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Covering Relations Between Ball-Quotient Orbifolds

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Abstract. Some ball-quotient orbifolds are related by covering maps. We exploit these coverings to find infinitely many orbifolds on $\mathbb{P}^2$ uniformized by the complex 2-ball $\mathbf{B_2}$ and some orbifolds over K3 surfaces uniformized by $\mathbf{B_2}$. We also give, along with infinitely many reducible examples, an infinite series of irreducible curves along which $\mathbb{P}^2$ is uniformized by the product of 1-balls $\mathbf{B_1} \times \mathbf{B_1}$.

1. Introduction

Let M be a smooth algebraic surface of general type. By the Hirzebruch proportionality theorem [H1] if M is covered by the complex 2-ball $\mathbf{B_2}$ then its Chern numbers satisfies the equality $c_1^2(M) = 3e(M)$. Conversely if the Chern numbers of M satisfy this equality then the universal covering of M is $\mathbf{B_2}$ by Yau's theorem [Ya].

One way to discover surfaces of general type M with $c_1^2(M) = 3e(M)$ is to construct them as finite branched Galois coverings $\varphi : M \to X$ where X is the blow-up of $\mathbb{P}^2$ at some points. This approach was first used by Hirzebruch [H2], [H3], developed further in [Hö], [BHH] and more recently in [Ho],[HV]. Corresponding lattices acting on $\mathbf{B_2}$ turned out to be commensurable with the ones obtained from the study of the hypergeometric differential equations (see [DeMo] and [Yo2]).

A branched Galois covering $\varphi : M \to X$ endows X with a map $\beta_\varphi : X \to \mathbb{N}$ sending $p \in X$ to the order of the isotropy group above p. The pair (X, β_φ) is an orbifold, and M is a *uniformization* of (X, β_φ). In case the degree of φ is finite, M is also called a *finite uniformization*, and if M is simply connected, it is called the *universal uniformization* of (X, β_φ). Alternatively, when finiteness or universality can be understood from the context, one says: (X, β_φ) *is uniformized by* M.

For an orbifold (X, β) it is possible to define orbifold Chern numbers $c_1^2(X, \beta)$ and $e(X, \beta)$ in such a way that for any finite uniformization

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$\varphi : M \to X$ with $\beta_\varphi = \beta$ one has

$$c_1^2(M) = \deg(\varphi)c_1^2(X,\beta), \qquad e(M) = \deg(\varphi)e(X,\beta)$$

On the other hand, by results of [KNS] an orbifold of general type (X,β) is uniformized by $\mathbf{B_2}$ if the equality

$$c_1^2(X,\beta) = 3e(X,\beta) \tag{1}$$

is satisfied. Hence, in order to discover lattices acting on $\mathbf{B_2}$ it suffices to find orbifolds satisfying (1). In this article, this idea is applied to rediscover some orbifolds discovered in [HV]. A study of some coverings (in the orbifold sense) of these orbifolds gave the following result.

Theorem 1. *There exists infinite series of pairwise non-isomorphic orbifolds uniformized by the 2-ball $\mathbf{B_2}$. Corresponding lattices in* $\mathrm{SU}(1,2)$ *are all arithmetic.*

In contrast with many known examples of orbifolds $(\mathbb{P}^2,\beta)$ uniformized by $\mathbf{B_2}$, to the author's knowledge there is just one orbifold $(\mathbb{P}^2,\beta)$ (alleged to Hirzebruch in [Yo1]) that is known to be uniformized by $\mathbf{B_1}\times\mathbf{B_1}$. In Section 4 below, infinitely many examples of such orbifolds are given. In particular, the following result is proved.

Theorem 2. *For $m > 0$ odd, let Q_m be the irreducible curve given by the equation $x^{m/2} + y^{m/2} + z^{m/2} = 0$. Define $\beta_m : \mathbb{P}^2 \to \mathbb{N}$ by $\beta_m(x) = 2$ if $x \in Q_m \backslash \mathrm{sing}(Q_m)$, $\beta_m(x) = 2m$ if $x \in \mathrm{sing}(Q_m)$ and $\beta_m(x) = 1$ otherwise. Then the orbifold $(\mathbb{P}^2,\beta_m)$ is uniformized by $Q_m \times Q_m$. Hence, for $m > 3$ the universal uniformization of $(\mathbb{P}^2,\beta_m)$ is $\mathbf{B_1}\times\mathbf{B_1}$. The uniformization $Q_m \times Q_m \to (\mathbb{P}^2,\beta_m)$ is of degree $2m^2$.*

The curve Q_1 is a smooth quadric, and it is well known that $(\mathbb{P}^2,\beta_1)$ is uniformized by $\mathbb{P}^1\times\mathbb{P}^1$. The curve Q_3 is the nine-cuspidal sextic, it was shown in [KTY] that $\mathbb{C}^2$ is the universal uniformization of $(\mathbb{P}^2,\beta_3)$. For a generalization of Theorem 2 to higher dimensional projective spaces, see [U].

2. Orbifolds

We shall mostly follow the terminology settled in [YY]. Let M be a connected complex manifold, $G \subset \mathrm{Aut}(M)$ a properly discontinuous subgroup and put $X := M/G$. Then the projection $\varphi : M \to X$ is a branched Galois covering endowing X with a map $\beta_\varphi : X \to \mathbb{N}$ defined by $\beta_\varphi(p) := |G_q|$ where q is a point in $\varphi^{-1}(p)$ and G_q is the isotropy subgroup of G at q. In this setting, the pair (X,β_φ) is said to be uniformized by $\varphi : M \to (X, \beta_\varphi)$. An *orbifold* is a pair (X, β) of an irreducible normal analytic space X with a function $\beta : X \to \mathbb{N}$ such that the pair (X, β) is locally finitely uniformizable. Let (X, β) and (X, γ) be two orbifolds with $\gamma|\beta$, and let $\varphi : (X', 1) \to (X, \gamma)$ be a uniformization of (X, γ), e.g. $\beta_\varphi = \gamma$. Then $\varphi : (X', \beta') \to (X, \beta)$ is called an *orbifold covering*, where $\beta' := \beta \circ \varphi / \gamma \circ \varphi$. The orbifold (X', β') is called the *lifting of* (X, β) *to the uniformization* X' *of* (X, γ).

Let (X, b) be an orbifold, $B_\beta := \operatorname{supp}(\beta - 1)$ and let $B_1, \ldots, B_n$ be the irreducible components of B_β. Then β is constant on $B_i \backslash \operatorname{sing}(B_\beta)$; so let b_i be this number. The *orbifold fundamental group* $\pi_1^{orb}(X, \beta)$ of (X, β) is the group defined by $\pi_1^{orb}(X, \beta) := \pi_1(X \backslash B_\beta) / \langle\langle \mu_1^{b_1}, \ldots, \mu_n^{b_n} \rangle\rangle$ where $\mu_i^{b_i}$ is a meridian of B_i and $\langle\langle\rangle\rangle$ denotes the normal closure. An orbifold (X, β) is said to be *smooth* if X is smooth. In case (X, β) is a smooth orbifold the map β is determined by the numbers b_i; in fact $\beta(p)$ is the order of the local orbifold fundamental group at p. In case $\dim X = 2$ the orbifold condition (i.e. locally finitely uniformizability) is equivalent to the finiteness of the local fundamental groups. For example if $x = B_i \cap B_j$ is a simple node of B_β then $\beta(p) = b_i b_j$. For some other singularities of B_β one has

Lemma 1. *Let (X, β) be an orbifold where X is a smooth complex surface and $p \in X$. (i) If $p = B_i \cap B_j \cap B_k$ is a transversal intersection of smooth branches of B_β then $b_i^{-1} + b_j^{-1} + b_k^{-1} > 1$ and $\beta(p) = 4[b_i^{-1} + {b_j}^{-1} + {b_k}^{-1} - 1]^{-2}$ (ii) If $p = B_i \cap B_j$ is a tacnode of B_β then $b_i^{-1} + b_j^{-1} > 1/2$ and $\beta(p) = 2[b_i^{-1} + b_j^{-1} - 2^{-1}]^{-2}$ (iii) If $p \in B_i$ is a simple cusp of B_β then $b_i^{-1} > 6^{-1}$ and $\beta(p) = \frac{2}{3}[b_i^{-1} - 6^{-1}]^{-2}$.*

Proof. Part (i) is well known, see e.g. [Yo1]. Now let B_1, B_2, $B_3 \subset \mathbb{C}^2$ be respectively the lines $x = 0$, $x = y$ and $y = 0$. By part (i) the pair $(\mathbb{C}^2, \beta)$ is an orbifold where $\beta : \mathbb{C}^2 \to \mathbb{N}$ is the function

$$\beta(p) = \begin{cases} b_i, & p \in B_i \backslash \{(0,0)\} \\ 4[b_1^{-1} + {b_2}^{-1} + {b_3}^{-1} - 1]^{-2}, & p = (0,0) \\ 1, & \text{otherwise} \end{cases}$$

for integers b_1, b_2, b_3 satisfying $b_1^{-1} + b_2^{-1} + b_3^{-1} > 1$. To prove (ii) put $b_3 = 2$ and consider the branched Galois covering $\varphi : (x, y) \in \mathbb{C}^2 \to (x, y^2) \in \mathbb{C}^2$. One has $\beta_\varphi(p) = b_3 = 2$ for $p \in B_3$ and $\beta_\varphi(p) = 1$ otherwise. Let $B_1', B_2' \subset \mathbb{C}^2$ be respectively the line $x = 0$ and the curve $x = y^2$. Then $\varphi(B_1') = B_1$ and $\varphi(B_2') = B_2$. Let $\beta' : \mathbb{C}^2 \to \mathbb{N}$ be the function which takes the value b_i on $B_i' \backslash \{(0,0)\}$, the value $2[b_i^{-1} + b_j^{-1} - 2^{-1}]^{-2}$ on $(0,0)$ and the value 1 otherwise. Then $\varphi : (\mathbb{C}^2, \beta') \to (\mathbb{C}^2, \beta)$ is an orbifold covering, which proves (ii). To prove (iii) one applies the above argument with $\varphi : (x, y) \in \mathbb{C}^2 \to (x^3, y^2)$, $b_1 = 3$ and $b_2 = 2$. (Setting B_1' to be the curve $x^3 = y^2$, one has $\varphi(B_2') = B_2$). $\square$

Conventions. We shall almost exclusively be concerned with orbifolds (X, β) with X being a smooth algebraic surface. In most cases X will be the projective plane $\mathbb{P}^2$. Since in this case β is determined by its values b_i on $B_i \backslash \mathrm{sing}(B_\beta)$, a smooth orbifold can alternatively be defined as a pair (X, B) where $B := b_1 B_1 + \ldots b_n B_n$ is a divisor on X with $b_i \geq 1$. For an orbifold (X, B) the corresponding map $X \to \mathbb{N}$ will be denoted by β_B. The *locus* of the orbifold (X, B) is the hypersurface $B \subset X$.

2.1. Orbifold Chern numbers.

Let $(\mathbb{P}^2, B)$ be an orbifold where $B = b_1 B_1 + \cdots + b_n B_n$ with B_i being an irreducible curve of degree d_i.

Definition 1. *The orbifold Chern numbers of* $(\mathbb{P}^2, B)$ *are defined as*

$$c_1^2(\mathbb{P}^2, B) := \left[-3 + \sum_{1 \leq i \leq n} d_i \left(1 - {b_i}^{-1}\right)\right]^2$$

$$e(\mathbb{P}^2, B) := 3 - \sum_{1 \leq i \leq n} \left(1 - {b_i}^{-1}\right) e(B_i \backslash \mathrm{sing}(B)) - \sum_{p \in \mathrm{sing}(B)} \left(1 - \beta_B(p)^{-1}\right)$$

If $(M, 1) \to (\mathbb{P}^2, \beta)$ is a finite uniformization of degree d then the Chern numbers of M are given by $e(M) = de(\mathbb{P}^2, B)$ and $c_1^2(M) = dc_1^2(\mathbb{P}^2, B)$.

In dimension 2, an orbifold (X, β) is said to be of *general type* if it is uniformized by a surface of general type. In the context of the following theorem, this simply means that (X, β) is *not* uniformized by $\mathbb{P}^2$.

Theorem 3 (Kobayashi, Nakamura, Sakai [KNS]). *Let* $(\mathbb{P}^2, B)$ *be an orbifold of general type. Then* $c_1^2(\mathbb{P}^2, B) \leq 3e(\mathbb{P}^2, B)$, *the equality holding if and only if* $(\mathbb{P}^2, B)$ *is uniformized by* $\mathbf{B_2}$.

In fact, the KNS theorem is proved in greater generality then its version stated above; in particular it is valid for orbifolds $(\mathbb{P}^2, \beta)$ with at worst "log-canonical singularities" and β being a function $\mathbb{P}^2 \to \mathbb{N} \cup \infty$. This implies that in Lemma 1 (i) one may have $b_i^{-1} + b_j^{-1} + b_k^{-1} = 1$ with $\beta(p) = \infty$, in (ii) one may have $b_i^{-1} + b_j^{-1} = 1/2$ with $\beta(p) = \infty$ and in (iii) one may have $b_i^{-1} = 1/6$ with $\beta(p) = \infty$.

Let M be an algebraic surface with $\mathbf{B_1} \times \mathbf{B_1}$ as the universal covering. Then by the Hirzebruch proportionality [H1] one has $c_1^2(M) = 2e(M)$. Similarly if an orbifold (X, β) is uniformized by $\mathbf{B_1} \times \mathbf{B_1}$ then by Selberg's theorem the corresponding transformation group has a torsion-free normal subgroup of finite index, which implies that (X, β) admits a finite uniformization $M \to X$ of degree d. Since M is uniformized by $\mathbf{B_1} \times \mathbf{B_1}$, one has $dc_1^2(X, \beta) = c_1^2(M) = 2e(M) = 2de(X, \beta)$.

3. The Apollonius configuration

Let $A_n := Q \cup T_1 \cup \cdots \cup T_n$ be an arrangement consisting of a smooth quadric Q with n distinct tangent lines of Q. Since there are only two tangent lines to a quadric from a point $\in \mathbb{P}^2 \backslash Q$, the lines $T_1, \ldots, T_n$ meets each other one by one. The configuration space of A_n's is naturally identified with the configuration space $\mathcal{M}_n$ of n distinct points in $\mathbb{P}^1$, via the contact points of the tangent lines with the quadric $Q \simeq \mathbb{P}^1$. Since the space $\mathcal{M}_n$ is connected, any two arrangements A_n with n fixed are isotopic in $\mathbb{P}^2$. In particular fundamental groups of their complements are isomorphic, see Theorem 6 for a presentation of this group. In [HV], the configuration A_3 was named the *Apollonius configuration* and studied from the orbifold point of view.

Lemma 2. *Let $Q \subset \mathbb{P}^2$ be a smooth quadric. Then there is a uniformization $\psi : Q \times Q \to (\mathbb{P}^2, 2Q)$. Let $p \in Q$ and put $T_p^v := \{p\} \times Q$, $T_p^h := Q \times \{p\}$. Then $T_p := \psi(T_p^h) = \psi(T_p^v) \subset \mathbb{P}^2$ is a line tangent to Q at the point $p \in Q$.*

Proof. Since any two smooth quadrics are projectively equivalent, it suffices to prove this for a given quadric. Consider the $\mathbb{Z}/(2)$-action defined by $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \to (y, x) \in \mathbb{P}^1 \times \mathbb{P}^1$. The diagonal $Q = \{(x, x) : x \in \mathbb{P}^1\}$ is fixed under this action. Let $x = [a : b] \in \mathbb{P}^1$ and $y = [c : d]$, then the symmetric polynomials $\sigma_1([a : b], [c : d]) := ad + bc$, $\sigma_2([a : b], [c, d]) := bd$, $\sigma_3([a : b], [c : d]) := ac$ are invariant under this action, and the Viéte map

$$\psi : (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow [\sigma_1(x, y) : \sigma_2(x, y) : \sigma_3(x, y)] \in \mathbb{P}^2$$

is a branched covering map of degree 2. The branching locus $\subset \mathbb{P}^2$ can be found as the image of Q. Note that the restriction of ψ to the diagonal Q is one-to-one, so that one can denote $\psi(Q)$ by the letter Q again. One has $\psi(Q) = [2ab : b^2 : a^2]$ $([a : b] \in \mathbb{P}^1)$, so that Q is a quadric given by the equation $4yz = x^2$. One can identify the surface $\mathbb{P}^1 \times \mathbb{P}^1$ with $Q \times Q$, via the projections of the diagonal $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$. Let $p \in Q$, and put $T_p^h := Q \times \{p\}$, $T_p^v : \{p\} \times Q$. Then $T_p := \psi(T_p^h) = \psi(T_p^v) \subset \mathbb{P}^2$ is a line tangent to Q. Indeed, if $p = [a : b]$, then $\psi(T_p^h)$ is parametrized as $[cb + da : db : ca]$ $([c : d] \in \mathbb{P}^1)$, and can be given by the equation $b^2 z + a^2 y - abx = 0$, which shows that T_p is tangent to Q at the point $[2ab : b^2 : a^2]$. $\square$

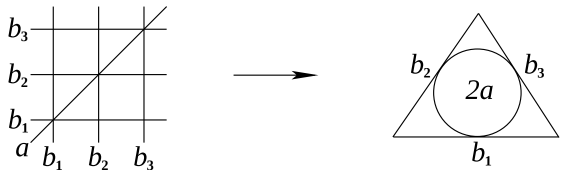

Fig. 1. The covering $\mathcal{B}(a;b_1,\dots,b_n) \to \mathcal{A}(2a;b_1,\dots,b_n)$

Consider the pair $(\mathbb{P}^2, aQ + b_1T_1 + \cdots + b_nT_n)$. By Lemma 1 this is an orbifold provided $1/a + 1/b_i \geq 1/2$. Denote

$$\mathcal{A}(a;b_1,\dots,b_n) := (\mathbb{P}^2, aQ + b_1T_1 \cdots + b_nT_n)$$

Theorem 4. *Suppose that $n > 1$ and if $n = 2$ then $b_1 = b_2$. Then there is a finite uniformization $\xi : R \times R \to \mathcal{A}(2;b_1,\dots,b_n)$, the Riemann surface R being a uniformization of $(Q, b_1p_1 + \cdots + b_np_n)$, where $p_i := T_i \cap Q$. Moreover;*
(i) If $n = 2$ and $b := b_1 = b_2 < \infty$ or $n = 3$ and $b_1^{-1} + b_2^{-1} + b_3^{-1} > 1$ then $R \simeq \mathbb{P}^1$. Furthermore, one has $|\pi_1^{orb}(\mathcal{A}(2;b,b))| = 2b^2$ and $|\pi_1^{orb}(\mathcal{A}(2;b_1,b_2,b_3))| = 8[b_1^{-1} + b_2^{-1} + b_3^{-1} - 1]^{-2}$ (see Figure 2)
(ii) If $n = 2$, $b_1 = b_2 = \infty$ or $n = 3$ and $b_1^{-1} + b_2^{-1} + b_3^{-1} = 1$ or $n = 4$ and $b_1 = b_2 = b_3 = b_4 = 2$ then R is elliptic. In this case the universal uniformization of $\mathcal{A}$ is $\mathbb{C} \times \mathbb{C}$ (see Figure 3)
(iii) Otherwise R is of genus> 1 and the universal uniformization of $\mathcal{A}$ is $\mathbf{B_1} \times \mathbf{B_1}$.

Proof. By Lemma 2, the lifting of $\mathcal{A}(2a;b_1,\dots,b_n)$ to the uniformization of $\mathcal{A}(2) = (\mathbb{P}^2, 2Q)$ is the orbifold (see Figure 1)

$$\mathcal{B}(a,b_1,\dots,b_n) := \big(Q \times Q, aQ + \sum_{1\leq i\leq n} b_i(T^v_{p_i} + T^h_{p_i})\big)$$

Consequently, there is a covering $\psi : \mathcal{B}(a;b_1,\dots,b_n) \to \mathcal{A}(2a;b_1,\dots,b_n)$. Consider the case $a = 2$, and denote $\mathcal{B}'(b_1,\dots,b_n) := \mathcal{B}(1;b_1,\dots,b_n)$. Then $\mathcal{B}' = \mathcal{S} \times \mathcal{S}$, where $\mathcal{S}$ is the one-dimensional orbifold $\mathcal{S}(b_1,\dots b_n) := (Q, b_1p_1 + \cdots + b_np_n)$. Recall that $Q \simeq \mathbb{P}^1$. Assume that $n > 1$ and if $n = 2$, then $b_1 = b_2$. Then $\mathcal{S}$ admits a finite uniformization $f : R \to \mathcal{S}$ by the Bundgaard-Nielsen-Fox theorem ([BuNi],[Fo]). The Riemann surface R is of genus 0, 1 or > 1 according to the conditions stated in the theorem. It is well known that in case $R \simeq \mathbb{P}^1$ the degree of f is the order of the triangle group $\pi_1^{orb}(\mathcal{S})$, which is b^2 if $n = 2$, $b := b_1 = b_2 < \infty$ and is $2[b_1^{-1} + b_2^{-1} + b_3^{-1} - 1]^{-1}$ if $n = 3$, $b_1^{-1} + b_2^{-1} + b_3^{-1} > 1$.

The map $\zeta : (x,y) \in R \times R \to (f(x), f(y)) \in \mathcal{S} \times \mathcal{S} \simeq \mathcal{B}'$ is a uniformization of $\mathcal{B}'$ with $\deg(\zeta) = \deg(f)^2$. Since ζ is compatible with

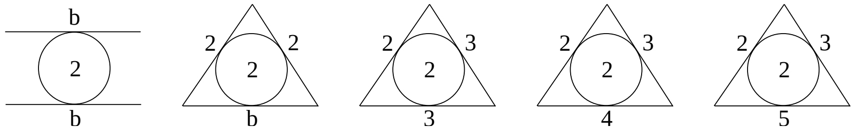

Fig. 2. Orbifolds $\mathcal{A}$ uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$

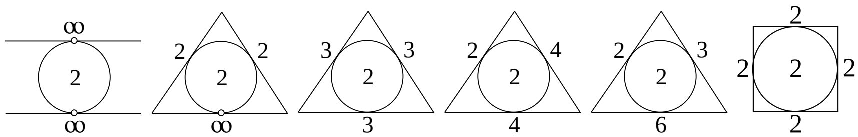

Fig. 3. Orbifolds $\mathcal{A}$ uniformized by $\mathbb{C} \times \mathbb{C}$

the involution on $\mathcal{B}'$, the composition $\xi := \psi \circ \zeta : R \times R \to \mathcal{A}$ is a Galois uniformization of $\mathcal{A}$ with $\deg(\xi) = \deg(\psi)\deg(\zeta) = 2\deg(f)^2$. □

The orbifolds in Figure 3 were discovered in [KTY], where a complete classification of the orbifolds $(\mathbb{P}^2, D)$ uniformized by $\mathbb{C}\times\mathbb{C}$ was also given. Besides these ones there are two other orbifolds, which we shall rediscover in Section 4.

3.1. Chern numbers of the orbifolds $\mathcal{A}(a; b_1, \ldots, b_n)$

By definition, the Chern numbers of $\mathcal{A}(a; b_1, \ldots, b_n)$ are given by

$$c_1^2(\mathcal{A}) = \left[n - 1 - 2a^{-1} - \sum_{1 \le i \le n} {b_i}^{-1}\right]^2$$

$$e(\mathcal{A}) = \frac{(n-1)(n-2)}{2} + (2-n)a^{-1} + \sum_{1 \le i \le n} (2-n) b_i^{-1} + \sum_{1 \le i \ne j \le n} b_i^{-1} b_j^{-1} + \frac{1}{2} \sum_{1 \le i \le n} \left[{b_i}^{-1} + a^{-1} - 2^{-1}\right]^2$$

Proposition 1. *For the orbifold $\mathcal{A}(a; b_1, \ldots, b_n)$, one has*
(i) $2e = c^2$ if and only if $a = 2$ or $n = 4$, $b_1 = b_2 = b_3 = b_4 = 2$ or $n = 3$, $(a; b_1, b_2, b_3) = (3; 2, 3, 4)$.
(ii) $e = c^2 = 0$ if and only if $n = 2$, $a = 2$; $b_1 = b_2 = \infty$ or $n = 3$, $a = 2$; $b_1^{-1} + b_2^{-1} + b_3^{-1} = 1$ or $n = 4$, $a = 2$; $b_1 = b_2 = b_3 = b_4 = 2$.
(iii) $3e = c_1^2 > 0$ if and only if $n = 3$ and $(a; b_1, b_2, b_3)$ is one of $(4; 4, 4, 4)$, $(3; 3, 4, 4)$, $(3; 6, 6, 2)$ or $(3; 6, 3, 3)$.
(iv) $c_1^2 = 0$ and $e > 0$ if and only if $n = 2$, and $(a; b_1, b_2)$ is one of

$(4;4,4)$, $(3;6,6)$, $(6;3,3)$ *or* $n=3$ *and* $(a;b_1,b_2,b_3)$ *is one of* $(4;2,2,2)$ *or* $(3;3,2,2)$.

Proof. Put $\beta=\sum_{1\le i\le n} b_i^{-1}$ and $\alpha:=\beta-a^{-1}$. We shall prove the following splitting formulas for the difference of the Chern numbers of $\mathcal{A}$, the claims of the proposition follows easily from these formulas.

$$2(2e-c_1^2)(\mathcal{A})=(a^{-1}-2^{-1})\left[n(2a^{-1}+3)-4\beta-4(2a^{-1}+1)\right] \tag{2}$$

$$8(3e-c_1^2)(\mathcal{A})=(2\alpha+5-2n)^2+3(n-3)\left(2a^{-1}-1\right)^2 \tag{3}$$

To prove (2), first note that

$$c_1^2(\mathcal{A})=(n-1)^2+4a^{-2}+\beta^2-4(n-1)a^{-1}-2(n-1)\beta+4a^{-1}\beta$$

and

$$2e(\mathcal{A})=(n-1)(n-2)+2(2-n)a^{-1}+2\beta(2-n)+2\sum_{1\le i\ne j\le n} b_i^{-1}b_j^{-1}+$$
$$\sum_{1\le i\le n}\left[b_i^{-2}+a^{-2}+2^{-2}+2b_i^{-1}a^{-1}-b_i^{-1}-a^{-1}\right]^2$$

$$=(n-1)(n-2)+\frac{n}{4}+2(2-n)a^{-1}+2\beta(2-n)+\beta^2+na^{-2}-na^{-1}+2\beta a^{-1}-\beta$$

which gives

$$(2e-c_1^2)(\mathcal{A})=1-\frac{3n}{4}+na^{-1}+\beta+(n-4)a^{-2}-2a^{-1}\beta \tag{4}$$

The formula (2) is easily seen to be equivalent to (4). The formula (3) is proved similarly, by using the above expressions for $e(\mathcal{A})$ and $c_1^2(\mathcal{A})$. □

The orbifolds (i)-(ii) in Proposition 1 were shown to be uniformizable in Theorem 4. Orbifolds $\mathcal{A}(a;2,2,2,2)$ will be shown to be uniformizable in Proposition 2 below. We don't know if the orbifold $\mathcal{A}(3;2,3,4)$ in (i) is uniformizable. The uniformizability by $\mathbf{B_2}$ of the orbifolds (iii) follows from the KNS theorem [KNS] or from [Ho], where it is also shown that the corresponding lattices are arithmetic. As for the orbifolds (iv), it will be shown elsewhere that the orbifolds $\mathcal{A}(4;4,4)$, $\mathcal{A}(3;6,6)$ and $\mathcal{A}(6;3,3)$ are not uniformizable, and that the orbifold $\mathcal{A}(3;3,2,2)$ is uniformized by a K3 surface. The universal uniformization of the orbifold $\mathcal{A}(4;2,2,2)$ is a K3 surface, and the group $\pi_1^{orb}(\mathcal{A}(4;2,2,2))$ is finite of order 256, see Proposition 3.

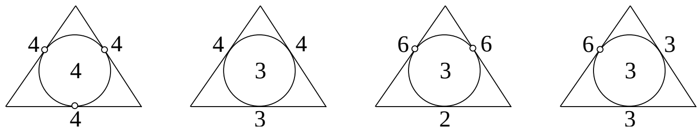

Fig. 4. Orbifolds $\mathcal{A}$ uniformized by $\mathbf{B_2}$

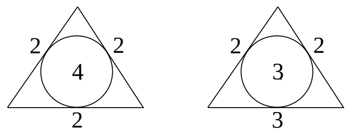

Fig. 5. Orbifolds $\mathcal{A}$ uniformized by K3 surfaces

3.2. Another orbifold covering of $\mathcal{A}$

Let $\mathcal{A} = (\mathbb{P}^2, aQ + \sum_{1\leq i\leq n} b_i T_i)$ be an orbifold over the Apollonius configuration with $n \geq 3$. Suppose without loss of generality that the lines T_1, T_2 and T_3 are defined by the equations $x = 0$, $y = 0$, and $z = 0$. Let $k \in \mathbb{N}$ and consider the orbifold $(\mathbb{P}^2, kT_1 + kT_2 + kT_3)$. This orbifold admits the uniformization $\varphi_k : [x, y, z] \in \mathbb{P}^2 \to [x^k, y^k, z^k] \in \mathbb{P}^2$. Denote by X, Y and Z the lines $\varphi_k^{-1}(T_1)$, $\varphi_k^{-1}(T_2)$ and $\varphi_k^{-1}(T_3)$ respectively. For $i \geq 4$, T_i is given by the equation $\alpha x + \beta y + \gamma z = 0$, then $T_{i,k} := \varphi_k^{-1}(T_i)$ is the Fermat curve given by the equation $\alpha x^k + \beta y^k + \gamma z^k = 0$.

The equation of the quadric Q, tangent to the lines T_1, T_2 and T_3 can be written in the form

$$Q : \alpha' x^{1/2} + \beta' y^{1/2} + \gamma' z^{1/2} = 0, \tag{5}$$

where $\alpha'\beta'\gamma' \neq 0$. This shows that for k odd, $Q_k := \varphi_k^{-1}(Q)$ is an irreducible curve given by the equation $\alpha' x^{k/2} + \beta' y^{k/2} + \gamma' z^{k/2} = 0$, and for $k = 2m$ even, $\varphi_k^{-1}(Q)$ consists of four Fermat curves $Q_{1,k}$, $Q_{2,k}$, $Q_{3,k}$, $Q_{4,k}$ given by the equations $\alpha' x^m \underline{+} \beta' y^m \underline{+} \gamma' z^m$.

For k odd, define the orbifolds

$$\mathcal{C}_k(a; e, f, g; b_4, \ldots, b_n) := (\mathbb{P}^2, aQ_k + eX + fY + gZ + \sum_{i=4}^{n} b_i T_{i,k}),$$

and for even $k = 2m$, define the orbifolds

$$\mathcal{C}_k(a_1, a_2, a_3, a_4; e, f, g; b_4, \ldots, b_n) :=$$
$$(\mathbb{P}^2, \sum_{j=1}^{4} a_j Q_{j,k} + eX + fY + gZ + \sum_{i=4}^{n} b_i T_{i,k})$$

With these notations, one has the following obvious lemma:

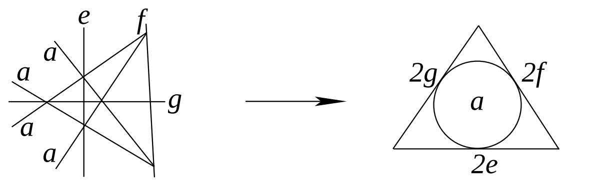

Fig. 6. The covering $\mathcal{C}_2(a,a,a,a;e,f,g) \to \mathcal{A}(a;2e,2f,2g)$

Lemma 3. *(i) For k odd, there is an orbifold covering*

$$\varphi_k : \mathcal{C}_k(a;e,f,g;b_4,\dots,b_n) \to \mathcal{A}(a;ke,kf,kg;b_4,\dots b_n).$$

(ii) For k even, there is an orbifold covering

$$\varphi_k : \mathcal{C}_k(a,a,a,a;e,f,g;b_4,\dots,b_n) \to \mathcal{A}(a;ke,kf,kg;b_4,\dots b_n).$$

The covering map φ_2 and the orbifold $\mathcal{C}_2$ are particularly interesting. In this case, $L_1 := Q_{1,2}$, $L_2 := Q_{2,2}$, $L_3 := Q_{3,2}$, $L_4 := Q_{4,2}$ are four lines given by the equations $\alpha' x \underline{+} \beta' y \underline{+} \gamma' z$. Since $abc \neq 0$, the points $L_i \cap L_j$ lie on the smooth points of $xyz = 0$. The curves $T_{i,2}$ are smooth quadrics for $i \geq 4$. Each $T_{i,2}$ has the four lines L_1, L_2, L_3, L_4 as tangents. In particular, for $n = 3$ one has the lines L_i ($i \in \{1,2,3,4\}$) and the lines X, Y, Z, forming an arrangement of 7 lines with 6 triple points and 3 nodes. For $n = 4$, the curve $T_{4,2}$ is a quadric, and the lines L_i ($i \in \{1,2,3,4\}$) are tangent to this quadric, in other words $\mathcal{C}_2(b,b,b,b;1,1,1;2) \simeq \mathcal{A}(2;b,b,b,b)$. This proves the following proposition.

Proposition 2. *There is a covering $\varphi_2 : \mathcal{A}(2;b,b,b,b) \to \mathcal{A}(b;2,2,2,2)$. Hence the orbifolds $\mathcal{A}(b;2,2,2,2)$ are uniformized by $\mathbf{B_1} \times \mathbf{B_1}$ for $b > 2$.*

For the orbifolds $\mathcal{A}(a;2,2,2)$ one has

Proposition 3. *The group $\pi_1^{orb}(\mathcal{A}(a;2,2,2))$ is finite of order $4a^3$. There is a universal uniformization $M_a \to \mathcal{A}(a;2,2,2)$ such that*

$$e(M_a) = a(a^2 - 4a + 6), \quad c_1^2(M_a) = a(4-a)^2.$$

In particular, $\mathcal{A}(4;2,2,2)$ is uniformized by a $K3$ surface.

Proof. The orbifold $\mathcal{C}_2(a,a,a,a;1,1,1)$ admits a universal uniformization of degree a^3 since its locus consists of the lines L_i $(1 \leq i \leq 4)$ in general position and $\pi_1^{orb}(\mathcal{C}_2(a,a,a,a;1,1,1)) \simeq \mathbb{Z}/(a) \oplus \mathbb{Z}/(a) \oplus \mathbb{Z}/(a)$. Composing this with the covering $\mathcal{C}_2(a,a,a,a;1,1,1) \to \mathcal{A}(a;2,2,2)$ gives the desired result.

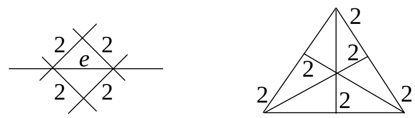

Fig. 7. Orbifolds $\mathcal{C}_2$ uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$ and by $\mathbb{C} \times \mathbb{C}$

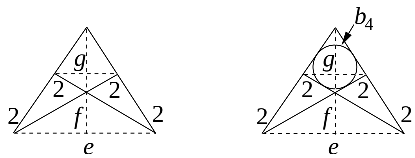

Fig. 8. Some orbifolds $\mathcal{C}_2$ uniformized by $\mathbf{B_1} \times \mathbf{B_1}$

4. Coverings of the orbifolds uniformized by $\mathbf{B_1} \times \mathbf{B_1}$.

Theorem 5. *(i) The orbifolds $\mathcal{C}_2(2,2,2,2;e,1,1)$ are uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 7)*
(ii) The orbifolds $\mathcal{C}_2(2,2,2,2;2,2,1)$ and $\mathcal{C}_3(2;1,1,1)$ are uniformized by $\mathbb{C} \times \mathbb{C}$ (see Figure 7)
(iii) Otherwise, $\mathcal{C}_k(2,2,2,2;e,f,g;b_4,\ldots,b_n)$ is uniformized by $\mathbf{B_1} \times \mathbf{B_1}$ (see Figure 8).

Proof. By Lemma 3, for k even there is a covering

$$\mathcal{C}_k(2,2,2,2;e,f,g;b_4,\ldots,b_n) \to \mathcal{A}(2;ek,fk,gk;b_4,\ldots,b_n),$$

and for k odd there is a covering

$$\mathcal{C}_k(2;e,f,g;b_4,\ldots,b_n) \to \mathcal{A}(2;ek,fk,gk;b_4,\ldots,b_n).$$

On the other hand, by Theorem 4, the orbifolds $\mathcal{A}(2;ek,fk,gk,b_4,\ldots,b_n)$ are uniformized by $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{C} \times \mathbb{C}$, or $\mathbf{B_1} \times \mathbf{B_1}$ according to the conditions stated in the theorem. □

Consider the line arrangement $\{L_1, L_2, L_3, L_4, X, Y, Z\}$. The lines $\{L_1, L_2, L_3, L_4, X, Y\}$ forms a complete quadrilateral, i.e. an arrangement of six lines with four triple points and three nodes. The triple points of the complete quadrilateral can be given as $L_1 \cap L_2 \cap Y$, $L_2 \cap L_3 \cap X$, $L_3 \cap L_4 \cap Y$ and $L_4 \cap L_1 \cap X$, in this case the nodes are $X \cap Y$, $L_1 \cap L_3$ and $L_2 \cap L_4$. The line Z passes through the nodes $L_1 \cap L_3$ and $L_2 \cap L_4$. Take the lines $\{L_1, L_2, X\}$, which do not meet at a triple point, and consider the uniformization map $\gamma_2 : \mathbb{P}^2 \to (\mathbb{P}^2, mL_1 + mL_2 + mX)$. Then $\gamma_2^{-1}(L_3)$ (or $\gamma_2^{-1}(L_4)$ or $\gamma_2^{-1}(Y)$) will consist of m lines, forming an arrangement of $3m$ lines with m^2 triple points and three points of multiplicity m. The

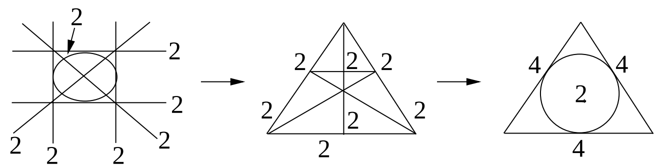

Fig. 9. Coverings $\mathcal{D}(1,1;2,2;2,2;1;2,2;2) \to \mathcal{C}_2(2,2,2,2;2,2,2) \to \mathcal{A}(2;4,4,4)$

curve $Z' := \gamma_2^{-1}(Z)$ will be a smooth Fermat curve of degree m. In the particular case where $m = 2$, let

$$\{L_3', L_3''\} := \gamma_2^{-1}(L_3), \quad \{L_4', L_4''\} := \gamma_2^{-1}(L_4), \quad \{Y', Y''\} := \gamma_2^{-1}(Y),$$

and define the orbifold $\mathcal{D}(a_1, a_2; a_3', a_3''; a_4', a_4''; e; f', f''; g)$ as

$$(\mathbb{P}^2, a_1 L_1' + a_2 L_2' + a_3' L_3' + a_3'' L_3'' + a_4' L_4' + a_4'' L_4'' + eX' + f'Y' + f''Y'' + gZ')$$

so that there is a covering of orbifolds

$$\mathcal{D}(a_1, a_2; a_3, a_3; a_4, a_4; e; f, f; g) \to \mathcal{C}_2(2a_1, 2a_2, a_3, a_4; 2e, f, g) \qquad (6)$$

In particular there is a covering

$$\mathcal{D}(1,1;2,2;2,2;1;2,2;1) \to \mathcal{C}_2(2,2,2,2;2,2) \qquad (7)$$

The locus of the orbifold $\mathcal{D}(1,1;2,2;2,2;1;2,2;1)$ consists of the lines $\{L_3', L_3'', L_4', L_4'', Y', Y''\}$. Since the lifting $\gamma_2^{-1}(L_3 \cap L_4 \cap Y)$ consist of four triple points, these lines forms an arrangement of 6 lines with 4 triple points, which is the complete quadrilateral. In other words, one has an isomorphism $\mathcal{D}(1,1;2,2;2,2;1;2,2;1) \simeq \mathcal{C}_2(2,2,2,2;2,2,1))$. The orbifolds $\mathcal{C}_2(2,2,2,2;2e,2f,g;b_4,\dots,b_n)$ can be lifted recursively by these coverings $\mathcal{C}_2(2,2,2,2;2,2,1) \to \mathcal{C}_2(2,2,2,2;2,2,1)$. The lifted orbifolds provide many examples of orbifolds $(\mathbb{P}^2, D)$ uniformized by $\mathbf{B_1} \times \mathbf{B_1}$. In Figure 9, we have shown an orbifold obtained this way, namely the orbifold $\mathcal{D}(1,1;2,2;2,2;1;2,2;2)$. Now one can take another set of three lines that do not meet at a triple point and lift this orbifold to the corresponding covering $\mathbb{P}^2 \to \mathbb{P}^2$.

Consider the covering $\mathcal{C}_4(2,2,2,2) \to \mathcal{A}(2;4,4,4)$. The curves $Q_{4,1}$, $Q_{4,2}$, $Q_{4,3}$, $Q_{4,4}$ are smooth quadrics, such that $Q_{4,i} \cap Q_{4,j}$ consists of exactly two points, and $Q_{4,i} \cap Q_{4,j} \cap Q_{4,k} = \emptyset$ for $1 \leq i \neq j \neq k \leq 4$. Explicitly, these quadrics can be given by the equations $\alpha' x^2 \underline{+} \beta' y^2 \underline{+} \gamma' z^2 = 0$. Since the orbifold $\mathcal{A}(2;4,4,4)$ is uniformized by $\mathbf{B_1} \times \mathbf{B_1}$, so is the orbifold $\mathcal{C}_4(2,2,2,2)$.

The orbifolds $\mathcal{A}(2; b_1, \dots, b_n)$ can be lifted to the uniformizations by K3 surfaces X_i of the orbifolds given in the following lemma, yielding infinitely many examples of orbifolds (X_i, D) uniformized by $\mathbf{B_1} \times \mathbf{B_1}$, with X_i $(i \in \{1,2,3\})$ being a K3 surface.

Lemma 4. *Let T_i $(1 \leq i \leq 6)$ be six lines in $\mathbb{P}^2$ in general position. Then the orbifolds*

$$\begin{aligned} \mathcal{E}_1 &:= (\mathbb{P}^2, 6T_1 + 6T_2 + 6T_3 + 2T_4), \\ \mathcal{E}_2 &:= (\mathbb{P}^2, 4T_1 + 4T_2 + 4T_3 + 4T_4), \\ \mathcal{E}_3 &:= (\mathbb{P}^2, 2T_1 + 2T_2 + 2T_3 + 2T_4 + 2T_5 + 2T_6) \end{aligned}$$

are uniformized by K3 surfaces.

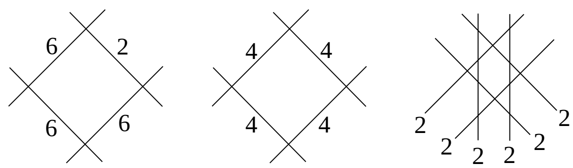

Fig. 10. Orbifolds uniformized by K3 surfaces

Proof. Lifting $\mathcal{E}_1$ to the uniformization of $(\mathbb{P}^2, 6T_1 + 6T_2 + 6T_3)$ yields an orbifold $(\mathbb{P}^2, 2T_{4,6})$ where $T_{4,6}$ is a smooth sextic. This latter orbifold clearly admits a universal uniformization by a K3 surface. Similarly, lifting $\mathcal{E}_2$ to the uniformization of $(\mathbb{P}^2, 4T_1 + 4T_2 + 4T_3)$ yields an orbifold $(\mathbb{P}^2, 4T_{4,4})$ where $T_{4,4}$ is a smooth quartic, which clearly admits a universal uniformization by a K3 surface. The case of the orbifold $\mathcal{E}_3$ is well known, see e.g. [Hu]. $\square$

Proof of Theorem 2. The orbifold $(\mathbb{P}^2, 2Q_m)$ is a lifting of the orbifold $\mathcal{A}(2; m, m, m)$ to the uniformization of $(\mathbb{P}^2, mT_1 + mT_2 + mT_3)$. For $i \in \{1, 2, 3\}$, let $p_i := Q \cap T_i$. Then the orbifold $(\mathbb{P}^2, mT_1 + mT_2 + mT_3)$ restricts to Q as the orbifold $\mathcal{S}(m, m, m) := (Q, mp_1 + mp_2 + mp_3)$, provided that m is an odd integer. The restriction of φ_m to Q shows that there is a uniformization $Q_m \to \mathcal{S}(m, m, m)$ by an abelian map of degree m^2, and consequently there is a uniformization of degree m^4

$$\zeta : Q_m \times Q_m \to \mathcal{S}(m, m, m) \times \mathcal{S}(m, m, m)$$

On the other hand, with the notations of Section 3 there is a covering

$$\psi : \mathcal{S}(m, m, m) \times \mathcal{S}(m, m, m) \to \mathcal{A}(2; m, m, m)$$

Taking the above uniformization of $\mathcal{S}(m, m, m)$, we see that there is a Galois covering $\psi \circ \zeta : Q_m \times Q_m \to \mathcal{A}(2; m, m, m)$. The map $\psi \circ \zeta$ is of degree $2m^4$. Let G be the corresponding Galois group.

$$\begin{array}{ccc}
(\mathbb{P}^2, 2Q_m) & \xleftarrow{\sigma} & Q_m \times Q_m \\
\varphi_m \downarrow & & \downarrow \zeta \\
\mathcal{A}(2; m, m, m) & \xleftarrow{\psi} & \mathcal{S}(m, m, m) \times \mathcal{S}(m, m, m)
\end{array}$$

We want to show that the uniformization $Q_m \times Q_m \to \mathcal{A}(2; m, m, m)$ lifts to a uniformization of $(\mathbb{P}^2, 2Q_m)$. This is equivalent to showing that

$$H := (\psi \circ \zeta)_*(\pi_1(Q_m) \times \pi_1(Q_m)) \lhd K := (\varphi_m)_*(\pi_1^{orb}(\mathbb{P}^2, 2Q_m))$$

The quotient of $\pi_1^{orb}(\mathcal{A}(2; m, m, m))$ by the normal subgroup generated by the meridians of Q gives the group

$$\pi_1^{orb}(\mathbb{P}^2, mT_1 + mT_2 + mT_3) \simeq \mathbb{Z}/(m) \oplus \mathbb{Z}/(m)$$

Since this is the Galois group of φ_m, the group K is the normal subgroup of $\pi_1^{orb}(\mathcal{A}(2; m, m, m))$ generated by the meridians of Q. Note that since $\pi_1^{orb}(\mathcal{A}(2; m, m, m))/K$ is abelian, K should contain the commutators $[\tau_i, \tau_j]$, for all meridians τ_i of T_i and τ_j of T_j. On the other hand, the quotient of $\pi_1^{orb}(\mathcal{A}(2; m, m, m))$ by the normal subgroup generated by the meridians of T_1, T_2 and T_3 gives the group $\pi_1^{orb}(\mathbb{P}^2, 2Q) \simeq \mathbb{Z}/(2)$, which shows that $\psi_*(\pi_1^{orb}(S(m, m, m) \times S(m, m, m)))$ is the normal subgroup of $\mathcal{A}(2; m, m, m)$ generated by the meridians of T_1, T_2 and T_3. Since

$$\zeta : Q_m \times Q_m \to S(m, m, m) \times S(m, m, m)$$

is the maximal abelian covering, the group H is the normal subgroup of $\mathcal{A}(2; m, m, m)$ generated by the commutators of the meridians of T_1, T_2 and T_3. This proves that H is a normal subgroup of K. □

Double covers of $\mathbb{P}^2$ branched along the curves Q_m were also studied in [Pe]. For a discussion of the groups $\pi_1(\mathbb{P}^2 \backslash Q_m)$ see Section 7.

5. Coverings of the orbifolds $\mathcal{A}$ uniformized by $\mathbf{B}_2$

Coverings of $\mathcal{A}(3; 6, 3, 3)$.
There is a covering $\varphi_3 : \mathcal{C}_3(3; 2, 1, 1) \to \mathcal{A}(3; 3, 3, 6)$. More explicitly, $\mathcal{C}_3(3, 2, 1, 1)$ is the orbifold $(\mathbb{P}^2, 3Q_m + 2X)$, where Q_m is the nine-cuspidal sextic, and X is a line passing through three of its cusps.

Coverings of $\mathcal{A}(3;6,6,2)$.
There is a covering $\varphi_2 : \mathcal{C}_2(3,3,3,3;3,3,1) \to \mathcal{A}(3;6,6,2)$. More explicitly, one has, as in Section 4,

$$\mathcal{C}_2(3,3,3,3;3,3,1) := (\mathbb{P}^2, 3L_1 + 3L_2 + 3L_3 + 3L_4 + 3X + 3Y),$$

where the set of lines $\{L_1, L_2, L_3, L_4, X, Y\}$ forms a complete quadrilateral. Keeping the notations of Section 4, consider the uniformization map $\gamma_3 : \mathbb{P}^2 \to (\mathbb{P}^2, 3X + 3Y + 3L_1)$. The lifting $\gamma_3^{-1}(L_2)$ consists of three lines $L_{2,1}$, $L_{2,2}$ and $L_{2,3}$ which meet at the point $\gamma_3^{-1}(L_{1,2} \cap L_{2,2} \cap Y)$. Similarly, the lifting $\gamma_3^{-1}(L_{4,2})$ consists of three lines $L_{4,1}$, $L_{4,2}$, $L_{4,3}$ which meet at the point $\gamma_3^{-1}(L_{4,2} \cap L_{1,2} \cap X)$. The lifting $\gamma_3^{-1}(L_{3,2})$ is a smooth cubic $L_{3,1}$, and it is readily seen by local considerations that for $i \in \{1,2,3\}$ the lines $L_{2,i}$, $L_{4,i}$ are tangent to $L_{3,1}$ with multiplicity 3. The points $\gamma_3^{-1}(L_{2,2} \cap L_{4,2})$ lifts as the nine points of intersection $L_{2,i} \cap L_{2,j}$ $(i,j \in \{1,2,3\})$. This shows that the lift of $\mathcal{C}_2(3,3,3,3;3,3,1)$ along γ_3 is the orbifold

$$(\mathbb{P}^2, 3L_{3,1} + 3L_{2,1} + 3L_{2,2} + 3L_{2,3} + 3L_{4,1} + 3L_{4,2} + 3L_{4,3})$$

with 2 points of type $x^3 = y^3$, nine points of type $x^2 = y^2$, and 6 points of type $x^6 = y^2$.

Lifting this orbifold once more to the uniformization of $(\mathbb{P}^2, 3L_{2,1} + 3L_{2,2} + 3L_{4,1})$ yields an arrangement of a smooth curve $L_{3,1,1}$ of degree 9 with two smooth cubics $L_{4,2,1}$ and $L_{4,3,1}$ and three lines $L_{2,3,1}$, $L_{2,3,2}$, $L_{2,3,3}$. These lines meet at a point, and intersects the two cubics at 18 nodes. The cubics are tangent to each other at 3 points with multiplicity 3. Cubics and lines altogether intersect the degree-9 curve at 27 distinct points of type $x^6 = y^2$.

Returning again to the orbifold $\mathcal{C}_2(3,3,3,3;3,3,1)$, consider the uniformization map $\sigma_3 : \mathbb{P}^2 \to (\mathbb{P}^2, 3L_1 + 3L_2 + 3X)$. One has

$$\sigma_3 - lift(\mathcal{C}(3,3,3,3;3,3,1)) = (\mathbb{P}^2, 3K_1 + 3K_2 + \cdots + 3K_9),$$

where the lines $K_1, \ldots, K_9$ forms a Ceva arrangement, which can be given by the equation $(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$. Suppose that K_1, K_2, K_3 do not meet at a triple point. Lifting this orbifold to the uniformization by $\mathbb{P}^2$ of $(\mathbb{P}^2, 3K_1 + 3K_2 + 3K_3)$ yields an arrangement of nine lines with three smooth cubics. It is left to the reader to verify that this arrangement can be lifted once more to $\mathbb{P}^2$ in two different ways.

Coverings of $\mathcal{A}(3;3,4,4)$.
This orbifold (and the orbifold $\mathcal{A}(3;6,6,2)$) can be lifted to a K3 - uniformization of $\mathcal{A}(3;3,2,2)$. An example of a ball quotient orbifold over a K3 surface was also given in [Na]. Recall that $\mathcal{A}(3;3,4,2)$ satisfy $c_1^2 = 2e$,

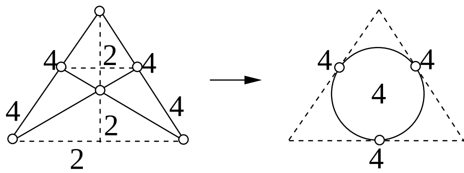

Fig. 11. The covering $\mathcal{C}_2(4,4,4,4;2,2,2) \to \mathcal{A}(4;4,4,4)$

but we don't know whether it admits a uniformization. In case it does, $\mathcal{A}(3;3,4,4)$ lifts to this uniformization.

Coverings of $\mathcal{A}(4;4,4,4)$**.**
Lifting $\mathcal{A}(4;4,4,4)$ by φ_4 gives the orbifold $\mathcal{C}_4(4,4,4,4)$ over four mutually tangent quadrics. It can also be lifted to the uniformization of $\mathcal{A}(4;2,2,2)$, which is a K3 surface. Note that by Theorem 4 the orbifold $\mathcal{A}(2;4,4,4)$ is uniformized by a product of two Riemann surfaces. The orbifold $\mathcal{A}(4;4,4,4)$ can also be lifted to this product.

Proof of Theorem 1. There is an orbifold covering

$$\mu_1 : \mathcal{C}_2(4,4,4,4;2,2,2) \to \mathcal{A}(4;4,4,4),$$

where μ_1 is a bicyclic covering of degree 4, branched along the dashed lines in the locus of $\mathcal{A}(4,4,4,4)$ (see Figure 11). The lattices corresponding to both of the orbifolds $\mathcal{A}(4;4,4,4)$ and $\mathcal{O}_1 := \mathcal{C}_2(4,4,4,4;2,2,2)$ are known to be arithmetic, see [Ho] and [DeMo]. In the locus of the orbifold $\mathcal{O}_1$, take three dashed lines and mark the remaining lines with red and blue as in Figure 12 (where "r" means red and "b" means blue). Consider the degree-4 bicyclic covering $\mu_2 : \mathbb{P}^2 \to \mathbb{P}^2$ branched along the dashed lines.

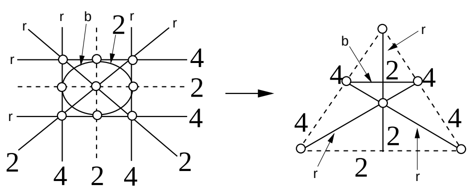

Fig. 12. The covering $\mathcal{O}_2 \to \mathcal{O}_1 := \mathcal{C}_2(4,4,4,4;2,2,2)$

Since the orbifold $\mathcal{O}_1$ is uniformized by $\mathbf{B_2}$, so is the orbifold $\mathcal{O}_2 := \mu_2 - lift\,(\mathcal{O}_1)$. Since the red lines pass through the intersection points of the dashed lines, the red lines will be lifted as 6 lines forming a complete quadrilateral. Since the blue line intersects the dashed lines at three distinct points, its lifting will be a smooth quadric. Two dashed lines with weight 4 also lifts as two lines, and their weight becomes 2 (see Figure 12). Now redraw the locus of the orbifold and mark the curves in this locus as in Figure 13 (where "r" means red, "b" means blue, "p" means pink, and "g" means green).

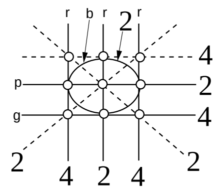

Fig. 13. New marking of the orbifold $\mathcal{O}_2$

Consider the degree-4 bicyclic covering $\mu_3 : \mathbb{P}^2 \to \mathbb{P}^2$ branched along the dashed lines. Since the orbifold $\mathcal{O}_2$ is uniformized by $\mathbf{B_2}$, so is the orbifold $\mathcal{O}_3 := \mu_3 - lift\,(\mathcal{O}_2)$. Let us describe the locus of $\mathcal{O}_3$. Since the red lines pass through the intersection points of the dashed lines, they will lift as 6 lines, forming a complete quadrilateral. Denote these lines by $R_1, \ldots, R_6$. The pink line also pass through an intersection point of the dashed lines, so it will lift as two lines P_1 and P_2. Green line intersects the dashed lines at three distinct points, so its lifting will be a smooth quadric G. The blue quadric will be lifted as a quartic curve B (which is in fact irreducible with two nodes). The dashed line with weight four will also lift as a line D with weight 2. Hence, $\mathcal{O}_3$ is the orbifold

$$(\mathbb{P}^2, 4R_1 + 4R_2 + 4R_3 + 4R_4 + 2R_5 + 2R_6 + 2P_1 + 2P_2 + 4G + 2B + 2D)$$

Consider the complete quadrilateral formed by the lines $R_1, \ldots, R_6$. Suppose that the lines R_1, R_3, R_5 meets at a point. Then the lines R_2, R_4, R_6 intersect each other at three distinct points. Let μ_4 be the degree-4 bicyclic covering $\mu_4 : \mathbb{P}^2 \to \mathbb{P}^2$ branched along R_2, R_4, R_6. Consider the orbifold $\mathcal{O}_4 := \mu_4 - lift\,(\mathcal{O}_3)$. Since the lines R_1, R_3, R_5 pass through the intersection points of R_2, R_4, R_6, they fill be lifted as 6 lines forming a complete quadrilateral, so that the locus of $\mathcal{O}_4$ will contain a complete quadrilateral. Now one can recursively apply this proces to get an infinite series $\mathcal{O}_r$ of orbifolds uniformized by $\mathbf{B_2}$. Let $\mu_r : \mathcal{O}_r \to \mathcal{O}_{r-1}$ be the $r+1$th covering in the recursion. Consider the blue quadric in the locus of $\mathcal{O}_2$. Mark a curve in the locus of $\mathcal{O}_r$ with blue if $\mu_r(C)$ is blue. During the recursion, one may assume that μ_r is never branched along a blue curve. Such a covering has the effect of multiplying the total degree of blue curves by two. Hence, the degree of the locus of $\mathcal{O}_r$ is $\geq 2^r$, which shows that among $\mathcal{O}_r$ there are infinitely many distinct orbifolds.

Remarks. (1) For $r \geq 2$, the uniformizations $\mathbf{B_2} \to \mathcal{O}_r$ take place outside of finitely many cusp points. Indeed, the orbifold $\mathcal{O}_2$ has cusp points only and locally an orbifold-covering of a cusp point is also a cusp point. But since $\mathcal{O}_r \to \mathcal{O}_{r-1}$ is an orbifold covering, singular points of $\mathcal{O}_r$ are locally coverings of the singular points of $\mathcal{O}_{r-1}$.
(2) One may apply the recursion described above in many different ways, for example taking three lines in the locus of $\mathcal{O}_2$, which are tangent to the blue quadric and lifting $\mathcal{O}_2$ to the bicyclic covering $\mathbb{P}^2 \to \mathbb{P}^2$ branched along these lines gives another infinite series of ball-quotient orbifolds.
(3) The orbifold $\mathcal{A}(4;4,4,4)$ is invariant under a Σ_3-action on $\mathbb{P}^2$, and the quotient orbifold $\mathcal{Q} := \mathcal{A}(4;4,4,4)/\Sigma_3$ (with a singular base space $\mathbb{P}^2/\Sigma_3$) is the "queen" of the orbifolds constructed above. Its locus consists of two rational curves, one is the image of the quadric, and the other is the image of the tangent lines.

6. Fundamental groups

A presentation of the group $\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_n)$ is known, see [ATU] and see [Be1], [Be2], [La] for generalizations.

Theorem 6. *One has the presentation*

$$\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_n) \simeq \left\langle \begin{matrix} \tau_1, \dots, \tau_n, \\ \kappa_1, \dots, \kappa_n \end{matrix} \left| \begin{matrix} \kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1}, \ 2 \leq i \leq n \\ (\kappa_i \tau_i)^2 = (\tau_i \kappa_i)^2, \ 1 \leq i \leq n \\ \left[\kappa_i^{-1} \tau_i \kappa_i, \tau_j\right] = 1, \ 1 \leq i < j \leq n \\ \tau_n \cdots \tau_1 \kappa_1^2 = 1 \end{matrix} \right. \right\rangle \tag{8}$$

where κ_i are meridians of Q and τ_i is a meridian of T_i for $1 \leq i \leq n$.

Note that $\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_n)$ is the braid group on two strands of the punctured sphere $Q \backslash \{p_1, \dots, p_n\}$. For small n, its presentation can be simplified:

Corollary 1. *[ATU] (i) The group $\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_1)$ is abelian.*
(ii) [DOZ] The group $\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_2)$ admits the presentation $\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_2) \simeq \langle \tau, \kappa \,|\, (\tau\kappa)^2 = (\kappa\tau)^2 \rangle$, where κ is a meridian of Q and τ is a meridian of T_1. A meridian of T_2 is given by $\kappa^{-2}\tau^{-1}$.
(iii) [Deg] The group $\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_3)$ admits the presentation $\pi_1(\mathbb{P}^2 \backslash \mathcal{A}_3) \simeq \langle \tau, \sigma, \kappa \,|\, (\tau\kappa)^2 = (\kappa\tau)^2,\ (\sigma\kappa)^2 = (\kappa\sigma)^2,\ [\sigma, \tau] = 1 \rangle$, where σ, τ are meridians of T_1 and T_3 respectively, and κ is a meridian of Q. A meridian of T_2 is given by $(\kappa\tau\kappa\sigma)^{-1}$.

Corollary 2. *One has the presentation*

$$\pi_1^{orb}(\mathcal{A}(a; b_1, b_2, b_3)) \simeq \left\langle \kappa, \tau, \sigma \left| \begin{matrix} (\tau\kappa)^2 = (\kappa\tau)^2, \quad (\sigma\kappa)^2 = (\kappa\sigma)^2, \\ [\sigma, \tau] = \kappa^a = \tau^{b_1} = \sigma^{b_2} = (\kappa\tau\kappa\sigma)^{b_3} = 1 \end{matrix} \right. \right\rangle$$

Fundamental groups of almost all the curves or arrangements appearing in this article is a subgroup of this group. In particular, there is an exact sequence

$$0 \to \pi_1(\mathbb{P}^2\backslash Q_m) \to \pi_1^{orb}(\mathcal{A}(\infty; m, m, m)) \to \mathbb{Z}/(m) \oplus \mathbb{Z}/(m) \to 0$$

In [Co], a presentation of $\pi_1(\mathbb{P}^2\backslash Q_m)$ was computed from this exact sequence by using the Reidemeister-Schreier algorithm. A presentation of the group $\pi_1(\mathbb{P}^2\backslash Q_3)$ was found by Zariski [Za], see also [Ka].

7. Final remarks

Let $C \subset \mathbb{P}^2$ be an irreducible curve with κ simple cusps ν nodes and no other singularities. The Euler number of the orbifold $(\mathbb{P}^2, bC)$ ($b \in \{2,3,4,5,6\}$) is

$$e(\mathbb{P}^2, bC) = e(\mathbb{P}^2\backslash C) + \frac{e(C\backslash \mathrm{Sing}(C))}{b} + \frac{3\kappa}{2}\left[\frac{1}{b} - \frac{1}{6}\right]^2$$

Considering C as a subset of $\mathbb{P}^2$, one has $e(C) = -d^2 + 3d + 2\kappa + \nu$, so that $e(\mathbb{P}^2\backslash C) = 3 + d^2 - 3d - 2\kappa - \nu$. Setting $e(C\backslash \mathrm{Sing}(C)) = e(C) - \kappa - \nu$ gives

$$e(\mathbb{P}^2, mC) = 3 + d^2 - 3d - 2\kappa - \nu + \frac{-d^2 + 3d + \kappa}{b} + \frac{\nu}{b^2} + \frac{3\kappa}{2}\left[\frac{1}{b} - \frac{1}{6}\right]^2$$

On the other hand, the first Chern number of this orbifold is

$$c_1^2(\mathbb{P}^2, bC) = \left[-3 + d\left(1 - \frac{1}{b}\right)\right]^2$$

Let g be the genus of C. It is easy to show that there exists infinitely many five-tuples (d, κ, ν, b, g) with $(3e - c_1^2)(\mathbb{P}^2, bC) = 0$. Some examples are given in the table below. The first curve in the table is the 9-cuspidal sextic. We don't know whether the other curves exists.

	d	κ	ν	b	g		d	κ	ν	b	g		d	κ	ν	b	g
1	6	9	0	2	1	**11**	12	27	24	2	4	**21**	15	54	19	2	18
2	7	11	3	2	1	**12**	13	44	7	2	15	**22**	15	45	35	2	11
3	8	17	0	2	4	**13**	13	35	23	2	8	**23**	15	36	51	2	4
4	9	18	7	2	3	**14**	13	26	39	2	1	**24**	15	40	51	6	0
5	9	18	8	4	2	**15**	14	56	0	2	22	**25**	16	74	0	2	31
6	10	23	8	2	5	**16**	14	47	16	2	15	**26**	16	65	16	2	24
7	11	32	3	2	10	**17**	14	38	32	2	8	**27**	16	56	32	2	17
8	11	23	19	2	3	**18**	14	29	48	2	1	**28**	16	47	48	2	10
9	12	36	8	2	11	**19**	15	63	3	2	25	**29**	16	38	64	2	3
10	12	36	10	4	9	**20**	15	64	3	3	24	**30**	17	80	7	2	33

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