More Zariski Pairs and Finite Fundamental Groups of Curve Complements

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Abstract. We give a recipe for finding new examples of plane curves with a finite non-abelian fundamental group of the complement and new examples of Zariski pairs of plane curves.

1. Introduction. Let $C \subset \mathbb{P}^2$ be an irreducible algebraic curve of degree d; we are interested in the fundamental group $\pi_1(\mathbb{P}^2 \backslash C)$ of its complement. This group will be called *the group of* C in the sequel.

There are many examples of irreducible curves with abelian groups: By the Zariski conjecture proved by Deligne and Fulton, $\pi_1(\mathbb{P}^2 \backslash C) \simeq \mathbb{Z}/d\mathbb{Z}$ whenever C is nodal (see [5]). On the other hand, most of the known non-abelian curve groups turns out to be infinite (see [6] for a survey of known $\pi_1(\mathbb{P}^2 \backslash C)$'s). Oka raised the problem of finding examples of finite non-abelian curve groups (see [12]).

The first example of a curve with a finite non-abelian group, namely the 3-cuspidal quartic, was given by Zariski [17]. The group of the rational quintic with three double cusps was found to be non-abelian of order 320 by Degtyarev [4]. Several infinite series of curves with a finite non-abelian group were obtained by Degtyarev [4], Oka [11], and Shimada [13], [12]. The main result of this article is the following theorem, supplying further evidence that there are many curves with finite non-abelian groups.

Theorem 1 *Let G be a curve group. Then, for any $n \in \mathbb{N}$, there is a central extension H of G by the cyclic group of order n, such that H is also a curve group. Hence, if G is a finite non-abelian group of order $|G|$, then H is a*

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finite non-abelian group of order $|H| = n|G|$. If G is almost solvable, then so is H.

One important reason for studying the group $\pi_1(\mathbb{P}^2\backslash C)$ is that this group is a strong invariant of the curve $C \subset \mathbb{P}^2$, and it can distinguish curves having the same singularities. The first example of this phenomenon is due to Zariski [17]: The group of a sextic C_1 with six cusps lying on a conic is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, whereas, if C_2 is another sextic with six simple cusps *not* lying on a conic, then the group of C_2 is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. A pair of curves of the same degree and with the same singularities, but with non-homeomorphic complements is called a *Zariski pair.* Many examples of Zariski pairs are known now (see [2] [3], [10], [14]), but probably the strongest result up-to date has appeared in [9], where it is shown that there exists an infinite family of Zariski k-tuples $(C_1, C_2, \ldots, C_k)$ for each $k \in \mathbb{N}$. Cremona transformations we describe below can be used to obtain more examples of Zariski pairs from the known ones (see Theorem 3).

The technique we employ has been introduced by Degtyarev [4] and elaborated by Artal [1]. The underlying idea is the following: If a curve $\widetilde{C}$ is obtained from a simpler curve C by means of a Cremona transformation $\phi : \mathbb{P}^2 \to \mathbb{P}^2$, then ϕ induces a biholomorphism $\mathbb{P}^2\backslash(C \cup A) \stackrel{\simeq}{\to} \mathbb{P}^2\backslash(\widetilde{C} \cup B)$, where A, B are some line arrangements. Hence $\pi_1(\mathbb{P}^2\backslash(C \cup A)) \simeq \pi_1(\mathbb{P}^2\backslash(\widetilde{C} \cup B))$ and if the former group is known, then $\pi_1(\mathbb{P}^2\backslash\widetilde{C})$ can be obtained by adding the relations corresponding to the gluing of B. We use this method in the opposite sense, i.e. given a curve C with a known group, we apply some Cremona transformations in a way that the group $\pi_1(\mathbb{P}^2\backslash(C \cup A))$ and the relations corresponding to the gluing of B are fairly simple. In the form we use them, these Cremona transformations are taken from [7], where certain series of rational cuspidal curves have been constructed and classified.

2. Recipe. Let X be a surface, $C \subset X$ be a curve, with a smooth point $p \in C$, and $* \in X\backslash C$ be a base point. Define a *meridian* μ of C at p to be a loop drawn as follows: Let Δ be a smooth analytic branch meeting C transversally at p. Connect $*$ to the boundary of Δ by a path ω in $X\backslash C$, and put $\mu := \omega \cdot \delta \cdot \omega^{-1}$, where δ is the boundary of Δ, oriented counterclockwise. The homotopy class of μ in $X\backslash C$ defines an element of the group $\pi_1(X\backslash C, *)$, but we shall take the liberty to consider μ directly as an element of $\pi_1(X\backslash C)$, and we shall omit base points when this do not lead to any confusion.

It is well known that if p, q lie on an irreducible component of C, then the

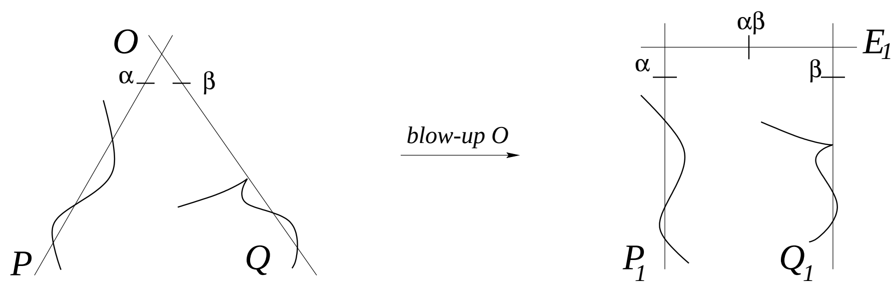

Figure 1

corresponding meridians are conjugate elements of the fundamental group, and, moreover, if such meridians $\mu_1, \ldots \mu_n$ are taken one for each irreducible component of C, then one has

$$\pi_1(X) = \pi_1(X \backslash C)/\langle\langle \mu_1, \ldots, \mu_n \rangle\rangle,$$

with $\langle\langle \mu_1, \ldots, \mu_n \rangle\rangle$ being the smallest normal subgroup of $\pi_1(X \backslash C)$ generated by $\mu_1, \ldots, \mu_n$. If a presentation of $\pi_1(X \backslash C)$ is given, then passing to the quotient above amounts to adding the relations $\mu_1 = \ldots \mu_n = 1$ to this presentation.

Theorem 1 will be obtained as a corollary of the following one:

Theorem 2 *For a curve $C \subset \mathbb{P}^2$ and a line $Q \subset \mathbb{P}^2$, let μ be a meridian of Q in $\mathbb{P}^2 \backslash C$. Then for each $n \in \mathbb{N}$, there exists a plane curve $\widetilde{C} \subset \mathbb{P}^2$ with*

$$\pi_1(\mathbb{P}^2 \backslash \widetilde{C}) = \pi_1(\mathbb{P}^2 \backslash (C \cup Q))/\langle\langle \mu^{n+1} \rangle\rangle.$$

The curve $\widetilde{C}$ is obtained as the image of C by a Cremona transformation $\mathbb{P}^2 \to \mathbb{P}^2$.

Proof. Let $O \in Q \backslash C$ be a point and take another line P passing through O. Blowing-up $\mathbb{P}^2$ at O, we get the Hirzebruch surface $\mathbb{F}_1$ (see Figure 1; the other figures show the construction we give below for $n = 1$). Denote the exceptional section of this blow-up by E_1, and the proper transforms of C, P, Q by C_1, P_1, and Q_1. Now perform an elementary transformation at the point $q_1 := Q \cap E_1$. Recall that this transformation consists in blowing-up the point q_1, followed by the contraction of Q_1 (see Figure 2). The resulting surface is the Hirzebruch surface $\mathbb{F}_2$, the proper transform E_2 of E_1 is the

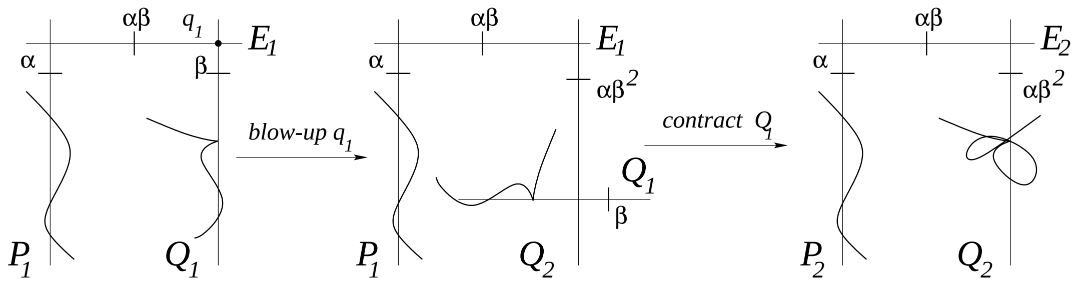

Figure 2

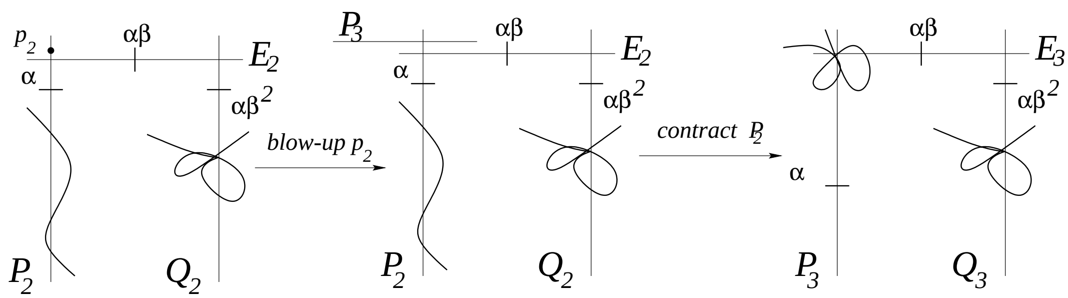

Figure 3

exceptional section of $\mathbb{F}_2$ with $E_2^2 = -2$. Denote by Q_2 the blow-up of the point q_1. Then Q_2 is a fiber of the ruling $\mathbb{F}_2 \to \mathbb{P}^1$. Let P_2 and C_2 be the proper transforms of P_1 and C_1. Applying another elementary transformation at $q_2 := Q_2 \cap E_2$, and continuing in this manner n times, we end up with the Hirzebruch surface $\mathbb{F}_{n+1}$ with $E_{n+1}, P_{n+1}, Q_{n+1}, C_{n+1} \subset \mathbb{F}_{n+1}$, the exceptional section satisfying $E_{n+1}^2 = -n-1$.

We proceed by applying an elementary transformation at a point $p_{n+1} \in P_{n+1} \backslash (E_{n+1} \cup C_{n+1})$ (see Figure 3). This increases by 1 the self-intersection number of the exceptional section, so one obtains the Hirzebruch surface $\mathbb{F}_n$. Denote by P_{n+2} the fiber replacing P_{n+1} (i.e. P_{n+2} is the blow-up of p_{n+1}), and put Q_{n+2}, E_{n+2}, C_{n+2} for the proper transform of Q_{n+1}, E_{n+1} and C_{n+1}, where $E_{n+2}^2 = -n$. Pick a point $p_{n+2} \in P_{n+2} \backslash (E_{n+2} \cup C_{n+2})$ and apply another elementary transformation at p_{n+2}. Iterating this procedure n times, we get the Hirzebruch surface $\mathbb{F}_1$ with $E_{2n+1}, P_{2n+1}, Q_{2n+1}, C_{2n+1} \subset \mathbb{F}_1$. The exceptional section satisfies $E_{2n+1}^2 = -1$, so it can be contracted, this transforms $\mathbb{F}_1$ into the projective plane $\mathbb{P}^2$. Denote the images of P_{2n+1}, Q_{2n+1}, C_{2n+1} under this contraction by $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{C}$. Notice that if P and Q are transversal to C, then in addition to the singularities of C, the curve $\widetilde{C}$ has two bad singularities, one at the point $\widetilde{P} \cap \widetilde{Q}$, the other one on the line $\widetilde{Q}$. (In general, the curve $\widetilde{C}$ may have less singular points then C, as the above process glues all the singularities of C lying on P (or Q) into one singular point.)

The composition of the birational maps transforming C to $\widetilde{C}$ yields a biholomorphism

$$\mathbb{P}^2 \backslash (C \cup P \cup Q) \xrightarrow{\simeq} \mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q}),$$

which in turn induces an isomorphism

$$\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q)) \simeq \pi_1(\mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q})).$$

Hence, the group $\pi_1(\mathbb{P}^2 \backslash \widetilde{C})$ can be recovered from $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$ by adding the relations which correspond to gluing the lines $\widetilde{P}$ and $\widetilde{Q}$.

In order to find these relations, let us first make some observations. Let $p \in C$ be a singular point of a curve $C \subset \mathbb{P}^2$, and Δ be a smooth analytical branch meeting C transversally at p. Let $* \in \mathbb{P}^2 \backslash C$ be a base point. Take a path ω joining $*$ to a boundary point of Δ, and put $\mu_p := \omega \cdot \delta \cdot \omega^{-1}$, where δ is the boundary of Δ, oriented in the positive sense. Let us call such a loop a *meridian of C at the singular point p*. It is easily seen that two meridians of C at a singular point p are conjugate elements in the group $\pi_1(\mathbb{P}^2 \backslash C)$.

Now let $\sigma_p : X \to \mathbb{P}^2$ be the blow-up of the plane at p. Denote the proper transform of C by the same letter and the exceptional section by E. As the proper transform of the branch Δ meets E transversally, at a point $q \in E \backslash C$, we have the following claim:

Claim. *The loop $\sigma_p^{-1}(\mu_p)$ in $X \backslash (C \cup E)$ is a meridian of E at a point $q \in E \backslash C$.*

Since $\pi_1(\mathbb{P}^2 \backslash C) \cong \pi_1(X \backslash (C \cup E))$, there is no confusion in denoting $\sigma_p^{-1}(\mu_p)$ by μ_p.

In virtue of the following simple observation, there is one particular case where the loop μ_p can be found easily, this is the content of the following lemma:

Lemma (Fujita [8]) *Let B be a ball centered at the origin O of $\mathbb{C}^2$, and consider the curve C defined by $x^2 - y^2 = 0$. Evidently, C has an ordinary double point at the origin, and $\pi_1(B\backslash C, *) = \mathbb{Z}^2$, where $* \in B\backslash C$ is a base point. Take meridians α, β of C on the branches $x = y$ and $x = -y$ respectively. Then $\alpha\beta$ is a meridian of C at the node O.*

Returning to the computation of the group of the curve $\widetilde{C}$, we let B be a small ball around the intersection point $O = P \cap Q$, and take meridians α of P and β of Q as in Fujita's lemma. By this lemma, $\alpha\beta$ is a meridian of E_1 in the surface $\mathbb{F}_1$, the blow-up of $\mathbb{P}^2$ at O (see Figure 1). Now we apply the Fujita lemma to the points q_i recursively, and get that $(\alpha\beta)^n\beta$ is a meridian of $\widetilde{Q}$ (see Figure 2). Other elementary transformations are performed at non-singular points of $P_i \cup C_i$, which leaves the meridian α of P_i unchanged. Hence, α is a meridian of $\widetilde{P}$. We conclude that

$$\pi_1(\mathbb{P}^2\backslash\widetilde{C}) \simeq \pi_1(\mathbb{P}^2\backslash(C \cup P \cup Q))/\langle\langle\alpha,\ \beta^{n+1}\rangle\rangle.$$

Since α is a meridian of P, passing to the quotient by the normal subgroup $\langle\langle\alpha\rangle\rangle \subset \pi_1(\mathbb{P}^2\backslash(C \cup P \cup Q))$ amounts to gluing P to $\mathbb{P}^2\backslash(C \cup P \cup Q)$. On the other hand being both meridians of Q, the elements μ and β are conjugate in the group $\pi_1(\mathbb{P}^2\backslash(C \cup Q))$, whence

$$\pi_1(\mathbb{P}^2\backslash\widetilde{C}) \simeq \pi_1(\mathbb{P}^2\backslash(C \cup Q))/\langle\langle\beta^{n+1}\rangle\rangle \simeq \pi_1(\mathbb{P}^2\backslash(C \cup Q))/\langle\langle\mu^{n+1}\rangle\rangle.$$

This proves Theorem 2. □

Proof of Theorem 1. Now suppose that the line Q intersects C transversally. Then the meridian β is a central element of the group $\pi_1(\mathbb{P}^2\backslash(C \cup Q))$. This result can be traced back to Zariski, and is proved by applying the Zariski-Van Kampen algorithm [16] to the projection $\phi : \mathbb{P}^2\backslash(C \cup Q \cup \{O\}) \to L$. Here, the point O is the center of projection with $O \in Q\backslash C$, and L is a line not passing through the point O and not contained in C. Take a fiber F of ϕ close to Q, and pick a geometric basis $B := \{b_1, \ldots, b_k\}$ for $\pi_1(F\backslash C, *)$, where $* := L \cap F$. For a meridian of Q, one can take a simple loop γ in L around the point $Q \cap L$. Because of the transversality, the monodromy around Q gives the relations $\gamma^{-1}b_i\gamma = b_i$, i.e. $[\gamma, b_i] = 1$. But $\pi_1(\mathbb{P}^2\backslash(C \cup Q))$ is generated by B, whence γ belongs to the center of this group. The meridian μ being conjugate to γ, we obtain $\gamma = \mu$.

It follows that the normal subgroup of $\pi_1(\mathbb{P}^2\backslash(C\cup Q))$ generated by μ is isomorphic to $\mathbb{Z}$, so one has the exact sequences

$$0\to\mathbb{Z}\to\pi_1(\mathbb{P}^2\backslash(C\cup Q))\to\pi_1(\mathbb{P}^2\backslash C)\to 0,$$

$$0\to(n+1)\mathbb{Z}\to\pi_1(\mathbb{P}^2\backslash(C\cup Q))\to\pi_1(\mathbb{P}^2\backslash\widetilde{C})\to 0.$$

These yield the exact sequence

$$0\to\mathbb{Z}/(n+1)\mathbb{Z}\to\pi_1(\mathbb{P}^2\backslash\widetilde{C})\to\pi_1(\mathbb{P}^2\backslash C)\to 0,$$

which proves Theorem 1. □

Remark. Curves with finite groups constructed above provide new examples of affine curves $\widetilde{C}\backslash\widetilde{Q}\subset\mathbb{P}^2\backslash\widetilde{Q}\simeq\mathbb{C}^2$, intersecting the line at infinity non-transversally, with a non-abelian, virtually abelian group, e.g. with $\mathbb{Z}$ being a finite index subgroup of $\pi_1(\mathbb{C}^2\backslash\widetilde{C})$.

Finally, Cremona transformations as in the proof of Theorem 2 can be used to obtain new Zariski pairs from the known ones as follows: Suppose that (C_1,C_2) is a Zariski pair, with $\pi_1(\mathbb{P}^2\backslash C_1)$ abelian, and $\pi_1(\mathbb{P}^2\backslash C_2)$ non-abelian (as a concrete example one can take C_1, C_2 the six-cuspidal sextics discussed by Zariski). Then an application of the Cremona transformations as in the proof of Theorem 2 produces two curves $\widetilde{C_1}$, $\widetilde{C_2}$ with the same singularities (provided that the lines P, Q have been chosen generically). Then $(\widetilde{C_1},\widetilde{C_2})$ will still be a Zariski pair, since the group of $\widetilde{C_1}$ is abelian, whereas the group of $\widetilde{C_2}$ is not. We summarize this in the following theorem.

Theorem 3 *Suppose that (C_1,C_2) is a Zariski pair, such that the group of C_1 is abelian, whereas the group of C_2 is non-abelian. If the lines P, Q are taken to be generic with respect to C_1 and C_2 then $(\widetilde{C_1},\widetilde{C_2})$ is also a Zariski pair.*

Zariski pairs of curves with many complicated singularities can be obtained by a recursive application of the above Cremona transformations.

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