the plane.

Any pair $(\mathbb{P}^1, D)$ not considered above has negative orbifold euler characteristic. The question of existence of finite coverings branched at D in this case is known as Fenchel's problem. It amounts to finding finite quotients of $\pi_1^{orb}(\mathbb{P}^1, D)$ satisfying the Branching Condition and is of combinatorial group theoretical in nature. Fenchel's problem has been solved by Bundgaard-Nielsen [2] and was generalized by Fox [6] to pairs (R, D) where R is a Riemann surface. These pairs are covered by Riemann surfaces of genus > 1 and their universal covering is the Poincaré disc. Summing up, we have

Theorem 1.4. *(Bundgaard-Nielsen, Fox) Let $k \geq 2$ and let $D := \Sigma_0^k m_i p_i$ be a divisor on $\mathbb{P}^1$. Then there exists a finite Galois covering $M \to \mathbb{P}^1$ which is branched at D; and M is*
(i) **(elliptic case)** $\mathbb{P}^1$ *if $k = 1$ and $m_0 = m_1$ or $k = 2$ and $\Sigma_0^2 1/m_i > 1$,*
(ii) **(parabolic case)** *a Riemann surface of genus 1 if $k = 2$ and $\Sigma_0^2 1/m_i = 1$ or*

$k = 3$ *and* $m_0 = \cdots = m_3 = 2$ *and*
(iii) **(hyperbolic case)** *a Riemann surface of genus* > 1 *otherwise.*

1.2. Fenchel's problem

In the last part of this section we present some results on branched coverings, which are of independent interest.

The natural generalization of Fenchel's problem to higher dimensions is: given a complex manifold X and a divisor with coefficients in $\mathbb{Z}_{>1}$ on X, decide whether the there exists a Galois covering $M \to X$ branched at D, regardless of the question of desingularization. There is no hope for a complete solution of the generalized Fenchel's problem as in Theorem 1.4, since the group $\pi_1(X \backslash \mathrm{supp}(D))$ does not admit a simple presentation, and it can be trivial, abelian, finite non-abelian, or infinite. However, there are some partial results obtained by several authors.

Theorem 1.5. *(Kato) Let* $H := H_0 \cup \cdots \cup H_k$ *be an arrangement of lines in* $\mathbb{P}^2$ *such that any line contains a point of multiplicity at least* 3. *Let* $m_0, \ldots, m_k \in \mathbb{Z}_{>1}$ *and put* $D := \Sigma_0^k m_i H_i$. *Then there exists a finite Galois covering of* $\mathbb{P}^2$ *branched at* D.

Kato also describes the resolution of singularities of the covering surfaces, and this resolution is compatible with the blowing-up of points of multiplicity > 2 of the branch locus. There is a generalization of the Kato theorem to conic arrangements given by Namba [16]. At the other extreme there is the following result concerning irreducible curves. Recall that for p, q coprime integers Oka [17] constructed an irreducible curve $C_{p,q}$ of degree pq and with $\pi_1(\mathbb{P}^2 \backslash C_{p,q}) \simeq \mathbb{Z}/(p) \star \mathbb{Z}/(q)$. For a proof of the following theorem see [23].

Theorem 1.6. *If* $C_{p,q}$ *is an Oka curve, then for any* $m \geq 1$ *there exists a finite Galois covering of* $\mathbb{P}^2$ *branched at* $mC_{p,q}$.

Given a projective manifold X, which groups can appear as the Galois group of a branched covering of X ? This question has the following solution.

Theorem 1.7. *(Namba* [15]*) (i) For any projective manifold* X *and any finite group* G *there is a finite branched Galois covering* $M \to X$ *with* G *as the Galois group.* *(ii) For* $n \geq 2$ *there exists a covering of the germ* $(\mathbb{C}^n, 0)$ *with a given finite Galois group.*

2. Orbifolds

In the previous section we studied branched Galois coverings of complex manifolds, which are possibly singular. Under which conditions a finite branched covering of a complex manifold is smooth? Loosely speaking, an orbifold is a pair (X, D) that locally admits a branched covering by a smooth manifold.

2.1. Transformation groups

A *transformation group* is a pair (G, M) where M is connected complex manifold and G is a group of holomorphic automorphisms of M acting properly discontinuously, in particular for any $z \in M$ the isotropy group

$$G_z := \{g \in G : gz = z\}$$

is finite. The most important example of a transformation group is (G, M), where M is a symmetric space such as the polydisc Δ^n or the n-ball B_n. Let (G, M) be a transformation group and X its orbit space with the projection $\varphi : M \to X$. The orbit space X is an irreducible normal analytic space endowed with a *b-map* defined as

$$b_\varphi : x \in X \to |G_z| \in \mathbb{Z}_{>0}$$

where $z \in \varphi^{-1}(x)$. In dimension 1 the orbit space X is always smooth. In higher dimensions X may have singularities of quotient type.

The product of two transformation groups (G_1, M_1) and (G_2, M_2) is the transformation group $(G_1 \times G_2, M_1 \times M_2)$ where $G_1 \times G_2$ acts in the obvious way.

Example 2.1. (The power map) The model example of a transformation group is $(\mathbb{Z}/(m), \mathbb{C})$, where $m \in \mathbb{Z}_{>0}$ and the element $[j] \in \mathbb{Z}/(m)$ acts by

$$\psi_{[j]} : z \in \mathbb{C} \to \omega^j z \in \mathbb{C},$$

ω being a primitive m-th root of unity. The orbit space of $(\mathbb{Z}/(m), \mathbb{C})$ is $\mathbb{C}$. The orbit map is the power map $\varphi_m : z \in \mathbb{C} \to z^m \in \mathbb{C}$. The isotropy group of the origin is the full group $\mathbb{Z}/(m)$, whereas the isotropy group of any other point is trivial. Hence the b-map is

$$b_\varphi(x) = \begin{cases} m & x = 0 \\ 1 & x \neq 0 \end{cases} \tag{2.1}$$

More generally, consider the product transformation group $(\oplus_{i=1}^n \mathbb{Z}/(m_i), \mathbb{C}^n)$. Obviously $\mathbb{C}^n$ is the orbit space of $(\oplus_{i=1}^n \mathbb{Z}/(m_i), \mathbb{C}^n)$, and the orbit map is $\varphi_{\vec{m}} : (z_1, \dots, z_n) \to (z_1^{m_1}, \dots, z_n^{m_n})$. Let H_i be the hyperplane defined by $z_i = 0$. The b-map of $\varphi_{\vec{m}}$ is

$$b_{\varphi_{\vec{m}}}(p) = \prod_{p \in H_i} m_i$$

Example 2.2. (The projective power map) Let as above $(G, \mathbb{C}^{n+1})$ be the product of $n+1$ copies of the transformation group $(\mathbb{Z}/(m), \mathbb{C})$, where $G := \oplus_{i=0}^n \mathbb{Z}/(m)$. Let ω be a primitive mth root of unity, the element $([j_0], \dots [j_n]) \in G$ acts by

$$\psi_{([j_0],\dots,[j_n])} : (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \to (\omega^{j_0} z_0, \omega^{j_1} z_1 : \cdots : \omega^{j_n} z_n) \in \mathbb{C}^{n+1}$$

Projectivizing $\mathbb{C}^{n+1}$, we get the projective space $\mathbb{P}^n$. The diagonal $\Delta := \{(g, \dots, g) \mid g \in \mathbb{Z}/(m)\}$ of G acts trivially on $\mathbb{P}^n$. The quotient $G/\Delta \simeq (\mathbb{Z}/(m))^n$ acts faithfully on $\mathbb{P}^n$. The orbit space of $(G/\Delta, \mathbb{P}^n)$ is $\mathbb{P}^n$ itself. The orbit map is

$$\varphi_m : [z_0 : \cdots : z_n] \in \mathbb{P}^n \to [z_0^m : \cdots : z_n^m] \in \mathbb{P}^n$$

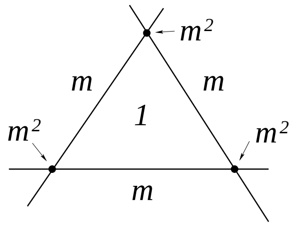


FIGURE 2.3. The b-map of the bicyclic covering $\varphi_2 : \mathbb{P}^2 \to \mathbb{P}^2$

The map φ_m is called a polycyclic covering of $\mathbb{P}^n$. Let $H_i := \{z_i = 0\}$. For any point $p \in \mathbb{P}^n$ denote by $\alpha(p)$ the number of hyperplanes H_i through p. Then the b-map of φ_m is (see Figure 2.3 for the case $n = 2$).

$$b(p) = m^{\alpha(p)} \tag{2.2}$$

Example 2.3. (A singular orbit space) Consider the action of $[j] \in \mathbb{Z}/(m)$ on $\mathbb{C}^2$ by

$$\psi_{[j]} : (x, y) \in \mathbb{C}^2 \to (\omega^j x, \omega^{-j} y) \in \mathbb{C}^2$$

The orbit space of $(\mathbb{Z}/(m), \mathbb{C}^2)$ is the hypersurface in $\mathbb{C}^3$ defined by $z^m = xy$, since the quotient map is $\psi : (x, y) \to (x^m, y^m, xy)$. This hypersurface has a cyclic quotient singularity at the origin.

2.2. Transformation groups and branched coverings

A transformation group is a locally finite branched Galois covering, as we now proceed to explain. Let (G, M) be transformation group with the orbit space X. Let $\varphi : M \to X$ be the orbit map and put

$$R_\varphi := \{z \in M \ : \ |G_z| > 1\} \ \text{ and } \ B_\varphi := \{x \in X \,|\, b_\varphi(x) > 1\} \ (= \varphi(R_\varphi)),$$

where G_z is the stabilizer of z. Let $\bar{X} := X \backslash \mathrm{Sing}(X)$ be the smooth part of X. (It can happen that a singularity of X lies on B_φ). Let $x \in \bar{X}$ and $z \in \varphi^{-1}(x)$. Let M_z be the germ of M at z and X_x the germ of X at x. Then G_z acts on M_z, and the orbit space is X_x. Since $|G_z|$ is finite and X_x is smooth, the orbit map of germs

$$\varphi_z : M_z \to X_x$$

is a finite Galois covering branched along $B_{\varphi,x}$ Therefore locally one can define the branch divisor $D_{\varphi,x}$, and the local branch divisors patch and yield a global branch divisor D_φ supported by B_φ. Let $D_\varphi = \Sigma m_i H_i$, where $H_1, H_2 \dots$ are the irreducible components of B_φ. The divisor D_φ is always locally finite and in all of the cases considered in this article, it is a finite sum. Thus D_φ is defined on the smooth part $\bar{X}$ of X - in what follows its closure in X will be denoted by D_φ again.

Let us turn our attention to the covering-germ $\varphi_z : M_z \to X_x$, which is a finite Galois covering branched *at* $D_{\varphi,x}$. Since M_z is a smooth germ, it is simply connected. Hence φ_z must be the universal covering branched at $D_{\varphi,x}$, in other words the Galois group of φ_z is

$$G_z \simeq \pi_1^{orb}(X, D_\varphi)_x,$$

where we denote the germ-pair (X_x, D_x) by $(X, D)_x$. In particular, one has

$$b(x) = |G_x| = |\pi_1^{orb}(X, D_\varphi)_x|$$

which also shows that the latter groups must be finite.

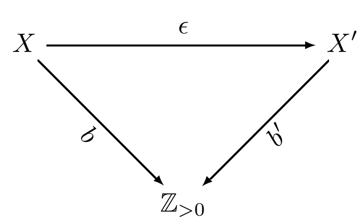
What is said above is in fact true for a singular point $x \in X$. For simplicity, assume that $x \notin B_\varphi$. Since M_z is a smooth germ it is simply connected and thus the covering germ $\varphi_z : M_z \to X_x$ must be universal. In other words the Galois group is $G_z \simeq \pi_1(X_x)$. For example, if $X \subset \mathbb{C}^3$ is defined by $z^m = xy$, then $\pi_1(X_O)$ is cyclic of order m, see Example 2.3.

2.3. b-spaces and orbifolds

Recall that a transformation group (G, M) induce a b-map on its orbit space X. Conversely, let X be a normal complex space and b a map $X \to \mathbb{Z}_{>0}$. The pair (X, b) is called a b-space. The basic question related to a b-space is the *uniformization problem:* Under which conditions does there exist a (finite) transformation group (G, M) with X as the orbit space and with the quotient map $\varphi : M \to X$ such that $b = b_\varphi$? In case such a transformation group exist, it is called a *uniformization* of (X, b) and (X, b) is said to be uniformized by (G, M). Observe that these definitions can be localized. Obviously, if (X, b) is uniformizable then it is locally finitely uniformizable, that is the germs $(X, b)_x$ must admit finite uniformization.

Definition 2.1. A locally finite uniformizable b-space (X, b) is called an *orbifold.* The space X is said to be the *base space* of (X, b), and (X, b) is said to be an orbifold *over* X. The set $\{x \in X \mid b(x) > 1\}$ is called the *locus* of the orbifold. An orbifold with a two-dimensional base is called an *orbiface.*

Orbifolds (X, b) and (X', b') are said to be equivalent if there is a biholomorphism $\epsilon : X \to X'$ such that the following diagram commutes.



The product of two b-spaces (X_1, b_1) and (X_2, b_2) is the b-space $(X_1, b_1) \times (X_2, b_2)$ which is defined as $(X_1 \times X_2, b)$ where $b(x, y) := b_1(x)b_2(y)$. If (X_i, b_i) is an uniformized by (G_i, M_i) for $i = 1, 2$, then the product orbifold is uniformized by $(G_1, M_1) \times (G_2, M_2)$.

Let (X, b) be an orbifold. Then by locally finite uniformizability its locus B is a locally finite union of hypersurfaces $H_1, H_2 \ldots$, and b must be constant along $H_i \backslash (\mathrm{Sing}(B) \cup \mathrm{Sing}(X))$. Let m_i be this number, and put $D_b := \Sigma m_i H_i$ (in most cases of interest this will be a finite sum). The orbifold fundamental group of (X, b) is defined as the orbifold fundamental group of the pair (X, D_b).

Lemma 2.2. *If (X, b) is an orbifold, then $b(x) = |\pi_1^{orb}(X, b)_x|$ for any $x \in X$.*

Proof. Let $x \in X$. Since (X, b) is an orbifold, the germ $(X, b)_x$ admits a finite uniformization. Hence there is a (unique) transformation group (G_z, M_z) with $(X, b)_x$ as the orbit space, such that $b_{\varphi_z} = b_x$, where $\varphi_z : M_z \to (X, b)_x$ is the quotient map and $\varphi_z^{-1}(x) = \{z\}$ (in other words G_z stabilizes z). By Lemma 1.3 one has the exact sequence

$$0 \to \pi_1(M_z) \to \pi_1^{orb}(X, b)_x \to G_z \to 0$$

Since M_z is smooth, it is simply connected, so that $G_z \simeq \pi_1^{orb}(X, b)_x$. Hence $b(x) = |G_z| = |\pi_1^{orb}(X, b)|$. □

Let (X, b) be an orbifold and let $D_b := \Sigma m_i H_i$ be the associated divisor. Since $\pi_1^{orb}(X, b)_x$ is defined as the group $\pi_1^{orb}(X, D_b)_x$, and since this latter group is determined by D_b, Lemma 2.2 implies that the b-function is completely determined by D_b. In other words the pair (X, D_b) determines the pair (X, b). On the other hand in dimensions ≥ 2 most pairs (X, D) do not come from an orbifold. The local uniformizability condition puts an important restriction on the possible pairs (X, D), in particular the local orbifold fundamental groups of (X, D) must be finite. In dimension 2 this latter condition is sufficient for local uniformizability, since by a theorem of Mumford a simply connected germ is smooth in dimension 2, see Theorem 3.1 below. This is no longer true in dimensions ≥ 3, see [3] for counterexamples.

Example 2.4. Consider the orbifold $(\mathbb{C}, b_m)$, where

$$b_m(z) = \begin{cases} m & z = 0 \\ 1 & z \neq 0 \end{cases}$$

This orbifold is uniformized by the transformation group $(\mathbb{Z}/(m), \mathbb{C})$, the uniformizing map is the power map. The (multivalued) inverse of a covering map is called a *developing map*. In this case the developing map is $\varphi_m^{-1} : [x : y] \in \mathbb{P}^1 \to [x^{1/m} : y^{1/m}] \in \mathbb{P}^1$.

Example 2.5. Let $p_0, \ldots, p_k$ be $k+1$ distinct points in $\mathbb{P}^1$ and let $m_0, \ldots, m_k$ be positive integers. Let $b : \mathbb{P}^1 \to \mathbb{Z}_{>0}$ be the function with $b(p_i) = m_i$ for $i \in [0, k]$ and $b(p) = 1$ otherwise. Around the point p_i the b-space $(\mathbb{P}^1, b)$ is uniformized by the transformation group $(\mathbb{Z}/(m_i), \mathbb{C})$. Hence, $(\mathbb{P}^1, b)$ is an orbifold, which can also be denoted by $(\mathbb{P}^1, \Sigma_0^k m_i p_i)$. Theorem 1.4 completely answers the question of uniformizability of these orbifolds.

Figure 2.4. Some orbiface germs with a smooth base

Example 2.6. (See Figure 2.4) Let p, q be two integers and consider the germ $(\mathbb{C}^2, b)_0$ where $b(0,0) = pq$, $b(x,0) = q$ for $x \neq 0$, $b(0,y) = p$ for $y \neq 0$ and $b(x,y) = 1$ for $xy \neq 0$. Put $H_1 := \{x = 0\}$ and $H_2 := \{y = 0\}$. The group $\pi_1(\mathbb{C}^2 \backslash (H_1 \cup H_2))_0$ is the free abelian group generated by the meridians of H_1, H_2 so that $\pi_1^{orb}(\mathbb{C}^2, b)_0 \simeq \mathbb{Z}/(p) \oplus \mathbb{Z}/(q)$ is finite. This is indeed an orbifold germ, the map $(x,y) \in \mathbb{C}^2 \to (x^p, y^q) \in \mathbb{C}$ is its uniformization. On the other hand, consider the germ of the pair $(\mathbb{C}^2, D)$ at the origin, where $D = pH_1 + qH_2 + rH_3$ with $H_1 := \{x = 0\}$, $H_2 := \{y = 0\}$ and $H_3 := \{x - y = 0\}$. One has

$$\pi_1(\mathbb{C}^2 \backslash (H_1 \cup H_2 \cup H_3)) \simeq \left\langle \mu_1, \mu_2, \mu_3 \,|\, [\mu_i, \mu_1\mu_2\mu_3] = 1 \ (i \in [1,3]) \right\rangle$$

where μ_i is a meridian of H_i for $i \in [1,3]$ (see [24]). The local orbifold fundamental group admits the presentation

$$\pi_1^{orb}(\mathbb{C}^2, D)_0 \simeq \left\langle \mu_1, \mu_2, \mu_3 \,|\, [\mu_i, \mu_1\mu_2\mu_3] = \mu_1^p = \mu_2^q = \mu_3^r = 1 \ (i \in [1,3]) \right\rangle$$

Obviously, adding the relation $\delta = 1$ to this group gives a triangle group. Hence this group is a central extension of the triangle group and is finite of order $4\rho^{-2}$ if $\rho := 1/p + 1/q + 1/r - 1 > 0$, infinite solvable when $\rho = 0$ and "big" otherwise. (Here, "big" means that the group contains non-abelian free subgroups.) Hence $(\mathbb{C}^2, D)_0$ do not come from an orbifold germ if $\rho < 0$. For $\rho > 0$ it comes from an orbifold germ, its uniformization will be described explicitly in Section 3.2.

2.4. Uniformizability

Let (X, b) be an orbifold and let D_b be the associated divisor. Recall that the group $\pi_1^{orb}(X, b)$ is by definition the group $\pi_1^{orb}(X, D_b)$. If

$$\rho : \pi_1^{orb}(X, b) \twoheadrightarrow G$$

is a surjection onto a finite group with $\mathrm{Ker}(\varphi)$ satisfying the branching condition, then there exists a Galois covering $\varphi : M \to X$ branched at D_b, where M is a possibly singular normal space.

Example 2.7. Let $(X, b) = (\mathbb{C}^2, b)$ with $b(0,0) = m^2$, $b(x,0) = m = b(0,y)$ $(x, y \neq 0)$ and $b(x,y) = 1$ otherwise, where $m \in \mathbb{Z}_{>1}$. Then $D_b = mH_1 + mH_2$, where $H_1 := \{x = 0\}$ and $H_2 := \{y = 0\}$. Consider the covering

$$\varphi : (x, y, z) \in \{z^m = xy\} \subset \mathbb{C}^3 \longrightarrow (x, y) \in \mathbb{C}^2$$

This is a $\mathbb{Z}/(m)-$Galois covering branched at D with $b_\varphi(0,0) = b_\varphi(0,y) = b_\varphi(x,0) = m$ and $b_\varphi(x,y) = 1$ otherwise. The covering space is singular. Note that $b_\varphi \neq b$. On the other hand, the Galois covering $\psi : (x,y) \in \mathbb{C}^2 \to (x^m, y^m) \in \mathbb{C}^2$ satisfies $b_\psi = b$, and it is smooth.

Lemma 2.3. *Let (X,b) be an orbifold, and $\varphi : M \to X$ a Galois covering branched at D_b. Then M is smooth if and only if $b_\varphi \equiv b$.*

Proof. For any $x \in X$, there is the induced branched covering of germs $\varphi_x : M_z \to X_x$, where $z \in \varphi^{-1}(x)$. The stabilizer G_z is the Galois group of φ_x. The germ M_z is smooth only if φ_x is the uniformization map of the germ $(X,b)_x$, which is the universal branched covering and has $\pi_1^{orb}(X,b)_x$ as its Galois group. In other words, M_z is smooth if and only if $G_z \simeq \pi_1^{orb}(X,b)_x$, if and only if

$$b_\varphi(x) = |G(z)| = |\pi_1^{orb}(X,b)_x| = b(x)$$

□

For a point $x \in X$, there is a natural map

$$\pi_1(X \backslash D_b)_x \longrightarrow \pi_1(X \backslash D_b)$$

and therefore a map $\iota_x : \pi_1^{orb}(X,b)_x \to \pi_1^{orb}(X,b)$, induced by the inclusion. The group G_z is the image of the composition map

$$\rho \circ \iota_x : \pi_1^{orb}(X,b)_x \longrightarrow \pi_1^{orb}(X,b) \to G$$

Theorem 2.4. *Let $\rho : \pi_1^{orb}(X,b) \twoheadrightarrow G$ be a surjection and let $\varphi : M \to X$ be the corresponding Galois covering of X branched along D_b. The pair (G,M) is a uniformization of the orbifold (X,b) if and only if for any $x \in X$, the map*

$$\rho \circ \iota_x : \pi_1^{orb}(X,b)_x \to G$$

is an injection.

Proof. One has $b_\varphi \equiv b$ if and only if for any $x \in X$ and $z \in \varphi^{-1}(x)$ the image G_z of $\rho \circ \iota_x$ is the full group $\pi_1^{orb}(X,b)_x$. The result then follows from Lemma 2.2. □

2.5. Sub-orbifolds and orbifold coverings

Let (X,b) be an orbifold. An orbifold (X,b') is said to be a *sub-orbifold* of (X,b) if $b'(x)$ divides $b(x)$ for any $x \in X$. Let $\varphi : Y \to X$ be a uniformization of (X,b'). Define the function $c : Y \to \mathbb{Z}_{>0}$ by

$$c(y) := \frac{b(\varphi(y))}{b'(\varphi(y))}$$

Then $\varphi : (Y,c) \to (X,b)$ is called an *orbifold covering*, and (Y,c) is called the *lifting of (X,b) to the uniformization Y of (X,b')*. The exact sequence of Lemma 1.3 can be generalized to the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1^{orb}(X,b') & \longrightarrow & G & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow \simeq & & \\ 0 & \longrightarrow & \pi_1^{orb}(Y,c) & \longrightarrow & \pi_1^{orb}(X,b) & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

Example 2.8. Let $m, n \in \mathbb{Z}_{>0}$ and consider the orbifold $(\mathbb{C}, b_{mn})$ defined in Example 2.5. Then $(\mathbb{C}, b_m)$ is a suborbifold of $(\mathbb{C}, b_{mn})$, which is uniformized via $\varphi_m : z \in \mathbb{C} \to z^m \in \mathbb{C}$. Hence φ is an orbifold covering $(\mathbb{C}, b_n) \to (\mathbb{C}, b_{mn})$.

Remark 2.5. If $Y \subset X$ is an irreducible subvariety of positive codimension, then an orbifold structure (X, b) on X induces an orbifold structure on the normalization of Y as follows (note that Y may belong to the locus of (X, b)): Let $y \in Y$, and take an irreducible branch $\widetilde{Y}_y$ of the germ Y_y. Since (X, b) is an orbifold, there is a finite uniformization $\varphi_z : M_z \to X_y$. The germ $\varphi_z^{-1}(\widetilde{Y}_y)$ may or may not be irreducible. The restriction of φ_z to an irreducible component of $\varphi_z^{-1}(\widetilde{Y}_y)$ is a branched Galois covering onto $\widetilde{Y}_y$. Let b'_y be its b-map. The b-maps b'_y for varying y patch together and yield a b-map b' on Y. Then (Y, b') is the induced orbifold structure on Y, which might also be called a suborbifold of (X, b). If $\varphi : M \to X$ is a uniformization of (X, b), then its restriction to an irreducible component of $\varphi^{-1}(Y)$ is a uniformization of (Y, b'). The induced orbifold has a significance in relative proportionality, if $\dim(X) = 2$ and $\dim(Y) = 1$ then (Y, b') is relatively proportional only if and only if the natural map $\pi_1^{orb}(Y, b') \to \pi_1^{orb}(X, b)$ is an injection.

2.6. Covering relations among triangle orbifolds

Convention. In order to present an orbifold (X, b) one has to specify its b-map. However, since by Lemma 2.2 the pair (X, D_b) determines the orbifold (X, b), an orbifold can be presented by a pair (X, D). Since the latter presentation is sometimes more practical, we shall use it in the sequel. To be precise, in what follows the expression "the orbifold (X, D)" refers to the pair (X, b), where the b-map is defined by $b(p) := |\pi_1^{orb}(X, D)_p|$ (it is implicitly assumed that (X, b) is indeed an orbifold, i.e. it is locally finite uniformizable).

Let us illustrate the notion of orbifold coverings in the simplest, one-dimensional setting. In this subsection, we fix three points $p_0 = [1 : 0]$, $p_1 = [0 : 1]$, $p_2 := [1 : 1]$ in $\mathbb{P}^1$. Consider first the orbifold $(\mathbb{P}^1, rm_0p_0 + rm_1p_1)$. Then $(\mathbb{P}^1, rp_0 + rp_1)$ is a suborbifold, which is uniformized by $(\mathbb{Z}/(r), \mathbb{P}^1)$ via $\varphi_r : [x : y] \to [x^r : y^r]$. Hence, there is an orbifold covering

$$\varphi_r : (\mathbb{P}^1, m_0p_0 + m_1p_1) \to (\mathbb{P}^1, rm_0p_0 + rm_1p_1)$$

Coverings of triangle orbifolds, elliptic case. Now consider the orbifold $(\mathbb{P}^1, 2p_0 + 2p_1 + mp_2)$. Then $(\mathbb{P}^1, 2p_0 + 2p_1)$ is a suborbifold, which is uniformized by $\mathbb{P}^1$ via φ_2. Hence, there is a covering as in Figure 2.5, where $q_0 := [1 : 1]$, $q_1 := [1 : -1]$,

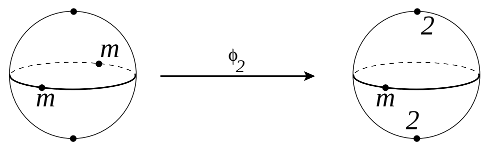


FIGURE 2.5. The covering $\varphi_2 : (\mathbb{P}^1, mq_0 + mq_1) \to (\mathbb{P}^1, 2p_0 + 2p_1 + mp_2)$

so that $\{q_0, q_1\} = \varphi_2^{-1}(p_2)$. One can map $(\mathbb{P}^1, mq_0 + mq_1)$ onto $(\mathbb{P}^1, mp_0 + mp_1)$ by a projective transformation. Since this latter orbifold is uniformized by φ_m, one has a chain of coverings

$$\mathbb{P}^1 \overset{\varphi_m}{\to} (\mathbb{P}^1, mq_0 + mq_1) \overset{\varphi_2}{\to} (\mathbb{P}^1, 2p_0 + 2p_1 + mp_2)$$

Then $\varphi_2 \circ \varphi_m$ is the uniformization of the dihedral orbifold $(\mathbb{P}^1, 2p_0 + 2p_1 + mp_2)$. (The covering $\varphi_2 \circ \varphi_m$ is Galois since it is universal). Now consider the octahedral orbifold $(\mathbb{P}^1, 2p_0 + 4p_1 + 3p_2)$. There is a covering

$$\varphi_2 : (\mathbb{P}^1, 2p_1 + 3q_0 + 3q_1) \to (\mathbb{P}^1, 2p_0 + 4p_1 + 3p_2)$$

Since any set of three points can be mapped to any set of three points on $\mathbb{P}^1$, one has $(\mathbb{P}^1, 2p_1 + 3q_0 + 3q_1) \simeq (\mathbb{P}^1, 3p_0 + 3p_1 + 2p_2)$. This latter orbifold admits the covering

$$\varphi_3 : (\mathbb{P}^1, 2r_0 + 2r_1 + 2r_2) \to (\mathbb{P}^1, 3p_0 + 3p_1 + 2p_2)$$

where $r_0 = [1 : 1]$, $r_1 := [1 : \omega]$, $r_2 := [1 : \omega^2]$ and ω being a primitive cubic root of unity, so that $\{r_0, r_1, r_2\} = \varphi_3^{-1}(p_2)$.

Exercice 2.1. Write down the uniformizing map of the octahedral orbifold explicitly.

Coverings of triangle orbifolds, parabolic case. Consider the orbifold $(\mathbb{P}^1, \Sigma_0^2 3p_i)$. The orbifold $(\mathbb{P}^1, 3p_0 + 3p_1)$ is a suborbifold uniformized by $\mathbb{P}^1$ via φ_3, and $\varphi_3^{-1}(p_2) = \{r_0, r_1, r_2\}$ as above. Hence, there is an orbifold covering as in Figure 2.6:

Since any two set of three points on $\mathbb{P}^1$ are projectively equivalent, we see that the orbifold $(\mathbb{P}^1, \Sigma_0^2 3p_i)$ admits a self-covering. This is not very surprising, since it is uniformized by the elliptic curve C which admits an automorphism of order 3, whose quotient is C.

Exercice 2.2. Discover the coverings of the remaining parabolic orbifolds with parameters $(2, 4, 4)$, $(2, 3, 6)$ and $(2, 2, 2, 2)$ (one can also add the parameters (∞, ∞) and $(2, 2, \infty)$ to this list)

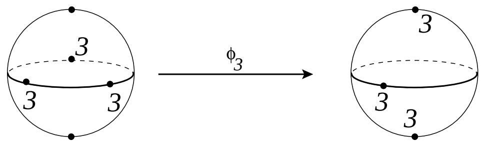


FIGURE 2.6. The covering $\varphi_3 : (\mathbb{P}^1, \Sigma_0^2 3r_i) \to (\mathbb{P}^1, \Sigma_0^2 3p_i)$

Coverings of triangle orbifolds, hyperbolic case. As an example, consider the orbifold $(\mathbb{P}^1, 5p_0 + 5p_1 + mp_2)$, which is hyperbolic for any $m \in \mathbb{Z}_{>1}$. The orbifold $(\mathbb{P}^1, 5p_0 + 5p_1)$ is a suborbifold uniformized by $\mathbb{P}^1$ via φ_5, and $\varphi_5^{-1}(p_2) = \{s_i := [1, \xi^i] \,|\, i \in [0,4]\}$, where ξ is a primitive fifth root of unity. Hence, there is an orbifold covering as in Figure 2.7.

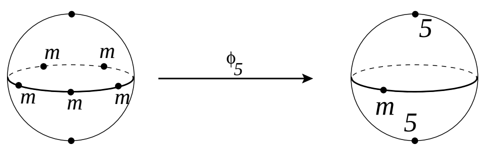


FIGURE 2.7. The covering $\varphi_5 : (\mathbb{P}^1, \Sigma_0^4 ms_i) \to (\mathbb{P}^1, 5p_0 + 5p_1 + mp_2)$

Now $(\mathbb{P}^1, ms_0 + ms_1)$ is a suborbifold of $(\mathbb{P}^1, \Sigma_0^4 ms_i)$, and it is clear how one can continue in this manner to get an infinite tower of hyperbolic orbifolds.

3. Orbifold Singularities

Recall that an orbifold germ $(X, b)_x$ is a germ that admits a finite uniformization by a transformation group (G_z, M_z), where M_z is a smooth germ and G_z is a finite group acting on M_z and fixes z. According to a classical result of Cartan [4], any orbifold germ $(X, b)_x$ is in fact equivalent to the quotient of the germ $\mathbb{C}_0^n$ by finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. In other words, any orbifold germ $(X, b)_x$ admits a finite uniformization by $(G, \mathbb{C}^n)$ where G is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. Observe that any finite group appears as a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for sufficiently large n. For small n these subgroups can be effectively classified.

Any finite subgroup of $\mathrm{GL}(\mathbb{C}, 1) \simeq \mathbb{C}^*$ is cyclic and is generated by a root of unity, its orbit space is $\mathbb{C}$ and the quotient map is the power map. Hence in dimension one, any orbifold germ $(X, b)_x$ is of the form $(\mathbb{C}, mO)_O$, where $O \in \mathbb{C}$ is the origin. In higher dimensions, an orbifold germ $(X, b)_x$ may have singularities. Resolution graphs of all orbiface singularities can be found in the appendix to [13]. Let us first consider orbifolds (X, b) with a smooth base space X.

Let $G \subset \mathrm{GL}(n, \mathbb{C})$. Then G acts on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ by

$$M(P)(x) := P(M^{-1}x)$$

The the ring of invariant polynomials under this action is denoted by $\mathbb{C}[x_1, \ldots, x_n]^G$. Recall that $M \in \mathrm{GL}(n, \mathbb{C})$ is called a *reflection* if one of its eigenvalues is a root of unity $\omega \neq 1$ and the remaining eigenvalues are all 1. A group $G \subset \mathrm{GL}(n, \mathbb{C})$ is called a *reflection group* if it is generated by reflections. By Chevalley's theorem [5] the ring $\mathbb{C}[x_1, \ldots, x_n]^G$ is generated by n algebraically independent homogeneous invariants if and only if G is a reflection group. In geometrical terms, the quotient $\mathbb{C}^n/G$ is isomorphic to $\mathbb{C}^n$ if and only if G is a reflection group. In other words, germs $(X, b)_x$ with a smooth base are in a one-to-one correspondence with finite reflection groups.

Irreducible finite reflection groups has been classified by Shepherd and Todd [19]. A group $G \subset \mathrm{GL}(n, \mathbb{C})$ is called *imprimitive* if $\mathbb{C}^n$ can be decomposed as a non-trivial direct sum of subspaces permuted by G, otherwise it is called *primitive*. Matrices permuting the coordinates of $\mathrm{GL}(n, \mathbb{C})$ generate the symmetric group S_n, which is primitive. Aside from S_n and an infinite family of imprimitive groups $G(m, p, n)$ there are only a finite number of primitive reflection groups, which are called exceptional reflection groups. There are no exceptional reflection groups in dimensions ≥ 9.

Observe that if G is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ then its projectivization $\mathrm{P}G$ is a subgroup of $\mathrm{PGL}(n, \mathbb{C})$. The extension $G \to PG$ is central, since its kernel is generated by the multiples of the identity matrix I. If G is finite, then the kernel of $G \to \mathrm{P}G$ is generated by ωI, where ω is a root of unity.

3.1. Orbiface singularities

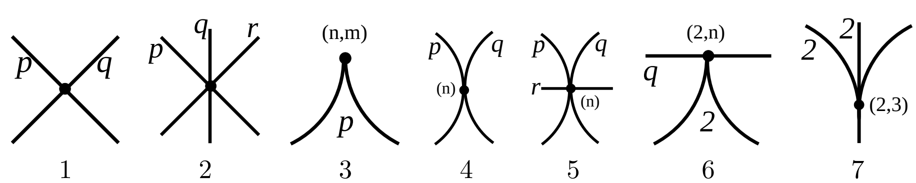
The following theorem gives a topological characterization of orbiface germs.

Theorem 3.1. *In dimension two, $(X, b)_x$ is an orbiface germ if and only its orbifold fundamental group $\pi_1^{orb}(X, b)_x$ is finite.*

Proof. We must show that $(X, b)_x$ admits a finite smooth uniformization. Since $\pi_1^{orb}(X, b)_x$ is finite, its universal covering is a finite covering by a simply connected germ. In dimension two, a simply connected germ is smooth by Mumford's theorem [14] (this is wrong in dimensions > 2, see [3] for a counterexample). The other direction is clear. ☐

We will mostly consider orbifaces with a smooth base. The following result characterizes their germs.

Theorem 3.2. *All orbiface germs with a smooth base are given in the table below.*



No	*Equation*	*Condition*	*Order*
1	xy	$--$	pq
2	$xy(x+y)$	$0<\rho:=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1$	$4\rho^{-2}$
3	x^n-y^m $(\gcd(n,m)=1)$	$0<\rho:=\frac{1}{n}+\frac{1}{m}+\frac{1}{p}-1$	$\frac{4}{nm}\rho^{-2}$
4	x^2-y^{2n} $(n\geq 2)$	$0<\rho:=\frac{1}{p}+\frac{1}{q}+\frac{1}{n}-1$	$\frac{4}{n}\rho^{-2}$
5	$y(x^2-y^{2n})$	$0<\rho:=\frac{1}{p}+\frac{1}{q}+\frac{1}{nr}-1$	$\frac{4}{n}\rho^{-2}$
6	$y(x^2-y^n)$ $(n\ \ odd)$	$--$	$2nq^2$
7	$x(x^2-y^3)$	$--$	96

Table 2. Orbiface germs with a smooth base

In dimension 2, Yoshida observed the following facts (see [24]): If $H \subset \mathrm{GL}(2,\mathbb{C})$ is a reflection group with a non-abelian PG, then among the reflection groups with the same projectivization there is a maximal one G containing H. Every reflection group K with $PK = PG$ is a normal subgroup of this maximal reflection group. In other words, the germ $\mathbb{C}^2/K$ is a Galois covering of $\mathbb{C}^2/G$. If G is maximal reflection group, then the quotient $\mathbb{C}^2/G$ is familiar from Example 2.6; it is the orbiface $(\mathbb{C}^2, pX+qY+rZ)$ for some (p,q,r) with $1/p+1/q+1/r-1>0$, where X, Y, Z are three lines meeting at the origin (recall our convention in 2.6). Hence any orbiface germ $(X,b)_x$ with a smooth base X_x is a covering of the germ $(\mathbb{C}^2, pX+qY+rZ)_0$.

3.2. Covering relations among orbiface germs

Below we give some examples of covering relations among orbiface germs.
The abelian germs $(\mathbb{C}^2, pX+qY)_0$**.** Abelian reflection groups are always reducible, and therefore isomorphic to a $\mathbb{Z}/(p)\oplus\mathbb{Z}/(q)$ for some p, q. Let us study some coverings of the quotient orbiface germ, which is equivalent to $(\mathbb{C}^2, pX+qY)_0$ where $X := \{x=0\}$ and $Y : \{y=0\}$. Any smooth sub-orbiface of this orbiface is of the form $(\mathbb{C}^2, rX+sY)_0$ where $r|p$ and $s|q$ and $r,s\in\mathbb{Z}_{\geq 1}$. This latter orbiface germ is uniformized by $\mathbb{C}^2_0$ via the map $\varphi_{r,s} : (x,y)\in\mathbb{C}^2 \to (x^r,y^s)\in\mathbb{C}^2$, with $\mathbb{Z}/(r)\oplus\mathbb{Z}/(s)$ as the Galois group. The lifting of $(\mathbb{C}^2, pX+qY)_0$ to this uniformization is the orbiface $(\mathbb{C}^2, \frac{p}{r}X, \frac{q}{s}Y)_0$. In other words, $\mathbb{Z}/(r)\oplus\mathbb{Z}/(s)$ acts on the orbiface germ $(\mathbb{C}^2, \frac{p}{r}X, \frac{q}{s}Y)_0$, and the quotient is $(\mathbb{C}^2, pX+qY)_0$.
The dihedral germ $(\mathbb{C}^2, 2X+2Y+mZ)_0$**.** Here we discuss the case where m is odd, the case of even m is left as an exercise. This orbiface has the suborbifaces $(\mathbb{C}^2, 2X)_0$, $(\mathbb{C}^2, 2Y)_0$, $(\mathbb{C}^2, mZ)_0$, $(\mathbb{C}^2, 2X+2Y)_0$, $(\mathbb{C}^2, 2Y+mZ)_0$ and $(\mathbb{C}^2, mZ+2X)_0$. Each one of these suborbifaces is uniformized by $\mathbb{C}^2_0$ via a bicyclic map $\varphi_{p,q}$, note that $\varphi_{r,s}\circ\varphi_{p,q}=\varphi_{rp,sq}$. The uniformizer of $(\mathbb{C}^2, 2X)_0$ is the map $\varphi_{2,1}$. Denote the branch $\varphi_{2,1}^{-1}(Y)=\{y=0\}$ by Y and the branch $\varphi_{2,1}^{-1}(Z)=\{x^2-y=0\}$ by Z'. Hence $\varphi_{2,1}$ is an orbiface covering

$$(\mathbb{C}^2, 2Y+mZ')_0 \to (\mathbb{C}^2, 2X+2Y+mZ)_0$$

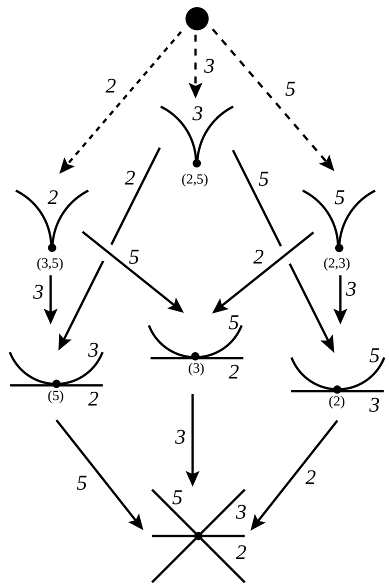


FIGURE 3.8. Coverings of the icosahedral orbiface germ

Now $\varphi_{1,2}$ is a covering of $(\mathbb{C}^2, 2Y + mZ')_0$ and one has $\varphi_{1,3}^{-1}(\mathbb{Z}') = \{x^2 - y^2 = 0\}$. Put $U := \{x + y = 0\}$ and $V := \{x - y = 0\}$. There is an orbiface covering

$$(\mathbb{C}^2, mU + mV)_0 \to (\mathbb{C}^2, 2Y + mZ')_0$$

which is related to the suborbiface $(\mathbb{C}^2, 2X + 2Y)_0$ of the initial orbiface $(\mathbb{C}^2, 2X + 2Y + mZ)_0$. The germ $(\mathbb{C}^2, mU + mV)_0$ is uniformized by $\mathbb{C}^2$ via $\varphi_{m,m}$. Hence $\varphi_{2,1} \circ \varphi_{m,m}$ is the uniformization of the dihedral germ $(\mathbb{C}^2, 2X + 2Y + mZ)_0$.

The icosahedral germ $(\mathbb{C}^2, 2X + 3Y + 5Z)_0$**.** This orbiface has the suborbifaces $(\mathbb{C}^2, 2X)_0$, $(\mathbb{C}^2, 3Y)_0$, $(\mathbb{C}^2, 5Z)_0$, $(\mathbb{C}^2, 2X + 3Y)_0$, $(\mathbb{C}^2, 3Y + 5Z)_0$ and $(\mathbb{C}^2, 5Z + 2X)_0$. Keeping the notations of the preceding paragraph, there is an orbiface covering

$$\varphi_{1,2} : (\mathbb{C}^2, 3Y + 5Z')_0 \to (\mathbb{C}^2, 2X + 3Y + 5Z)_0$$

Now $\varphi_{1,3}$ is a covering of $(\mathbb{C}^2, 3Y + 5Z')_0$, such that $\varphi_{1,3}^{-1}(\mathbb{Z}') = \{x^2 - y^3 = 0\}$, so that there is an orbiface covering

$$(\mathbb{C}^2, 5Z'')_0 \to (\mathbb{C}^2, 3Y + 5Z')_0$$

which is related to the suborbiface $(\mathbb{C}^2, 2X + 3Y)_0$ of the initial orbiface $(\mathbb{C}^2, 2X + 3Y + 5Z)_0$. For coverings corresponding to other suborbifaces, see Figure 3.8.

The black dot on top of Figure 3.8 represents the isolated surface (Du Val) singularity of type E_8 given by the equation $S := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = 0\}$. It is clear how the projection $(x, y, z) \to (x, y)$ defines a $\mathbb{Z}/(5)$-orbiface covering by this singularity of the the orbiface $(\mathbb{C}^2, 5Z'')_0$. Other coordinate projections define respectively $\mathbb{Z}/(2)$ and $\mathbb{Z}/(3)$-coverings by the same singularity of the orbifaces $(\mathbb{C}^2, 2X'')_0$ $(\mathbb{C}^2, 3Y'')_0$, defined in the same way as $(\mathbb{C}^2, 5Z'')$. The germ of S at the origin is the universal homology covering (i.e. the maximal abelian covering) of the germ $(\mathbb{C}^2, 2X + 3Y + 5Z)_0$. Notice that S_0 is an orbiface germ with a singular base space and empty branch divisor.

Exercice 3.1. Study the covering relations among other orbiface germs with a smooth base. More generally, study the covering relation among orbiface germs with a singular base and possibly with branch loci.

3.3. Orbifaces with cusps

Many transformation groups (G, M) encountered in practice are not cocompact. In many cases, the orbit space M/G admits a "nice" compactification. It is possible to incorporate the compactifications into the orbifold theory by considering pairs (X, b) with extended b-functions with values in $\mathbb{N} \cup \{\infty\}$, and by declaring that the points with infinite b-value are added in the compactification process. Outside the points with an infinite b-value, the pair (X, b) remains an orbifold. Lets consider the case where M is the 2-ball B_2, and G is a finite volume discrete subgroup of $\mathrm{Aut}(B_2)$. Let (X, b) be the quotient orbifold. Then the germs $(X, b)_p$ with $b(p) = \infty$ are called *ball-cusp points*. For smooth X, a classification of ball-cusp points was given in [24]. It turns out that any such germ is a covering of one of the germs (i) $(\mathbb{C}^2, pH_1 + qH_2 + rH_3)_0$ with $\rho := 1/p + 1/q + 1/r = 1$ and (ii) $(\mathbb{C}^2, 2H_1 + 2H_2 + 2H_3 + 2H_4)_0$ where H_1, H_2, H_3 and H_4 are smooth branches meeting transversally at the origin. These germs are uniformized by a transformation group $(\Gamma, \mathbb{C}^2)$, where Γ is a parabolic subgroup of $\mathrm{Aut}(\mathbb{C}^2)$ generated by reflections. The orbifold fundamental groups of these germs are infinite solvable. Note that many ball-cusp points (with singular base and branch loci) are coverings of the germs (i) and (ii) above. For example the germ at the origin of the isolated surface singularity $z^3 = xy(x - y)$ is a triple covering of the germ $(\mathbb{C}^2, 3H_1 + 3H_2 + 3H_3)_0$ where H_1, H_2, H_3 are given by the polynomials x, y and $x - y$. This is called (somewhat paradoxically) an elliptic singularity, since it is resolved by a blow up which replace the origin by an elliptic curve.

In case M is the bidisc, the germs $(X, b)_p$ with $b(p) = \infty$ are called *cusp points*. In [13] it was shown that the only cusp point with a smooth base is the germ $(\mathbb{C}^2, 2H_1 + 2H_2 + 2H_3 + 2H_4)_0$ where H_1,H_2, H_3 and H_4 are given by the polynomials x, y and $x - y$ and $x - y^2$. This germ also has an infinite solvable orbifold fundamental group, and admits several coverings by germs with a singular base.

4. Orbifaces

Let M be an algebraic surface and let K be its canonical class. The number

$$c_1^2(M) := K \cdot K$$

is an important numerical invariant of M, and is called the *first Chern number of* M. Let $e(M)$ be the the euler number of M (the euler number is also called the *second Chern number of* M and denoted by $c_2(M)$). Hirzebruch proved in 1958 the celebrated proportionality theorem: If M is a quotient of the two-ball B_2 then one has

$$c_1(M)^2 = 3e(M).$$

Similarly, if M is a quotient of the bidisc $\Delta \times \Delta$ then the proportionality $c_1(M)^2 = 2e(M)$ holds. In 1977 Miyaoka and Yau proved the inequality $c_1(M)^2 \leq 3e(M)$ for an arbitrary algebraic surface and the following converse to Hirzebruch's proportionality theorem: if M satisfies the $c_1(M)^2 = 3e(M) > 0$ then either M is $\mathbb{P}^2$ or its universal covering is B_2. The analogue of this result for surfaces with $c_1(M)^2 = 2e(M) > 0$ is not correct.

Chern numbers are invariants of algebraic surfaces, but they have orbifold versions. Below we introduce the Chern numbers for orbifolds over the base $\mathbb{P}^2$ only.

Definition 4.1. Let $(\mathbb{P}^2, b)$ be an orbiface with the associated divisor $D_b = \Sigma_1^k m_i B_i$, the curves B_i being irreducible of degree d_i for $i \in [1, k]$. The *orbifold Chern numbers of* $(\mathbb{P}^2, b)$ are defined as

$$c_1^2(\mathbb{P}^2, b) := \Big[-3 + \sum_{i \in [1,k]} d_i \big(1 - \frac{1}{m_i}\big)\Big]^2$$

$$e(\mathbb{P}^2, b) := 3 - \sum_{i \in [1,k]} \big(1 - \frac{1}{m_i}\big) e(B_i \backslash \mathrm{Sing}(B)) - \sum_{p \in \mathrm{Sing}(B)} \big(1 - \frac{1}{b(p)}\big)$$

(If $(\mathbb{P}^2, b)$ is an orbiface with cusp points set $1/b(p) = 0$ whenever $b(p) = \infty$).

The orbifold Chern numbers have the following property: if $M \to (X, b)$ is a finite uniformization with G as its Galois group, then

$$e(M) = |G| e(X, b) \text{ and } c_1^2(M) = |G| c_1^2(X, b) \tag{4.1}$$

The following orbiface analogue of the Miyaoka-Yau theorem was proved in 1989. We refer the reader to [13] for an introduction to metric uniformization theory of algebraic surfaces.

Theorem 4.2. *[Kobayashi, Nakamura, Sakai] Let* $(\mathbb{P}^2, b)$ *be an orbiface of general type, possibly with ball-cusp points. Then* $c_1^2(\mathbb{P}^2, b) \leq 3e(\mathbb{P}^2, b)$*, the equality holding if and only if* $(\mathbb{P}^2, b)$ *is uniformized by* B_2*.*

4.1. Orbifaces $(\mathbb{P}^2, D)$ with an abelian uniformization

Consider an orbifold $(\mathbb{P}^2, D)$ where $D = \Sigma_1^k m_i H_i$ where H_i are irreducible and let $B := \cup_1^k H_i$ be the support of D. Suppose $(\mathbb{P}^2, D)$ admits a uniformization with an abelian Galois group. Then for any point p, the local groups $\pi_1^{orb}(\mathbb{P}^2, D)_p$ must be abelian since these groups inject into the Galois group. Nodes are the only orbifold singularities with a smooth base and an abelian fundamental group. Hence B must be a nodal curve. Then by the Zariski conjecture proved by Deligne and Fulton, the group $\pi_1(\mathbb{P}^2\backslash B)$ is abelian and admits the presentation

$$\pi_1(\mathbb{P}^2\backslash B) \simeq \Big\langle \mu_1, \dots, \mu_k \,\Big|\, \sum_1^k d_i\mu_i = 0 \Big\rangle$$

where $d_i := \deg(H_i)$. Therefore the group $\pi_1^{orb}(\mathbb{P}^2, D)$ is finite abelian and admits the presentation

$$\pi_1^{orb}(\mathbb{P}^2, D) \simeq \Big\langle \mu_1, \dots, \mu_k \,\Big|\, m_1\mu_1 = \dots = m_k\mu_k = \sum_1^k d_i\mu_i = 0 \Big\rangle$$

Since the subgroup of the group $\langle \mu_1, \dots, \mu_k \,|\, m_1\mu_1 = \dots = m_k\mu_k = 0\rangle$ generated by $\langle \Sigma_1^k d_i\mu_i\rangle$ is of order $\mathrm{lcm}\{m_i/\gcd(m_i, d_i) \,|\, i \in [1, k]\}$, we find that

$$|\pi_1^{orb}(\mathbb{P}^2, D)| = \frac{\prod_1^k m_i}{\mathrm{lcm}\{b_i \,|\, i \in [1, k]\}} \tag{4.2}$$

where $b_i := m_i/\gcd(m_i, d_i)$.

We claim that if $(\mathbb{P}^2, D)$ admits a uniformization, then irreducible components of B must be smooth: Assume the contrary; e.g. suppose that H_i has a node at $p \in \mathbb{P}^2$. The local orbifold fundamental group of this node admits the presentation

$$\pi_1^{orb}(\mathbb{P}^2, D)_p \simeq \langle \mu_p, \mu_p' \,|\, m_i\mu_p = m_i\mu_p' = 0\rangle \simeq \mathbb{Z}/(m_i) \oplus \mathbb{Z}/(m_i)$$

where μ_p and μ_p' are meridians of the branches of H_i meeting at p. Since H_i is irreducible, μ_p and μ_p' are conjugate elements in $\pi_1^{orb}(\mathbb{P}^2, D)$. Since this latter group is abelian, one actually has $\mu_p = \mu_p'$. Hence, the subgroup of $\pi_1^{orb}(\mathbb{P}^2, D)$ generated by μ_p and μ_p' is at most $\mathbb{Z}/(m_i)$ and can not be isomorphic to the local orbifold fundamental group at p, which is $\mathbb{Z}/(m_i) \oplus \mathbb{Z}/(m_i)$.

Suppose that $(\mathbb{P}^2, D)$ is an orbiface with a nodal locus, whose irreducible components are all smooth. Since the group $\pi_1^{orb}(\mathbb{P}^2, D)$ is finite, either $(\mathbb{P}^2, D)$ is not uniformizable or there is a finite universal uniformization. Hence by Theorem 2.4, $(\mathbb{P}^2, D)$ is uniformizable if for every $p \in \mathbb{P}^2$, the image of the inclusion-induced map

$$\rho \circ \iota_* : \pi_1^{orb}(\mathbb{P}^2, D)_p \to \pi_1^{orb}(\mathbb{P}^2, D) \tag{4.3}$$

is an injection.

For a point in $\mathbb{P}^2\backslash B$ the local orbifold fundamental group is trivial, so that $\rho \circ \iota_*$ is always an injection. Now let $p \in H_i\backslash \mathrm{Sing}(B)$. Then $\pi_1^{orb}(\mathbb{P}^2, D)_p \simeq \mathbb{Z}/(m_i)$, and $\rho \circ \iota_*$ is an injection only if the condition below is satisfied:

Condition 1. For any $i \in [1, k]$, the subgroup $\langle \mu_i \rangle$ is of order m_i in $\pi_1^{orb}(\mathbb{P}^2, D)$.

(The notation $\langle A\rangle$ means the subgroup generated by the subset A). Finally, if p is a point of intersection of H_i and H_j, $(i \neq j)$ then

$$\pi_1^{orb}(\mathbb{P}^2, D)_p = \pi_1^{orb}(\mathbb{P}^2, m_iH_i + m_jH_j)_p \simeq \mathbb{Z}/(m_i) \oplus \mathbb{Z}/(m_j),$$

and $\rho \circ \iota_*$ is injective only if the following condition is satisfied:

Condition 2. For any pair of distinct integers $i, j \in [1, k]$, the subgroup $\langle \mu_i, \mu_j \rangle$ is of order $m_i m_j$ in $\pi_1^{orb}(\mathbb{P}^2, D)$.

Obviously, Condition 2 implies Condition 1 (since any two curves intersects in $\mathbb{P}^2$). Let $D - (m_i H_i + m_j H_j)$ be the divisor obtained from D by removing H_i and H_j. Then Condition 2 is equivalent to

$$|\pi_1^{orb}(\mathbb{P}^2, D)| = |\langle \mu_i, \mu_j \rangle| |\pi_1^{orb}(\mathbb{P}^2, D - (m_i H_i + m_j H_j))| \quad \forall i, j \in [1, k],\ (i \neq j)$$

By (4.2), this is equivalent to the condition

$$\frac{\prod_1^k m_i}{\operatorname{lcm}\{b_i \,|\, i \in [1, k]\}} = \frac{\prod_1^k m_i}{\operatorname{lcm}\{b_i \,|\, i \in [1, k] \backslash \{i, j\}\}} \quad \forall i, j \in [1, k],\ (i \neq j)$$

$$\Leftrightarrow \operatorname{lcm}\{b_i \,|\, i \in [1, k]\} = \operatorname{lcm}\{b_i \,|\, i \in [1, k] \backslash \{i, j\}\} \quad \forall i, j \in [1, k],\ (i \neq j) \qquad (4.4)$$

Finally, one has the following condition, equivalent to Condition 2:

Condition 3. Any prime power dividing one of $b_1, \ldots, b_k$ must divide at least two others.

We have proved the following theorem:

Theorem 4.3. *Let $D = \Sigma_1^k m_i H_i$ where H_i are irreducible of degree d_i and let $B := \cup_1^k H_i$. Then $(\mathbb{P}^2, D)$ is an orbiface with an abelian uniformization if and only if B is a nodal curve whose irreducible components are all smooth, and the numbers $b_1, \ldots, b_k$ satisfies Condition 3, where $b_i := m_i / \gcd(m_i, d_i)$.*

Let p be a prime, $\alpha \in \mathbb{Z}_{>0}$ and take numbers $\alpha_i \in [0, \alpha]$ for $i \in [4, k]$. Then the vector

$$[p^\alpha, p^\alpha, p^\alpha, p^{\alpha_4}, p^{\alpha_5}, \ldots, p^{\alpha_k}],$$

as well as any of its permutations, satisfies Condition 3. Any vector $[b_1, \ldots, b_k]$ satisfying Condition 3 admits a unique factorization into a product of such vectors with distinct p (where the product is taken component-wise).

For $k \leq 2$ Condition 3 is satisfied only if $b_1 = b_2 = 1$, that is when m_i divides d_i $(i = 1, 2)$. For $k = 3$, it is satisfied only if $b_1 = b_2 = b_3$. Some solutions for $k = 4$ can be given as

$$[b_1, b_2, b_3, b_4] = [p^3, p^3, p^3, p] \circledast [q^2, q^6, q^6, q^6] \circledast [r, r, r, r] \ldots$$

where p, q, r are distinct primes and $\circledast$ is the operation of component-wise multiplication. In general, Condition 3 is always satisfied if $k \geq 2$ and $b_1 = \cdots = b_k$.

Exercice 4.1. The study of algebraic surfaces from the point of view of possible values of $(c_1^2, e) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is called the *surface geography*. Study the geography of abelian uniformizations of $\mathbb{P}^2$.

Orbifaces with a linear locus. Now suppose that $B = \cup_1^k H_i$ is a line arrangement. By Theorem 4.3 the lines $H_1, \dots, H_k$ must be in general position. Then $d_i = 1$ for $i \in [1, k]$, so that $b_i = m_i$. Obviously Condition 3 is not satisfied unless $k \geq 3$, except the trivial case $b_1 = b_2 = 1$. As we have already seen, in case $k = 3$ and $m_1 = m_2 = m_3 =: m$ the uniformizing surface is $\mathbb{P}^2$ itself, with the polycyclic map

$$\varphi_m : [z_1 : z_2 : z_3] \in \mathbb{P}^2 \to [z_1^m : z_2^m : z_3^m] \in \mathbb{P}^2$$

as the uniformizing map, where we assumed $H_i = \{z_i = 0\}$ for $i = 1, 2, 3$.

The orbifold $(\mathbb{P}^2, \Sigma_1^4 2H_i)$ is uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, $(\mathbb{P}^2, \Sigma_1^3 2H_i)$ is a suborbifold which is uniformized by $\mathbb{P}^2$ via φ_2, and the lifting of $(\mathbb{P}^2, \Sigma_1^4 2H_i)$ to this uniformization is the orbifold $(\mathbb{P}^2, 2Q)$, where $Q \simeq \varphi_2^{-1}(H_4)$ is a smooth quadric. This latter orbifold is uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$ as we shall show below (see Theorem 4.5).

Note that the orbifaces $(\mathbb{P}^2, D)$ may admit intermediate uniformizations (e.g. uniformizations which are not universal). For example, consider the case $D = \Sigma_1^6 2H_i$. There is a surjection of degree 2

$$\pi_1^{orb}(\mathbb{P}^2, D) \simeq \left\langle \mu_1, \dots, \mu_6 \,\middle|\, 2\mu_1 = \dots = 2\mu_6 = \sum_1^6 2\mu_i = 0 \right\rangle \twoheadrightarrow$$

$$\left\langle \mu_0, \dots, \mu_5 \,\middle|\, 2\mu_1 = \dots = 2\mu_6 = \mu_1 + \mu_2 + \mu_3 = \mu_4 + \mu_5 + \mu_6 = 0 \right\rangle$$

Then the latter group G satisfies Condition 2, hence there is a uniformization with G as the Galois group. The uniformizing surface is an Enriques surface N. As we shall below, the universal uniformization of $(\mathbb{P}^2, \Sigma_1^6 2H_i)$ is a K3 surface, which is a double covering of N. Observe that the arrangement of hyperplanes $\cup_1^6 H_i$ is not projectively rigid, so that $(\mathbb{P}^2, \Sigma_1^6 2H_i)$ is in fact an orbiface family.

4.1.1. K3 orbifaces. A simply connected algebraic surface M with $c_1^2(M) = 0$ is called a $K3$ surface. It is known that all K3 surfaces have the same euler number, which is 24. An orbiface uniformized by a K3 surface M is called a *K3 orbiface.* Since M is simply connected, this uniformization must be universal. Let $(\mathbb{P}^2, D)$ be a K3 orbiface uniformized by the K3 surface M, where $D = \Sigma_1^k m_i H_i$ and H_i is an irreducible and reduced curve of degree d_i. Put $B = \cup_1^k H_i$, then B is of degree $d = \Sigma_1^k d_i$. Then

$$c_1^2(\mathbb{P}^2, D) = \frac{c_1^2(M)}{|\pi_1^{orb}(\mathbb{P}^2, D)|} = 0 \tag{4.5}$$

and

$$e(\mathbb{P}^2, D) = \frac{e(M)}{|\pi_1^{orb}(\mathbb{P}^2, D)|} = \frac{24}{|\pi_1^{orb}(\mathbb{P}^2, D)|} \tag{4.6}$$

Equation 4.5 implies that

$$\sum_{i \in [1,k]} d_i \big(1 - \frac{1}{m_i}\big) = 3 \Leftrightarrow \sum_{i \in [1,k]} \frac{d_i}{m_i} = d - 3 \tag{4.7}$$

which in turn implies that $4 \le d \le 6$. Equation 4.6 implies that $24/e(\mathbb{P}^2, D)$ must be an integer, which equals the order of the orbifold fundamental group. Under the assumption that $(\mathbb{P}^2, D)$ admits an abelian uniformization, this group order can be computed easily. It is possible to classify all “abelian” K3 orbifaces in this way, see [20] for details. Let us carry out this program for K3 orbifaces with a linear support.

Abelian K3 orbifaces with a linear locus. Suppose $k = 6$. Equation 4.7 implies $\Sigma_1^6 1/m_i = 3$, which forces $m_1 = \cdots = m_6 = 2$. This orbifold satisfies the conditions of Theorem 4.3 and is uniformizable. Hence, the universal uniformization is a K3 surface M_1. The orbifold fundamental group

$$\pi_1^{orb}(\mathbb{P}^2, D) \simeq \left\langle \mu_1, \dots, \mu_6 \;\middle|\; 2\mu_1 = \cdots = 2\mu_6 = \Sigma_0^6 \mu_i = 0 \right\rangle$$

is of order 32. Let us verify that $e(M_1) = 24$. For any H_i, there are 5 singular points of B lying on $H_i \simeq \mathbb{P}^1$, so that $e(H_i \backslash \mathrm{Sing}(B)) = e(H_i) - e(\mathrm{Sing}(B)) = 2 - 5 = -3$. Since the local orbifold fundamental group at the point $H_i \cap H_j$ is of order $m_i m_j$, one has

$$e(\mathbb{P}^2, D) = 3 + 3\sum_1^6 (1 - \frac{1}{m_i}) - \sum_{1 \le i \ne j \le 6} (1 - \frac{1}{m_i m_j}) = \frac{3}{4}$$

so that $e(M_1) = 32 e(\mathbb{P}^2, D) = 24$.

For $k = 5$ there are no abelian K3 orbifaces with a linear support, this can be proved by a case by case analysis. Suppose $k = 4$. Equation 4.7 implies

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} = 1 \tag{4.8}$$

For any H_i, there are 3 singular points of B lying on $H_i \simeq \mathbb{P}^1$, so that $e(H_i \backslash \mathrm{Sing}(B)) = e(H_i) - e(\mathrm{Sing}(B)) = 2 - 3 = -1$. Suppose without loss of generality that $m_1 \le m_2 \le m_3 \le m_4$. There are finitely many 4-tuples satisfying (4.8). It can be shown by case-by-case analysis that the only 4-tuples satisfying Condition 3 are $[4, 4, 4, 4]$ and $[2, 6, 6, 6]$. Hence, the universal uniformizations of these orbifolds are K3 surfaces, say M_2 and M_3 respectively. On the other hand, assumption 4.8 gives

$$e(\mathbb{P}^2, D) = \sum_{1 \le i \ne j \le 4} \frac{1}{m_i m_j}$$

By using the formula $|\pi_1^{orb}(\mathbb{P}^2, D)| = \prod_1^4 m_i / \mathrm{lcm}(m_1, \dots, m_4)$ one can verify that $e(M_2) = e(M_3) = 24$.

Let us now prove that the surface M_2 is the Fermat quartic surface, the hypersurface in $\mathbb{P}^3$ defined by the equation $M_2 : z_4^4 = z_1^4 + z_2^4 + z_3^4$. Since any two 4-line arrangements are projectively equivalent, one can assume that $H_i = \{z_i = 0\}$ for $i \in [1, 3]$, and $H_4 = \{z_1 + z_2 + z_3 = 0\}$. The suborbifold $(\mathbb{P}^2, \Sigma_1^3 4H_i)$ is uniformized by $\mathbb{P}^2$ via φ_4. Lifting the initial orbifold yields the orbifold $(\mathbb{P}^2, 4K)$, where K is the Fermat quartic curve $z^4 + z_2^4 + z_3^4 = 0$. Now it is easy to see that the restriction of the projection $[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^3 \to [z_1 : z_2 : z_3] \in \mathbb{P}^2$ to M_2 is a Galois covering branched at $4K$.

Exercice 4.2. Classify the abelian K3 orbifaces and study the covering relations between them.

4.2. Covering relations among orbifaces $(\mathbb{P}^2, D)$ uniformized by $\mathbb{P}^2$

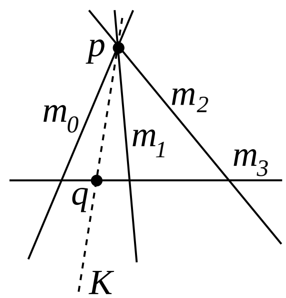


FIGURE 4.9. The orbiface $(\mathbb{P}^2, \Sigma_0^3 m_i H_i)$

Now let us consider the simplest orbiface with a non-abelian fundamental group. Let $H_0 := \{x = 0\}$, $H_1 := \{y = 0\}$, $H_2 := \{x = y\}$ and $H_3 := \{z = 0\}$ be four lines in $\mathbb{P}^2$. Observe that the arrangement $\cup_0^3 H_i$ is projectively rigid. Consider the orbiface $(\mathbb{P}^2, D)$ where $D = \Sigma_0^3 m_i H_i$ The point $p := [0 : 0 : 1]$ is an orbiface germ only if $\Sigma_0^2 1/m_i > 1$. Assume this is the case. Consider another line K through p. Take a base point $\star \in K$, a meridian $\mu_p \subset$ of $p \in K$ and a meridian $\mu_3 \subset K$ of the point $q := K \cap H_3 \in K$ (see Figure 4.9). Since K is topologically a sphere, the loop $\mu_p \mu_3$ is contractible in $K \backslash \{p, q\}$ and hence in $\mathbb{P}^2 \backslash (\cup_0^3 H_i)$. Hence, $\mu_p = \mu_3^{-1}$ in the group $\pi_1^{orb}(\mathbb{P}^2, D)$. In particular, these two meridians are of the same order. Now let m_p be the order of μ_p, considered as an element of $\pi_1^{orb}(\mathbb{P}^2, D)_p$. If the orbiface is uniformizable, this latter group injects into the global orbifold fundamental group. Hence, if $(\mathbb{P}^2, D)$ is uniformizable, the element μ_p, and therefore the element μ_3 must be of order m_p $\pi_1^{orb}(\mathbb{P}^2, D)$. In other words, $m_3 = m_p = 2(\Sigma_0^2 m_i - 1)^{-1}$. Hence, (m_0, m_1, m_2, m_3) must be one of $(2, 2, r, 2r)$, $(3, 3, 2, 12)$, $(2, 4, 3, 24)$ or $(2, 3, 5, 60)$.

Exercice 4.3. Compute the Chern numbers of these orbifolds and check that $c_1^2(\mathbb{P}^2, D) = 3e(\mathbb{P}^2, D)$.

The case $(2, 2, r, 2r)$**:** Observe that $(\mathbb{P}^2, 2H_0 + 2H_1 + 2H_3)$ is a suborbiface of $(\mathbb{P}^2, 2H_0 + 2H_1 + rH_2 + 2rH_3)$. This suborbiface is uniformized by $\mathbb{P}^2$ via the bicyclic covering

$$\varphi_2 : [x : y : z] \in \mathbb{P}^2 \to [x^2 : y^2 : z^2] \in \mathbb{P}^2$$

The lifting $\varphi_2^{-1}(H_2)$ consists of two lines given by the equation $x^2 = y^2$, which we denote by H_2^1 and H_2^2. Denote $\varphi_2^{-1}(H_3)$ by H_3 again. Hence φ_2 is an orbiface covering

$$\varphi_2 : (\mathbb{P}^2, rH_2^1 + rH_2^2 + rH_3) \to (\mathbb{P}^2, 2H_0 + 2H_1 + rH_2 + 2rH_3)$$

Obviously, the covering orbiface is uniformized by $\mathbb{P}^2$ via φ_r.

The case $(2, 4, 3, 24)$**:** Observe that $(\mathbb{P}^2, 2H_0 + 2H_1 + 2H_3)$ is a suborbiface of $(\mathbb{P}^2, 2H_0 + 4H_1 + 3H_2 + 24H_3)$. Let φ_2 be its uniformization, denote $\varphi_2^{-1}(H_1)$ by H_1 and $\varphi_2^{-1}(H_3)$ by H_3. As in the previous case, denote the lines $\varphi_2^{-1}(H_2)$ by H_2^1 and H_2^2. Hence there is an orbiface covering

$$\varphi_2 : (\mathbb{P}^2, 3H_1^1 + 3H_1^2 + 2H_2 + 12H_3) \to (\mathbb{P}^2, 2H_0 + 4H_1 + 3H_2 + 24H_3)$$

Observe that the covering orbiface is equivalent to the orbiface $(\mathbb{P}^2, 3H_0 + 3H_1 + 2H_2 + 12H_3)$

The case $(3, 3, 2, 12)$**:** Observe that $(\mathbb{P}^2, 3H_0 + 3H_1 + 3H_3)$ is a suborbiface of $(\mathbb{P}^2, 3H_0 + 3H_1 + 2H_2 + 12H_3)$. This suborbiface is uniformized by $\mathbb{P}^2$ via the bicyclic covering

$$\varphi_3 : [x : y : z] \in \mathbb{P}^2 \to [x^3 : y^3 : z^3] \in \mathbb{P}^2$$

The lifting $\varphi_3^{-1}(H_2)$ consists of two lines given by the equation $x^3 = y^3$, which we denote by H_2^1, H_2^2 and H_2^3. Denote $\varphi_2^{-1}(H_3)$ by H_3 again. Hence φ_3 is an orbiface covering

$$\varphi_3 : (\mathbb{P}^2, 2H_2^1 + 2H_2^2 + 2H_2^3 + 4H_3) \to (\mathbb{P}^2, 3H_0 + 3H_1 + 2H_2 + 12H_3)$$

The covering orbiface appeared in the first case with $r = 2$ and is uniformized by $\mathbb{P}^2$.

4.3. Orbifaces $(\mathbb{P}^2, D)$ uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{C} \times \mathbb{C}$ and $\Delta \times \Delta$

It is well known that the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ under the obvious action of the symmetric group Σ_2 is the projective plane. To put in another way, one has the following fact:

Lemma 4.4. *Let $Q \subset \mathbb{P}^2$ be a smooth quadric. Then there is a uniformization $\psi : Q \times Q \to (\mathbb{P}^2, 2Q)$. Let $p \in Q$ and put $T_p^v := \{p\} \times Q$, $T_p^h := Q \times \{p\}$. Then $T_p := \psi(T_p^h) = \psi(T_p^v) \subset \mathbb{P}^2$ is a line tangent to Q at the point $p \in Q$.*

Proof. Since any two smooth quadrics are projectively equivalent, it suffices to prove this for a special quadric. Consider the $\mathbb{Z}/(2)$-action defined by $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \to (y, x) \in \mathbb{P}^1 \times \mathbb{P}^1$. The diagonal $Q = \{(x, x) : x \in \mathbb{P}^1\}$ is fixed under this action. Let $x = [a : b] \in \mathbb{P}^1$ and $y = [c : d]$, then the symmetric polynomials $\sigma_1([a : b], [c : d]) := ad + bc$, $\sigma_2([a : b], [c, d]) := bd$, $\sigma_3([a : b], [c : d]) := ac$ are invariant under this action, and the Viéte map

$$\psi : (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow [\sigma_1(x, y) : \sigma_2(x, y) : \sigma_3(x, y)] \in \mathbb{P}^2$$

is a branched covering map of degree 2. The branching locus $\subset \mathbb{P}^2$ can be found as the image of Q. Note that the restriction of ψ to the diagonal Q is one-to-one, so that one can denote $\psi(Q)$ by the letter Q again. One has $\psi(Q) = [2ab : b^2 : a^2]$ ($[a : b] \in \mathbb{P}^1$), so that Q is a quadric given by the equation $4yz = x^2$. One

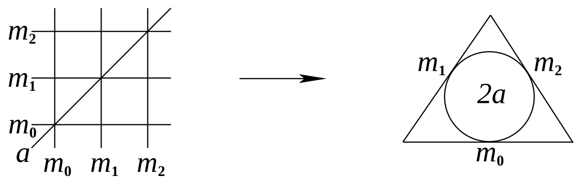


FIGURE 4.10. The covering $(\mathbb{P}^1 \times \mathbb{P}^1, aQ + \Sigma_0^n m_i(T_i^v + T_i^h)) \longrightarrow (\mathbb{P}^2, 2aQ + \Sigma_0^n m_i T_i)$

can identify the surface $\mathbb{P}^1 \times \mathbb{P}^1$ with $Q \times Q$, via the projections of the diagonal $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$. Let $p \in Q$, and put $T_p^h := Q \times \{p\}$, $T_p^v : \{p\} \times Q$. Then $T_p := \psi(T_p^h) = \psi(T_p^v) \subset \mathbb{P}^2$ is a line tangent to Q. Indeed, if $p = [a : b]$, then $\psi(T_p^h)$ is parametrized as $[cb + da : db : ca]$ ($[c : d] \in \mathbb{P}^1$), and can be given by the equation $b^2z + a^2y - abx = 0$, which shows that T_p is tangent to Q at the point $[2ab : b^2 : a^2]$. □

Now let $Q \subset \mathbb{P}^2$ be a smooth quadric and $T_0, \dots, T_n$ tangents to Q at distinct points $p_i := Q \cap T_i$, $i \in [0, n]$. The configuration $Q \cup T_0 \cup T_1 \cup T_2$ is called the *Apollonius configuration*. Consider the orbiface $(\mathbb{P}^2, aQ + \Sigma_0^n m_i T_i)$. By Theorem 3.2 this is an orbiface provided $1/a + 1/m_i \geq 1/2$. An immediate consequence of Lemma 4.4 is the following result.

Proposition 4.5. *There is an orbiface covering*

$$(\mathbb{P}^1 \times \mathbb{P}^1, aQ + \Sigma_0^n m_i(T_i^v + T_i^h)) \longrightarrow (\mathbb{P}^2, 2aQ + \Sigma_0^n m_i T_i)$$

In particular, when $a = 1$, there is an orbiface covering

$$(\mathbb{P}^1, \Sigma_0^n m_i p_i) \times (\mathbb{P}^1, \Sigma_0^n m_i p_i) \longrightarrow (\mathbb{P}^2, 2Q + \Sigma_0^n m_i T_i)$$

By Theorem 1.4, the covering orbiface above is uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$ if $n = 1$ and $m_0 = m_1$, or if $n = 2$ and $\Sigma_0^2 1/m_i > 1$. Hence the following orbifaces are uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$.

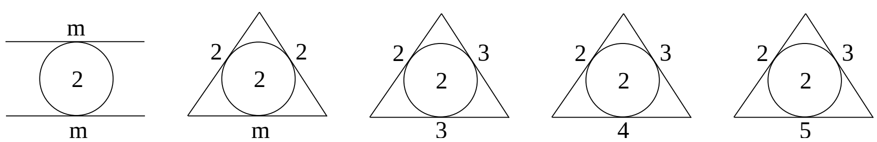


FIGURE 4.11. Orbifaces uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$

Similarly, the following orbifaces are uniformized by $\mathbb{C} \times \mathbb{C}$.

Otherwise, the orbifolds $(\mathbb{P}^2, 2Q + \Sigma m_i T_i)$ are uniformized by the bidisc $\Delta \times \Delta$. The orbifaces in Figure 4.11 were first discovered in 1982 by Kaneko, Tokunaga and Yoshida who also gave a complete classification of the orbifaces $(\mathbb{P}^2, D)$ uniformized by $\mathbb{C} \times \mathbb{C}$ (see [11]). Note that the Apollonius configuration is projectively rigid.

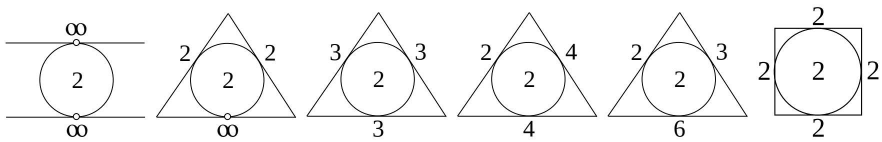


FIGURE 4.12. Orbifaces A uniformized by $\mathbb{C} \times \mathbb{C}$

Except the first and the fifth orbifolds in Figure 4.11 these orbifolds admits liftings to $\mathbb{P}^2$ and gives rise to new orbifaces uniformized by $\mathbb{C} \times \mathbb{C}$. Below we shall study the coverings of the fourth orbiface in detail.

Coverings of the orbiface $(\mathbb{P}^2, 2Q + 2T_0 + 4T_1 + 4T_2)$**.** This orbiface has the suborbifold $(\mathbb{P}^2, 2T_0 + 2T_1 + 2T_2)$, which is uniformized by $\mathbb{P}^2$ via φ_2. We can assume that in projective coordinates the tangent lines are given by $T_i := \{z_i = 0\}$, in these coordinates φ_2 is the map $[z_0 : z_1 : z_2] \to [z_0^2 : z_1^2 : z_2^2]$. A quadric tangent to both the lines $z_0 z_1 z_2 = 0$ is given by the equation $a\sqrt{z_0} + b\sqrt{z_1} + c\sqrt{z_2} = 0$. Hence $\varphi_2^{-1}(Q)$ is given by $\pm a z_0 \pm b z_1 \pm c z_2 = 0$, in other words the lifting of Q consists of four lines $\varphi_2^{-1}(Q) := Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ which meets two by two on the lines $z_0 z_1 z_2 = 0$. The arrangement $T_1 \cup T_2 \cup_1^4 Q_i$ is known as the complete quadrilateral, since it is the set of lines through two points among four points in general position in $\mathbb{P}^2$ (see Figure 4.13).

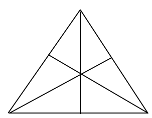


FIGURE 4.13. The complete quadrilateral

Hence the lifting of $(\mathbb{P}^2, 2Q + 2T_0 + 4T_1 + 4T_2)$ is the orbifold $(\mathbb{P}^2, 2T_1 + 2T_2 + \Sigma_1^4 2Q_i)$. Since any two sets of 4 points in general position in $\mathbb{P}^2$ are projectively equivalent, the complete quadrilateral is projectively rigid. Hence there are projective coordinates in which the locus of $(\mathbb{P}^2, 2T_1 + 2T_2 + \Sigma_1^4 2Q_i)$ is given by the equation $z_0 z_1 z_2 (z_0 - z_1)(z_1 - z_2)(z_2 - z_3) = 0$, which is another equation for the complete quadrilateral. Let us name these lines $L_1, \ldots, L_6$ respectively. Now $(\mathbb{P}^2, \Sigma_1^3 2L_i)$ is a suborbifold of $(\mathbb{P}^2, \Sigma_1^6 2L_i)$. This orbifold is uniformized by $\mathbb{P}^2$ via φ_2. The liftings of L_4, L_5, L_6 are given by the equation $(z_0^2 - z_1^2)(z_1^2 - z_2^2)(z_2^2 - z_3^2) = 0$. But this is another equation for the complete quadrilateral. This shows that the orbiface $(\mathbb{P}^2, 2T_1 + 2T_2 + \Sigma_1^4 2Q_i)$ admits self coverings and proves the following result.

Lemma 4.6. *The orbiface* $(\mathbb{P}^2, 2Q + 2T_0 + 4T_1 + 4T_2)$ *has an infinite tower of coverings.*

Observe the analogy with the one-dimensional case: The orbifold $(\mathbb{P}^1, 2p_0 + 3p_1 + 6p_2)$ is covered by $(\mathbb{P}^1, 3_0 + 3p_1 + 3_2)$, which admits self-coverings.

For a higher dimensional version of the results in this subsection, see [22].

4.4. Covering relations among ball-quotient orbifolds

The orbifaces $(\mathbb{P}^2, aQ + \Sigma_0^2 m_i T_i)$ supported by the Apollonius configuration were throughly studied in [10] and [21]. The Chern numbers of $(\mathbb{P}^2, aQ + \Sigma_0^2 m_i T_i)$ are given by

$$c_1^2 = \left[2 - \frac{2}{a} - \sum_1^3 \frac{1}{m_i}\right]^2$$

$$e = 1 - \frac{1}{a} - \sum_1^3 \frac{1}{m_i} + \sum_{1 \le i \neq j \le 3} \frac{1}{m_i m_j} + \frac{1}{2} \sum_1^3 \left[\frac{1}{m_i} + \frac{1}{a} - \frac{1}{2}\right]^2$$

One has

$$(3e - c_1^2)(\mathbb{P}^2, aQ + \Sigma_0^2 m_i T_i) = \frac{1}{2}\left[\sum_1^3 \frac{1}{m_i} - \frac{1}{a} - \frac{1}{2}\right]^2 \tag{4.9}$$

which vanishes for the following orbifaces

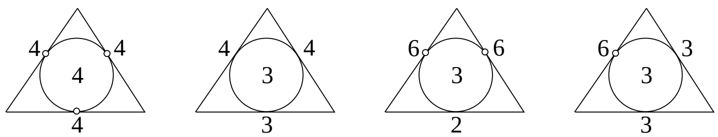


FIGURE 4.14. Orbifolds A uniformized by B_2

By Theorem 4.2 these orbifolds are uniformized by the 2-ball. Observe that the orbiface $(\mathbb{P}^2, 2Q + 2T_0 + 4T_1 + 4T_2)$ is a suborbifold of the first orbiface in Figure 4.14. By Lemma 4.6 this suborbifold admits an infinite tower of coverings. The orbiface $(\mathbb{P}^2, 4Q + 4T_0 + 4T_1 + 4T_2)$ can be lifted to these coverings, and these liftings give an infinite tower of orbifaces uniformized by the 2-ball. Since the group $\pi_1^{orb}(\mathbb{P}^2, 4Q + 4T_0 + 4T_1 + 4T_2)$ is Picard modular, this tower is called a *Picard modular tower.*

Exercice 4.4. Find the first three steps of the Picard modular tower.

Exercice 4.5. Study the coverings of the ball-quotient orbifolds in Figure 4.14.

Question 4.1. The orbifaces $(\mathbb{P}^2, 3Q+3T_0+4T_1+2T_2)$ and $(\mathbb{P}^2, 6Q+2T_0+3T_1+3T_2)$ satisfy $2e - c_1^2 = 0$. What is their universal uniformization?

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