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Fundamental groups of some quadric-line arrangements

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Abstract

In this paper we obtain presentations of fundamental groups of the complements of three quadric-line arrangements in $\mathbb{P}^2$. The first arrangement is a smooth quadric Q with n tangent lines to Q, and the second one is a quadric Q with n lines passing through a point $p \notin Q$. The last arrangement consists of a quadric Q with n lines passing through a point $p \in Q$.

Keywords: Fundamental groups; Complements of curve; Conic-line arrangements

1. Introduction

This is the first of a series of articles in which we shall study the fundamental groups of complements of some quadric-line arrangements. In contrast with the extensive literature on line arrangements and the fundamental groups of their complements, (see, e.g., [14,7, 15]), only a little known about the quadric-line arrangements (see [12,1,2]). The present article is dedicated to the computation of the fundamental groups of the complements of three infinite families of such arrangements. A similar analysis for the quadric-line arrangements up to degree six will be done in our next paper.

Let $C \subset \mathbb{P}^2$ be a plane curve and $* \in \mathbb{P}^2 \setminus C$ a base point. By abuse of language we will call the group $\pi_1(\mathbb{P}^2 \setminus C, *)$ the *fundamental group of C*, and we shall frequently omit base

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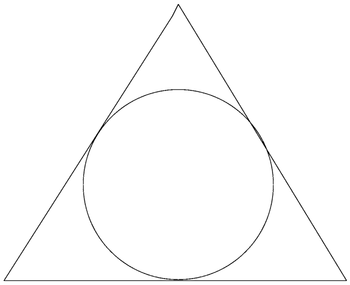

Fig. 1. The arrangement $\mathcal{A}_3$.

points and write $\pi_1(\mathbb{P}^2 \setminus C)$. One is interested in the group $\pi_1(\mathbb{P}^2 \setminus C)$ mainly for the study of the Galois coverings $X \to \mathbb{P}^2$ branched along C. Many interested surfaces have been constructed as branched Galois coverings of the plane, for example for the arrangement $\mathcal{A}_3$ in Fig. 1, there are Galois coverings $X \to \mathbb{P}^2$ branched along $\mathcal{A}_3$ such that $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, or X is an abelian surface, a K3 surface, or a quotient of the two-ball $\mathbb{B}_2$ (see [9,8,17]). Moreover, some line arrangements defined by unitary reflection groups studied in [13] are related to $\mathcal{A}_3$ via orbifold coverings. For example, if $\mathcal{L}$ is the line arrangement given by the equation

$$xyz(x+y+z)(x+y-z)(x-y+z)(x-y-z)=0$$

then the image of $\mathcal{L}$ under the branched covering map $[x:y:z] \in \mathbb{P}^2 \to [x^2:y^2:z^2] \in \mathbb{P}^2$ is the arrangement $\mathcal{A}_3$, see [17] for details.

The standard tool for fundamental group computations is the Zariski–van Kampen algorithm [19,18], see [3] for a modern approach. We use a variation of this algorithm developed in [16] for computing the fundamental groups of real line arrangements and avoids lengthy monodromy computations. The arrangements $\mathcal{B}_n$ and $\mathcal{C}_n$ discussed below are of fiber type, so presentations of their fundamental groups could be easily found as an extension of a free group by a free group. However, our approach has the advantage that it permits to capture the local fundamental groups around the singular points of these arrangements. The local fundamental groups are needed for the study of the singularities of branched of $\mathbb{P}^2$ branched along these arrangements.

In Section 2 below, we give fundamental group presentations and prove some immediate corollaries. In Section 3 we deal with the computations of fundamental group presentations given in Section 2.

2. Results

Let $C \subset \mathbb{P}^2$ be a plane curve and B an irreducible component of C. Recall that a *meridian* μ of B in $\mathbb{P}^2 \setminus C$ with the base point $* \in \mathbb{P}^2$ is a loop in $\mathbb{P}^2 \setminus C$ obtained by following a path ω with $\omega(0) = *$ and $\omega(1)$ belonging to a small neighborhood of a smooth point $p \in B \setminus C$, turning around C in the positive sense along the boundary of a small disc Δ intersecting B transversally at p, and then turning back to $*$ along ω. The meridian μ represents a homotopy class in $\pi_1(\mathbb{P}^2 \setminus C, *)$, which we also call a meridian of B. Any two meridians of B in $\mathbb{P}^2 \setminus C$ are conjugate elements of $\pi_1(\mathbb{P}^2 \setminus C)$ (see, e.g., [10, 7.5]), hence the meridians of irreducible components of C are supplementary invariants of the pair $(\mathbb{P}^2, C)$. These meridians are specified in presentations of the fundamental group given below, they will be used in orbifold-fundamental group computations in [17].

2.1. The arrangement $\mathcal{A}_n$

Theorem 1. *Let $\mathcal{A}_n := Q \cup T_1 \cup \cdots \cup T_n$ be an arrangement consisting of a smooth quadric Q with n distinct tangent lines $T_1, \ldots, T_n$. Then*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n) \simeq \left\{ \begin{array}{l|l} & \kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1},\ 2 \leqslant i \leqslant n \\ \tau_1, \ldots, \tau_n, & (\kappa_i \tau_i)^2 = (\tau_i \kappa_i)^2,\ 1 \leqslant i \leqslant n \\ \kappa_1, \ldots, \kappa_n & \left[\kappa_i^{-1} \tau_i \kappa_i, \tau_j\right] = 1,\ 1 \leqslant i < j \leqslant n \\ & \tau_n \cdots \tau_1 \kappa_1^2 = 1 \end{array} \right\} \tag{1}$$

where κ_i are meridians of Q and τ_i is a meridian of T_i for $1 \leqslant i \leqslant n$. Local fundamental groups around the singular points of $\mathcal{A}_n$ are generated by $\langle \kappa_i^{-1} \tau_i \kappa_i, \tau_j \rangle$ for the nodes $T_i \cap T_j$ and by $\langle \kappa_i, \tau_i \rangle$ for the tangent points $T_i \cap Q$.

Part (i) of the corollary below is almost trivial. Part (ii) appears in [6], and part (iii) was given in [4].

Corollary 2.

(i) *One has:* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_1) \simeq \mathbb{Z}$.

(ii) *The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ admits the presentation*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2) \simeq \langle \tau, \kappa \mid (\tau\kappa)^2 = (\kappa\tau)^2 \rangle, \tag{2}$$

where κ is a meridian of Q and τ is a meridian of T_1. A meridian of T_2 is given by $\kappa^{-2}\tau^{-1}$.

(iii) *The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ admits the presentation*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \simeq \langle \tau, \sigma, \kappa \mid (\tau\kappa)^2 = (\kappa\tau)^2,\ (\sigma\kappa)^2 = (\kappa\sigma)^2,\ [\sigma, \tau] = 1 \rangle \tag{3}$$

where σ, τ are meridians of T_1 and T_3 respectively, and κ is a meridian of Q. A meridian of T_2 is given by $(\kappa\tau\kappa\sigma)^{-1}$.

A group G is said to be *big* if it contains a non-abelian free subgroup, and *small* if G is almost solvable. In [6], it was proved by V. Lin that the group (2) is big. Below we give an alternative proof:

Proposition 3. *For $n > 1$, the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ is big.*

Proof. A group with a big quotient is big. Since τ_{n+1} is a meridian of T_{n+1} in $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{n+1})$, one has

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n) \simeq \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{n+1}) / \langle\langle \tau_{n+1} \rangle\rangle,$$

and it suffices to show that the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ is big. In the presentation (2), applying the change of generators $\alpha := \tau\kappa$, $\beta := \tau$ gives

$$\pi_1\left(\mathbb{P}^2 \setminus \mathcal{A}_2\right) \simeq \left\langle \alpha, \beta \mid \left[\alpha^2, \beta\right] = 1 \right\rangle.$$

Adding the relations $\alpha^2 = \beta^3 = 1$ to the latter presentation gives a surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2) \twoheadrightarrow \mathbb{Z}/(2) * \mathbb{Z}/(3)$. Since the commutator subgroup of $\mathbb{Z}/(2) * \mathbb{Z}/(3)$ is the free group on two generators (see [5]), we get the desired result. □

2.2. The arrangement $\mathcal{B}_n$

Theorem 4. *Let $\mathcal{B}_n := Q \cup T_1 \cup T_2 \cup L_1 \cup \cdots \cup L_n$ be an arrangement consisting of a smooth quadric Q with $n+2$ distinct lines $T_1, T_2, L_1, \ldots, L_n$ all passing through a point $p \notin Q$ such that T_1, T_2 are tangent to Q. Then one has*

$$\pi_1\left(\mathbb{P}^2 \setminus \mathcal{B}_n\right) \simeq \left\{ \tau, \kappa, \lambda_1, \ldots, \lambda_n \;\middle|\; \begin{array}{l} (\kappa\tau)^2 = (\tau\kappa)^2 \\ [\kappa, \lambda_i] = 1,\ 1 \leqslant i \leqslant n \\ \left[\tau^{-1}\kappa\tau, \lambda_i\right] = 1,\ 1 \leqslant i \leqslant n \end{array} \right\} \tag{4}$$

where τ is a meridian of T_1, λ_i is a meridian of L_i for $1 \leqslant i \leqslant n$, and κ is a meridian of Q. A meridian σ of T_2 is given by $\sigma := (\lambda_n \ldots \lambda_1 \kappa^2 \tau)^{-1}$. Local fundamental groups around the singular points of $\mathcal{B}_n$ are generated by $\langle \kappa, \lambda_i \rangle$ and $\langle \tau^{-1}\kappa\tau, \lambda_i \rangle$ for the nodes $L_i \cap Q$, by $\langle \kappa, \tau \rangle$ for the tangent point $T_1 \cap Q$, and by $\langle \kappa, \sigma \rangle$ for the tangent point $T_2 \cap Q$.

Corollary 5. (i) *Put $\mathcal{B}'_n := \mathcal{B}_n \setminus T_1$ and $\mathcal{B}''_n := \mathcal{B}'_n \setminus T_2$. Then*

$$\pi_1\left(\mathbb{P}^2 \setminus \mathcal{B}'_n\right) \simeq \pi_1\left(\mathbb{P}^2 \setminus \mathcal{B}''_{n+1}\right) \simeq \left\langle \kappa, \lambda_1, \ldots, \lambda_n \mid [\kappa, \lambda_i] = 1,\ 1 \leqslant i \leqslant n \right\rangle. \tag{5}$$

Proof. One has $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}'_n) \simeq \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)/\langle\langle \tau \rangle\rangle$. Setting $\tau = 1$ in presentation (4) gives

$$\pi_1\left(\mathbb{P}^2 \setminus \mathcal{B}'_n\right) \simeq \left\langle \kappa, \lambda_1, \ldots, \lambda_n \mid [\kappa, \lambda_i] = 1,\ 1 \leqslant i \leqslant n \right\rangle.$$

Setting $\tau = 1$ in the expression for a meridian σ of T_2 given in Theorem 4 shows that $(\lambda_n \ldots \lambda_1 \kappa^2)^{-1}$ is a meridian of T_2 in $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}'_n)$. In order to find $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}''_n)$, it suffices to set $\lambda_n \ldots \lambda_1 \kappa^2 = 1$ in the presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}'_n)$. Eliminating λ_n by this relation yields the presentation

$$\pi_1\left(\mathbb{P}^2 \setminus \mathcal{B}''_n\right) \simeq \left\langle \kappa, \lambda_1, \ldots, \lambda_{n-1} \mid [\lambda_i, \kappa] = \left[\lambda_{n-1} \ldots \lambda_1 \kappa^2, \kappa\right] = 1 \right\rangle.$$

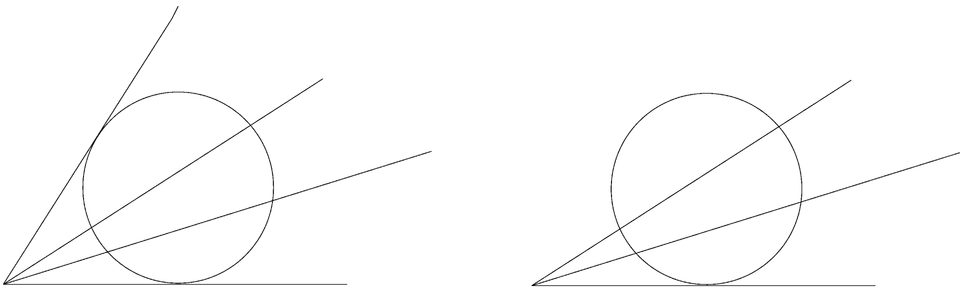

Fig. 2. Arrangements $\mathcal{B}_2$ and $\mathcal{B}'_2$.

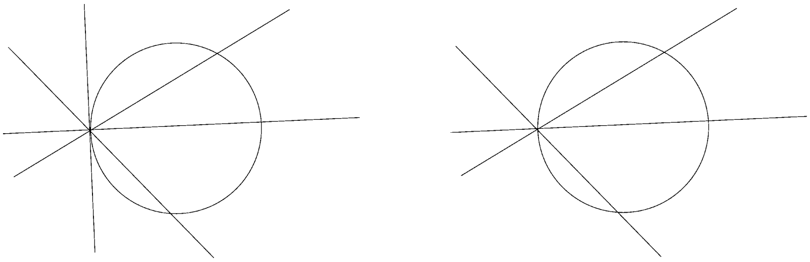

Fig. 3. Arrangements $\mathcal{C}_3$ and $\mathcal{C}'_3$.

Since the last relation above is redundant, we get the desired isomorphism $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}''_n) \simeq \pi_1(\mathbb{P}^2 \setminus \mathcal{B}'_{n+1})$. □

Note that the groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}''_i)$ are abelian for $i = 0, 1, 2$. Hence, the groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}'_i)$ are abelian for $i = 0, 1$. Otherwise, setting $\kappa = 1$ in presentation (5) gives the free group on $n - 1$ generators, which shows that these groups are big. The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ are always big, since the arrangement $\mathcal{B}_0$ is same as $\mathcal{A}_2$, and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ is big by Proposition 3.

2.3. The arrangement $\mathcal{C}_n$

Theorem 6. *Let $\mathcal{C}_n := Q \cup T \cup L_1 \cup \cdots \cup L_n$ be an arrangement consisting of a smooth quadric Q with $n + 1$ distinct lines $T, L_1, \ldots, L_n$, all passing through a point $p \in Q$ such that T is tangent to Q. Then one has*

$$\pi_1\left(\mathbb{P}^2 \setminus \mathcal{C}_n\right) \simeq \left\langle \kappa, \lambda_1, \ldots, \lambda_n \mid [\kappa, \lambda_i] = 1,\ 1 \leqslant i \leqslant n \right\rangle, \tag{6}$$

where κ is a meridian of Q and λ_i is a meridian of L_i for $1 \leqslant i \leqslant n$. A meridian τ of T is given by $\tau := (\lambda_n \ldots \lambda_1 \kappa^2)^{-1}$. Local fundamental groups around the singular points of $\mathcal{C}_n$ are generated by $\langle \kappa, \lambda_i \rangle$ for the nodes $L_i \cap Q$, and by $\langle \tau, \lambda_1, \ldots, \lambda_n, \kappa \rangle$ for the point p.

Note that the arrangement $\mathcal{C}_n$ is a degeneration (in the sense of Zariski) of the arrangement $\mathcal{B}'_n$ as the point p approaches to Q. By Zariski's "semicontinuity" theorem of the fundamental group [19] (see also [5]), there is a surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{B}'_n)$. In our case, this is also an injection:

Corollary 7.

(i) $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}'_n) \simeq \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n)$.
(ii) *Put $\mathcal{C}'_n := \mathcal{C}_n \setminus T$. Then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_n) \simeq \pi_1(\mathbb{P}^2 \setminus \mathcal{C}'_{n+1})$.*

Proof. Part (i) is obvious. The proof of part (ii) is same as the proof of Corollary 5, (ii). □

3. The arrangement $\mathcal{A}_n$

It is easily seen that any two arrangements $\mathcal{A}_n$ with fixed n are isotopic. In particular, the groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ are isomorphic. Hence one can take as a model of the arrangements $\mathcal{A}_n$ the quadric Q defined by $x^2 + y^2 = z^2$, where $[x : y : z] \in \mathbb{P}^2$ is a fixed coordinate system in $\mathbb{P}^2$. Pass to the affine coordinates in $\mathbb{C}^2 \simeq \mathbb{P}^2 \setminus \{z = 0\}$. Choose real numbers $x_1, \ldots, x_n$ such that $-1 < x_1 < x_2 < \cdots < x_n < 0$, and define y_i to be the positive solution of $x_i^2 + y_i^2 = 1$ for $1 \leqslant i \leqslant n$. Put $t_i := (x_i, y_i) \in Q$, and take T_i to be the tangent line to Q at the point t_i (see Fig. 4).

Let $\mathrm{pr}_1 : \mathbb{C}^2 \setminus \mathcal{A}_n \to \mathbb{C}$ be the first projection. The base of this projection will be denoted by B. Put $F_x := \mathrm{pr}_1^{-1}(x)$, and denote by S the set of singular fibers of pr_1. It is clear that if $F_x \in S$, then $x \in [-1, 1]$. There are three types of singular fibers:

(i) The fibers F_1 and F_{-1}, corresponding to the 'branch points' $(-1, 0)$ and $(1, 0)$.
(ii) The fibers F_{x_i} $(1 \leqslant i \leqslant n)$ corresponding to the 'tangent points' $t_i = (x_i, y_i) = T_i \cap Q$.
(iii) The fibers $F_{a_{i,j}}$ $(1 \leqslant i \neq j \leqslant n)$ corresponding to the nodes $n_{i,j} = (a_{i,j}, b_{i,j}) := T_i \cap T_j$. One can arrange the lines T_i such that

$$-1 < x_1 < a_{1,2} < a_{1,3} < \cdots < a_{1,n} < x_2 < a_{2,3} < \cdots < x_n < 1.$$

Identify the base B of the projection pr_1 with the line $y = -2 \subset \mathbb{C}^2$. Let N be the number of singular fibers and let $-1 = s_1 < s_2 < \cdots < s_{N-1} < s_N = 1$ be the elements of $S \cap B$ (so that $s_2 = x_1$, $s_3 = a_{1,2}$, $s_4 = a_{1,3}$, and so on). In B, take small discs Δ_i around the points s_i, and denote by c_i, d_i $(c_i < d_i)$ the points $\partial\Delta_i \cap \mathbb{R}$ for $1 \leqslant i \leqslant N$ (see Fig. 5).

Put $B_1 := [c_1, c_2] \cup \Delta_1$ and for $2 \leqslant i \leqslant N$ let $B_i := [c_1, c_{i+1}] \cup \Delta_1 \cup \cdots \cup \Delta_i$. Let $X_i := \mathrm{pr}^{-1}(B_i)$ be the restriction of the fibration pr to B_i. Let

$$A_i := \Delta_i \cup \partial\bigl(\bigl\{\Im(z) \leqslant 0,\ c_2 \leqslant \Re(z) \leqslant c_i\bigr\} \setminus (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{i-1})\bigr)$$

and let $Y_i := \mathrm{pr}^{-1}(A_i)$ be the restriction of the fibration pr to A_i (see Fig. 6).

Clearly, $X_i = X_{i-1} \cup Y_i$ for $2 \leqslant i \leqslant N$. We will use this fact to compute the groups $\pi_1(X_i, *)$ recursively, where $* := (c_2, -2)$ is the base point. For details of the algorithm we apply below, see [16].

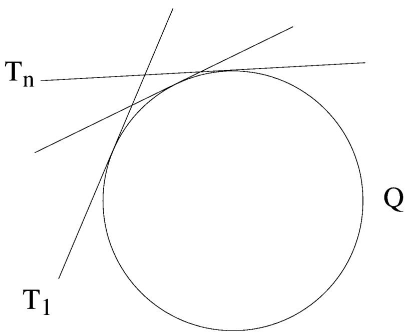

Fig. 4.

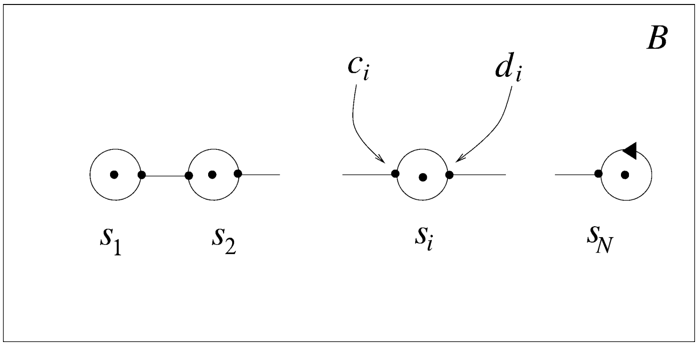

Fig. 5. The base B.

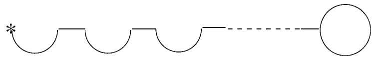

Fig. 6. The space A_i.

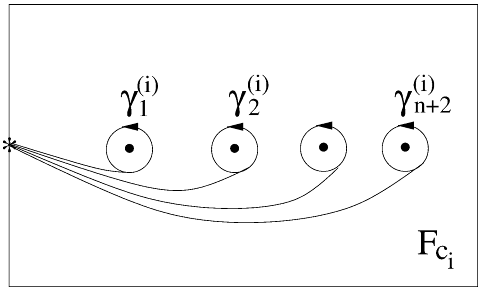

Fig. 7.

Identify the fibers of pr_1 with F_0 via the second projection $\mathrm{pr}_2 : (x, y) \in \mathbb{C}^2 \to y \in \mathbb{C}$. In *each one* of the fibers F_{c_i} (respectively F_{d_i}) take a basis for $\pi_1(F_{c_i}, -2)$ (respectively for $\pi_1(F_{d_i}, -2)$) as in Fig. 7 (for F_{d_i}, just replace γ's by θ's in Fig. 7). We shall denote these basis by the vectors $\Gamma_i := [\gamma_1^{(i)}, \ldots, \gamma_{n+2}^{(i)}]$ (respectively $\Theta_i := [\theta_1^{(i)}, \ldots, \theta_{n+2}^{(i)}]$).

Let $\nu_i \subset B_i \subset B$ be a path starting at $\nu_i(0) = c_2$, ending at $\nu_i(1) = c_i$ and such that

$$\nu_i\bigl([0,1]\bigr) = \partial\bigl(\bigl\{\Im(z) \leqslant 0,\ c_2 \leqslant \Re(z) \leqslant c_i\bigr\} \setminus (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{i-1})\bigr).$$

Similarly, let $\eta_i \subset B_i \subset B$ be a path starting at $\eta(0) = c_2$, ending at $\eta(0) = d_i$ and such that

$$\eta_i\bigl([0,1]\bigr) = \partial\bigl(\bigl\{\Im(z) \leqslant 0,\ c_2 \leqslant \Re(z) \leqslant d_i\bigr\} \setminus (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_i)\bigr).$$

For $2 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant n+2$ each loop $\tilde{\gamma}_j^{(i)} := \nu_i \cdot \gamma_j^{(i)} \cdot \nu_i^{-1}$ represents a homotopy class in $\pi_1(X_i, *)$, where $* := (c_2, -2)$ is the base point. Similarly, each loop $\tilde{\theta}_j^{(i)} := \eta_i \cdot \theta_i \cdot \eta_i^{-1}$ represents a homotopy class in $\pi_1(X_i, *)$. Denote $\widetilde{\Gamma}_i := [\tilde{\gamma}_1^{(i)}, \ldots, \tilde{\gamma}_{n+2}^{(i)}]$, and $\widetilde{\Theta}_i := [\tilde{\theta}_1^{(i)}, \ldots, \tilde{\theta}_{n+2}^{(i)}]$.

It is well known that the group $\pi_1(Y_i, *)$ has the presentation

$$\langle \tilde{\gamma}_1^{(i)}, \dots, \tilde{\gamma}_{n+2} \mid \tilde{\gamma}_j^{(i)} = M_i(\tilde{\gamma}_j^{(i)}),\ 1 \leqslant j \leqslant n+2 \rangle \tag{7}$$

where $M_i : \pi_1(F_{c_i}, -2) \to \pi_1(F_{c_i}, -2)$ is the monodromy operator around the singular fiber above s_i. It is also well known that if it is the branches of $\mathcal{A}_n$ corresponding to the loops $\tilde{\gamma}_k^{(i)}$ and $\tilde{\gamma}_{k+1}^{(i)}$ that meet above s_i, then the only non-trivial relation in (7) is $\tilde{\gamma}_k^{(i)} = \tilde{\gamma}_{k+1}^{(i)}$ in case of a branch point, $[\tilde{\gamma}_k^{(i)}, \tilde{\gamma}_{k+1}^{(i)}] = 1$ in case of a node, and $(\tilde{\gamma}_k^{(i)}\tilde{\gamma}_{k+1}^{(i)})^2 = (\gamma_k^{(i)}\gamma_{k+1}^{(i)})^2$ in case of a tangent point.

Now suppose that the group $\pi_1(X_{i-1}, *)$ is known, with generators $\widetilde{\Gamma}_2$. Recall that $X_i = X_{i-1} \cup Y_i$. In order to find the group $\pi_1(X_i, *)$, one has to express the base $\widetilde{\Gamma}_i$ in terms of the base $\widetilde{\Gamma}_i$. Adding to the presentation of $\pi_1(X_{i-1})$ the relation obtained by writing the relation of $\pi_1(Y_i)$ in the new base then yields a presentation of $\pi_1(X_i)$. Note that, since the space Y_i is eventually glued to X_{i-1}, it suffices to find an expression of $\widetilde{\Gamma}_i$ in terms of the base $\widetilde{\Gamma}_2$ *in* the group $\pi_1(X_{i-1}, *)$.

Since all the points of $\mathcal{A}_n$ above the interval $[d_{i-1}, c_i]$ are smooth and real, one has

Fact. *The loops $\tilde{\theta}_j^{(i-1)}$ and $\tilde{\gamma}_j^{(i)}$ are homotopic in X_i (or in Y_i) for $2 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant n+2$. In other words, the bases $\widetilde{\Theta}_{i-1}$ and $\widetilde{\Gamma}_i$ are homotopic.*

In order to express the base $\widetilde{\Theta}_i$ in terms of the base $\widetilde{\Gamma}_i$ the following lemma will be helpful.

Lemma 8. *Let C_k: $x^2 - y^{k+1} = 0$ be an A_k singularity, where $k = 1$ or $k = 3$. Put $D := \{(x, y)\colon |x| \leqslant 1, |y| \leqslant 1\}$ and let $\mathrm{pr}_1 := (x, y) \in D \setminus C_k \to (x, -1)$ be the first projection. Denote by F_x the fiber of pr_1 above $(x, -1)$. Identify the fibers of pr_1 via the second projection. Let $-1 < c < 0$ be a real number, and put $d := -c$. In F_c (respectively in F_d) take a basis $\Gamma := [\gamma_1, \gamma_2]$ for $\pi_1(F_c, -1)$ (respectively a basis $\Theta := [\theta_1, \theta_2]$ for $\pi_1(F_d, -1)$) as in Fig. 8. Let η be the path $\eta(t) := c\mathrm{e}^{\pi \mathrm{i}t}$, and put $\tilde{\theta}_i := \eta \cdot \theta \cdot \eta^{-1}$ for $i = 1, 2$. Then $\gamma_i, \tilde{\theta}_i$ are loops in $D \setminus C_k$ based at $* := (c, -1)$, and one has*

(i) *If $k = 1$, then $\tilde{\theta}_1$ is homotopic to γ_2, and $\tilde{\theta}_2$ is homotopic to γ_1, in other words, $\widetilde{\Theta} = [\gamma_2, \gamma_1]$.*
(ii) *If $k = 3$, then $\widetilde{\Theta} = [\gamma_2\gamma_1\gamma_2^{-1}, \gamma_1^{-1}\gamma_2\gamma_1]$.*

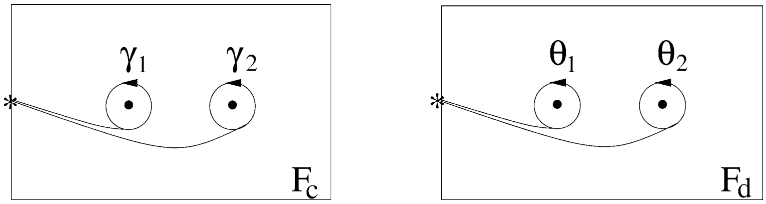

Fig. 8.

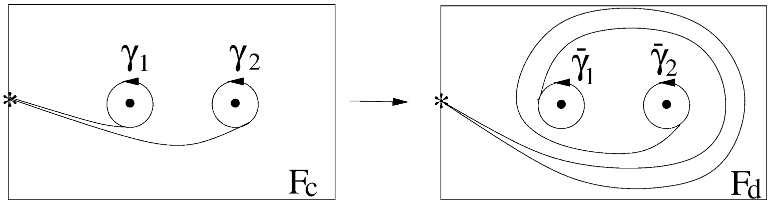

Fig. 9.

Proof. Since $\pi_1(D \setminus C_2)$ is abelian, part (i) is obvious. For part (ii), note that the points of intersection $F_{\eta(t)} \cap C_4$ are $y_1 := c^2 e^{2\pi i t}$ and $y_2 := -c^2 e^{2\pi i t}$. Hence, when we move the fiber F_c over F_d along the path η, y_1 and y_2 make one complete turn around the origin in the positive sense. The loops γ_1, γ_2 are transformed to loops $\overline{\gamma}_1, \overline{\gamma}_2 \subset F_d$ as in Fig. 9. It follows that the loop $\eta \cdot \overline{\gamma}_i \cdot \eta^{-1}$ is homotopic to γ_i for $i = 1, 2$. This homotopy can be constructed explicitly as follows: Let $\Phi_{\eta(t)} : F_c \to F_{\eta(t)}$ be the corresponding Leftschez homeomorphism (see [11]). Then

$$H(s,t) := \begin{cases} \eta(3s), & 0 \leqslant s \leqslant t/3, \\ \Phi_{\eta(t)}\big(\gamma_i\big(3(s - t/3)/(3 - 2t)\big)\big), & t/3 \leqslant s \leqslant 1 - t/3, \\ \eta\big(3(1-s)\big), & 1 - t/3 \leqslant s \leqslant 1 \end{cases}$$

gives a homotopy between γ_i and $\overline{\gamma}_i$. Expressing $\tilde{\theta}_i$ in terms of $\overline{\gamma}_i$, we get

$$\tilde{\theta}_1 = \overline{\gamma}_1^{-1}\overline{\gamma}_2^{-1}\overline{\gamma}_1\overline{\gamma}_2\overline{\gamma}_1 = \gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2\gamma_1,$$
$$\tilde{\theta}_2 = \overline{\gamma}_1^{-1}\overline{\gamma}_2\overline{\gamma}_1 = \gamma_1^{-1}\gamma_2\gamma_1.$$

Since from the monodromy one has the relation $(\gamma_1\gamma_2)^2 = (\gamma_2\gamma_1)^2$, the expression for $\tilde{\theta}_1$ can be simplified to get $\tilde{\theta}_1 = \gamma_2\gamma_1\gamma_2^{-1}$. □

Now we proceed with the computation of the groups $\pi_1(X_i)$. Clearly, the group $\pi_1(X_2)$ is generated by the base

$$\widetilde{\Gamma}_2 = \big[\gamma_1^{(2)}, \gamma_2^{(2)}, \ldots, \gamma_{n+2}^{(2)}\big]$$

with the only relations

$$\gamma_1^{(2)} = \gamma_2^{(2)} \tag{8}$$

and

$$\big(\gamma_2^{(2)}\gamma_3^{(2)}\big)^2 = \big(\gamma_3^{(2)}\gamma_2^{(2)}\big)^2. \tag{9}$$

Put

$$[\kappa_1, \kappa_1, \tau_1, \ldots, \tau_n] := \Gamma_2.$$

Then relation (9) becomes

$$(\kappa_1\tau_1)^2 = (\tau_1\kappa_1)^2. \tag{10}$$

By Lemma 8 and the above Fact, one has

$$\widetilde{\Gamma}_3 = \widetilde{\Theta}_2 = \big[\kappa_1, \tau_1\kappa_1\tau_1^{-1}, \kappa_1^{-1}\tau_1\kappa_1, \tau_2, \ldots, \tau_n\big].$$

Since s_3 corresponds to the node $T_1 \cap T_2$, the next relation is

$$\big[\kappa_1^{-1}\tau_1\kappa_1, t_2\big] = 1. \tag{11}$$

Hence,

$$\pi_1(X_3, *) \simeq \big\langle \kappa_1, \tau_1, \ldots, \tau_n \mid (\kappa_1\tau_1)^2 = (\tau_1\kappa_1)^2,\ \big[\kappa_1^{-1}\tau_1\kappa_1, \tau_2\big] = 1\big\rangle.$$

By Lemma 8, one has

$$\widetilde{\Gamma}_4 = \widetilde{\Theta}_3 = \big[\kappa_1, \tau_1\kappa_1\tau_1^{-1}, \tau_2, \kappa_1^{-1}\tau_1\kappa_1, \tau_3, \ldots, \tau_n\big].$$

Since s_4 corresponds to the node $T_1 \cap T_3$, one has the relation

$$\big[\kappa_1^{-1}\tau_1\kappa_1, \tau_3\big] = 1.$$

Hence,

$$\pi_1(X_4, *) \simeq \big\langle \kappa_1, \tau_1, \ldots, \tau_n \mid (\kappa_1\tau_1)^2 = (\tau_1\kappa_1)^2, \\ \big[\kappa_1^{-1}\tau_1\kappa_1, \tau_2\big] = \big[\kappa_1^{-1}\tau_1\kappa_1, \tau_3\big] = 1\big\rangle.$$

By Lemma 8, one has

$$\widetilde{\Gamma}_5 = \widetilde{\Theta}_4 = \big[\kappa_1, \tau_1\kappa_1\tau_1^{-1}, \tau_2, \tau_3, \kappa_1^{-1}\tau_1\kappa_1, \tau_4, \ldots, \tau_n\big].$$

Since s_k corresponds to the node $T_1 \cap T_{k-1}$ for $2 \leqslant k \leqslant n+1$, repeating the above procedure gives the presentation

$$\pi_1(X_{n+1}, *) \simeq \big\langle \kappa_1, \tau_1, \ldots, \tau_n \mid (\kappa_1\tau_1)^2 = (\tau_1\kappa_1)^2,\ \big[\kappa_1^{-1}\tau_1\kappa_1, \tau_k\big] = 1,\ 2 \leqslant k \leqslant n\big\rangle$$

and

$$\widetilde{\Gamma}_{n+2} = \widetilde{\Theta}_{n+1} = \big[\kappa_1, \kappa_2, \tau_2, \tau_3, \ldots, \tau_n, \kappa_1^{-1}\tau_1\kappa_1\big],$$

where we put $\kappa_{i+1} := \tau_i\kappa_i\tau_i^{-1}$ for $1 \leqslant i \leqslant n-1$.

The next point s_{n+2} corresponds to the tangent point $T_2 \cap Q$. This gives the relation

$$(\kappa_2\tau_2)^2 = (\tau_2\kappa_2)^2 \tag{12}$$

and

$$\widetilde{\Gamma}_{n+2} = \widetilde{\Theta}_{n+1} = \big[\kappa_1, \tau_2\kappa_2\tau_2^{-1}, \kappa_2^{-1}\tau_2\kappa_2, \tau_3, \ldots, \tau_n, \kappa_1^{-1}\tau_1\kappa_1\big].$$

Now comes the $n-2$ points s_k corresponding to the nodes $T_2 \cap T_{k-n}$ for $n+3 \leqslant 2n+1$. These give the relations

$$\big[\kappa_2^{-1}\tau_2\kappa_2, \tau_k\big] = 1, \quad 3 \leqslant k \leqslant n.$$

Hence, one has

$$\pi_1(X_{n+1}, *) \simeq \big\langle \kappa_1, \kappa_2, \tau_1, \ldots, \tau_n \mid \kappa_2 = \tau_1\kappa_1\tau_1^{-1},\ (\kappa_i\tau_i)^2 = (\tau_i\kappa_i)^2, \\ \big[\kappa_i^{-1}\tau_i\kappa_i, \tau_k\big] = 1,\ i < k \leqslant n,\ i = 1, 2\big\rangle.$$

We proceed in this manner until the last singular fiber s_N. Since this is a branch point, the final relation is

$$\kappa_n = \kappa_1. \tag{13}$$

This gives the presentation

$$\pi_1(X_N,*) \simeq \left\{ \tau_i, \kappa_i,\ 1 \leqslant i \leqslant n \;\middle|\; \begin{array}{l} \kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1},\ 2 \leqslant i \leqslant n \\ (\kappa_i \tau_i)^2 = (\tau_i \kappa_i)^2,\ 1 \leqslant i \leqslant n \\ \left[\kappa_i^{-1} \tau_i \kappa_i, \tau_j\right] = 1,\ 1 \leqslant i < j \leqslant n \\ \kappa_1 = \kappa_n \end{array} \right\}. \tag{14}$$

Adding to this presentation of $\pi_1(X_N,*)$ the projective relation $\tau_n \cdots \tau_1 \kappa_1^2 = 1$ gives the presentation

$$\pi_1\bigl(\mathbb{P}^2 \setminus \mathcal{A}_n\bigr) \simeq \left\{ \tau_i, \kappa_i,\ 1 \leqslant i \leqslant n \;\middle|\; \begin{array}{l} \kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1},\ 2 \leqslant i \leqslant n \\ (\kappa_i \tau_i)^2 = (\tau_i \kappa_i)^2,\ 1 \leqslant i \leqslant n \\ \left[\kappa_i^{-1} \tau_i \kappa_i, \tau_j\right] = 1,\ 1 \leqslant i < j \leqslant n \\ \tau_n \ldots \tau_1 \kappa_1^2 = 1,\ \kappa_1 = \kappa_n \end{array} \right\}. \tag{15}$$

Note that the relation $\kappa_1 = \kappa_n$ is redundant. Indeed, since $\kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1}$, one has

$$\kappa_n = (\tau_n \ldots \tau_1)\kappa_1(\tau_n \ldots \tau_1)^{-1}. \tag{16}$$

But $\tau_n \ldots \tau_1 = \kappa^{-2}$ by the projective relation. Substituting this in (16) yields the relation $\kappa_1 = \kappa_n$. This finally gives the presentation (1) and proves Theorem 1. Claims regarding the local fundamental groups around the singular points of $\mathcal{A}_n$ are direct consequences of the above algorithm.

3.1. *Proof of Corollary 2*

(i) *The arrangement* $\mathcal{A}_1$. Writing down the presentation (1) explicitly for $n = 1$ gives

$$\pi_1\bigl(\mathbb{P}^2 \setminus \mathcal{A}_1\bigr) \simeq \left\{ \kappa_1, \tau_1 \;\middle|\; \begin{array}{l} (\kappa_1 \tau_1)^2 = (\kappa_1 \tau_1)^2 \\ t_1 \kappa_1^2 = 1 \end{array} \right\}.$$

Eliminating τ_1 from the last relation shows that $\pi_1(\mathbb{P}^2 \setminus A_1) \simeq \mathbb{Z}$.

(ii) *The arrangement* $\mathcal{A}_2$. Writing down the presentation (1) explicitly for $n = 2$ gives

$$\pi_1\bigl(\mathbb{P}^2 \setminus \mathcal{A}_2\bigr) \simeq \left\{ \kappa_1, \kappa_2, \tau_1, \tau_2 \;\middle|\; \begin{array}{l} (1)\ \kappa_2 = \tau_1 \kappa_1 \tau_1^{-1} \\ (2)\ (\kappa_1 \tau_1)^2 = (\tau_1 \kappa_1)^2 \\ (3)\ (\kappa_2 \tau_2)^2 = (\tau_2 \kappa_2)^2 \\ (4)\ \left[\kappa_1^{-1} t_1 \kappa_1, t_2\right] = 1 \\ (5)\ \tau_2 \tau_1 \kappa_1^2 = 1 \end{array} \right\}.$$

Eliminating κ_2 by (1) and τ_2 by (5) one easily shows that the relations (3) and (4) are redundant. This leaves (2) and gives the desired presentation.

(iii) *The arrangement* $\mathcal{A}_3$. Writing down the presentation (1) explicitly for $n = 3$ gives

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \simeq \left\{ \begin{array}{c|l} \kappa_1, \kappa_2, \kappa_3, & (1)\ \kappa_2 = \tau_1\kappa_1\tau_1^{-1} \quad (2)\ \kappa_3 = (\tau_2\tau_1)\kappa_1(\tau_2\tau_1)^{-1} \\ \tau_1, \tau_2, \tau_3 & (3)\ (\kappa_1\tau_1)^2 = (\tau_1\kappa_1)^2 \quad (4)\ (\kappa_2\tau_2)^2 = (\tau_2\kappa_2)^2 \\ & (5)\ (\kappa_3\tau_3)^2 = (\tau_3\kappa_3)^2 \quad (6)\ [\kappa_1^{-1}\tau_1\kappa_1, \tau_2] = 1 \\ & (7)\ [\kappa_1^{-1}\tau_1\kappa_1, \tau_3] = 1 \quad (8)\ [\kappa_2^{-1}\tau_2\kappa_2, \tau_3] = 1 \\ & (9)\ \tau_3\tau_2\tau_1\kappa_1^2 = 1 \end{array} \right\}.$$

Eliminate κ_2 by (1), κ_3 by (2), and τ_2 by (9). It can be shown that the relations (4), (6) and (8) are consequences of the remaining relations. The relation (5) becomes $(\kappa_1\tau_3)^2 = (\tau_3\kappa_1)^2$. This gives the presentation

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \simeq \langle \kappa_1, \tau_1, \tau_3 \mid (\kappa_1\tau_1)^2 = (\tau_1\kappa_1)^2, \\ (\kappa_1\tau_3)^2 = (\tau_3\kappa_1)^2,\ [\kappa_1^{-1}\tau_1\kappa_1, \tau_3] = 1 \rangle.$$

Finally, put $\kappa := \kappa_1$, $\tau := \kappa_1^{-1}\tau_1\kappa_1$ and $\sigma := \tau_3$. Then $\tau_1 = \kappa\tau\kappa^{-1}$, and the first relation in the above presentation becomes $(\kappa^2\tau\kappa^{-1})^2 = (\kappa\tau)^2 \Longrightarrow (\kappa\tau)^2 = (\tau\kappa)^2$. This gives the desired presentation.

4. The arrangement $\mathcal{B}_n$

As in the case of the arrangements $\mathcal{A}_n$, it is readily seen that arrangements $\mathcal{B}_n$ are all isotopic to each other for fixed n, so one can compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ from the following model for $\mathcal{B}_n$'s (see Fig. 10): The quadric Q is given by the equation $x^2 + y^2 = 1$, and p is the point $(2, 0)$. The lines L_i intersect Q above the x-axis.

The projection to the x-axis has four types of singular fibers:

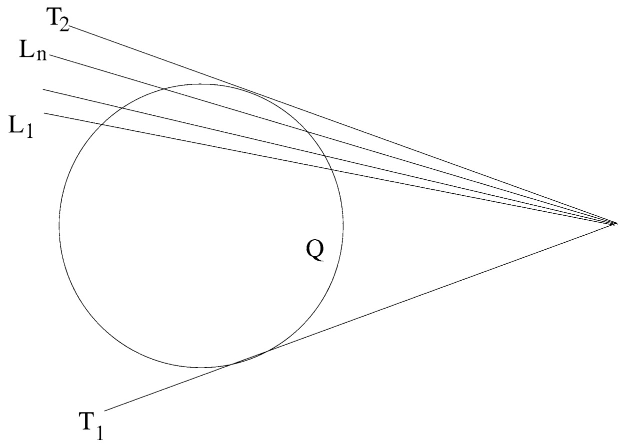

Fig. 10.

(i) The fibers F_1 and F_{-1}, corresponding to 'branch points'.
(ii) The fiber F_x corresponding to the 'tangent points' $(x, y) = t_1 := T_1 \cap Q$ and $(x, -y) = t_2 := T_2 \cap Q$.
(iii) The fibers $F_{a_1}, \dots, F_{a_n}$ $(-1 < a_n < \cdots < a_1 < x)$ corresponding to the nodes $L_i \cap Q$ lying on the right of the tangent points and the fibers $F_{b_1}, \dots, F_{b_n}$ $(x < b_1 < \cdots < b_n < 1)$ corresponding to the nodes $L_i \cap Q$ lying on the left of the tangent points.
(iv) The fiber F_2, corresponding to the point p.

In order to find the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$, we shall apply the same procedure as in the computation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$. Let $y \in \mathbb{R}$ be such that $-1 < y < a_n$, and take F_y to be the base fiber. Let $s_1 := -1$, $s_{i+1} := b_i$ for $1 \leqslant i \leqslant n$, $s_{n+2} := x$, $s_{n+2+i} := a_{n+1-i}$ for $1 \leqslant i \leqslant n$, and $s_{2n+3} := 1$, and $s_{2n+4} = 2$. Take a basis

$$\widetilde{\Gamma}_2 := [\tau, \kappa_1, \kappa_2, \lambda_1, \dots, \lambda_n, \sigma]$$

for F_y as in Fig. 7.

Since s_1 corresponds to a branch point, one has the relation $\kappa_1 = \kappa_2$. Put $\kappa := \kappa_1 = \kappa_2$. The point s_2 is a node, and yields the relation $[\kappa, \lambda_1] = 1$, and one has

$$\widetilde{\Gamma}_3 = [\tau, \kappa, \lambda_1, \kappa, \lambda_2, \dots, \lambda_n, \sigma].$$

Repeating this for the nodes $s_3, \dots, s_{n+1}$ gives the relations $[\kappa, \lambda_i] = 1$ for $1 \leqslant i \leqslant n$, and one has

$$\widetilde{\Gamma}_{n+2} = [\tau, \kappa, \lambda_1, \dots, \lambda_n, \kappa, \sigma].$$

The monodromy around the fiber F_x gives the relations $(\tau\kappa)^2 = (\kappa\tau)^2$ and $(\sigma\kappa)^2 = (\kappa\sigma)^2$. One has

$$\widetilde{\Gamma}_{n+3} = \left[\kappa\tau\kappa^{-1}, \tau^{-1}\kappa\tau, \lambda_1, \dots, \lambda_n, \sigma\kappa\sigma^{-1}, \kappa^{-1}\sigma\kappa\right].$$

Since the points $s_{n+3}, \dots, s_{2n+2}$ corresponds to nodes, one has the relations $[\lambda_i, \sigma\kappa\sigma^{-1}] = 1$, and

$$\widetilde{\Gamma}_{2n+3} = \left[\kappa\tau\kappa^{-1}, \tau^{-1}\kappa\tau, \sigma\kappa\sigma^{-1}, \lambda_1, \dots, \lambda_n, \kappa^{-1}\sigma\kappa\right].$$

The branch point corresponding to s_{2n+3} yields the relation

$$\tau^{-1}\kappa\tau = \sigma\kappa\sigma^{-1}.$$

Together with the projective relation $\sigma\lambda_n \dots \lambda_1\kappa^2\tau = 1$ these relations already gives a presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$, since one can always ignore one of the singular fibers when computing the monodromy (see [16]).

We obtained the presentation

$$\pi_1\left(\mathbb{P}^2 \setminus \mathcal{B}_n\right) \simeq \left\{ \Lambda, \tau, \kappa, \lambda_1, \dots, \lambda_n, \sigma \;\middle|\; \begin{array}{l} (1)\ (\kappa\tau)^2 = (\tau\kappa)^2 \\ (2)\ (\kappa\sigma)^2 = (\sigma\kappa)^2 \\ (3)\ \tau^{-1}\kappa\tau = \sigma\kappa\sigma^{-1} \\ (4)\ [\kappa, \lambda_i] = 1 \quad 1 \leqslant i \leqslant n \\ (5)\ \left[\sigma\kappa\sigma^{-1}, \lambda_i\right] = 1 \quad 1 \leqslant i \leqslant n \\ (6)\ \sigma\lambda_n \dots \lambda_1\kappa^2\tau = 1 \end{array} \right\}.$$

Put $\Lambda := \lambda_n \ldots \lambda_1$. Eliminating σ by (7), it is easily seen that (3) is redundant. Relation (2) becomes

$$\big(\Lambda\kappa^2\tau\kappa^{-1}\big)^2 = \big(\kappa^{-1}\Lambda\kappa^2\tau\big)^2 \implies \big[\tau^{-1}\kappa\tau, \Lambda\big] = 1.$$

But this relation is a consequence of (4), so that (2) is also redundant. Since $\tau^{-1}\kappa\tau = \sigma\kappa\sigma^{-1}$ by (3), the relation (5) can be written as $[\tau^{-1}\kappa\tau, \lambda_i] = 1$. This gives the presentation (4) and proves Theorem 4.

5. The arrangement $\mathcal{C}_n$

In order to compute the group, consider the model of $\mathcal{C}_n$ shown in Fig. 11, where Q is given by $x^2 + y^2 = 1$. Suppose that the second points of intersection of the lines L_i with Q lie above the x-axis. As in the previous cases, take an initial base

$$\widetilde{\Gamma}_2 := [\kappa_1, \kappa_2, \lambda_1, \ldots, \lambda_n, \tau].$$

The relation induced by the branch point is $\kappa_1 = \kappa_2 =: \kappa$. The nodes of $\mathcal{C}_n$ will give the relations $[\kappa, \lambda_i] = 1$ for $1 \leqslant i \leqslant n$, and one has

$$\widetilde{\Gamma}_{n+2} = [\kappa, \lambda_1, \ldots, \lambda_n, \kappa, \tau].$$

One can simplify the computation of the monodromy around the complicated singular fiber as follows: Put $\Lambda := \lambda_n \ldots \lambda_1$. By the projective relation one has $\tau\Lambda\kappa^2 = 1 \Longrightarrow \tau = \kappa^{-2}\Lambda^{-1}$. Hence, $[\kappa, \tau] = 1$. Since we also have $[\kappa, \lambda_i] = 1$ for $1 \leqslant i \leqslant n$, this means that when computing the monodromy around this fiber, one can ignore the branch Q. This leaves $n + 1$ branches intersecting transversally, and the induced relation is (see [16])

$$[\tau\Lambda, \lambda_i] = [\tau\Lambda, \tau] = 1, \tag{17}$$

and one has

$$\widetilde{\Gamma}_{n+3} = [\kappa, \kappa, \ldots].$$

The last relation induced by the branch point yields the trivial relation $\kappa = \kappa$, as expected.

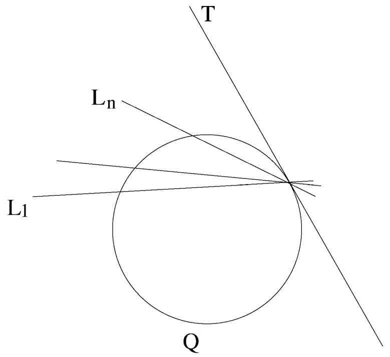

Fig. 11.

Eliminating τ shows that the relations (17) are redundant and gives the presentation

$$\pi_1(\mathbb{P}^2 \setminus C_n) \simeq \langle \kappa, \lambda_1, \ldots, \lambda_n \mid [\lambda_i, \kappa] = 1 \rangle. \qquad \square$$

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References

[1] M. Amram, Braid group and braid monodromy, M.Sc. Thesis, 1995.
[2] M. Amram, M. Teicher, Braid monodromy of special curves, J. Knot Theory Ramifications 10 (2) (2001) 171–212.
[3] D. Cheniot, Topologie du complémentaire d'un ensemble algébrique projectif, Enseign. Math., II. Ser. 37 (3/4) (1991) 293–402.
[4] A. Degtyarev, Quintics in $\mathbb{C}P^2$ with non-abelian fundamental group, St. Petersburg Math. J. 11 (5) (2000) 809–826.
[5] A. Dimca, Singularities and the Topology of Hypersurface Complements, in: Universitext, Springer-Verlag, New York, 1992.
[6] G. Detloff, S. Orevkov, M. Zaidenberg, Plane curves with a big fundamental group of the complement, Amer. Math. Soc. Transl. Ser. 2 184 (1998).
[7] D. Graber, M. Teicher, The fundamental group's structure of the complement of some configurations of real line arrangements, in: Complex Analysis and Algebraic Geometry. A Volume in Memory of Michael Schneider, de Gruyter, Berlin, 2000, pp. 173–223.
[8] R.P. Holzapfel, N. Vladov, Quadric-line configurations degenerating plane Picard–Einstein metrics I–II, in: Proceedings to 60th Birthday of H. Kurke, Math. Ges., Berlin, 2000, to appear.
[9] J. Kaneko, S. Tokunaga, M. Yoshida, Complex crystallographic groups II, J. Math. Soc. Japan 34 (4) (1982).
[10] K. Lamotke, The topology of complex projective varieties after S. Leftschez, Topology 20 (1979) 15–51.
[11] B. Moishezon, M. Teicher, Braid group technique in complex geometry I, Line arrangements in $\mathbb{CP}^2$, Contemp. Math. 78 (1988) 425–555.
[12] B. Moishezon, M. Teicher, Braid group technique in complex geometry II, From arrangements of lines and conics to cuspidal curves, in: Algebraic Geometry, in: Lecture Notes in Math., Vol. 1479, Springer-Verlag, Berlin, 1990.
[13] P. Orlik, L. Salomon, Arrangements defined by unitary reflection groups, Math. Ann. 261 (1982) 339–357.
[14] P. Orlik, H. Terao, Arrangements of Hyperplanes, in: Grundlagen Mat. Wiss., Vol. 300, Springer-Verlag, Berlin, 1992.
[15] A. Suciu, Fundamental groups of line arrangements: Enumerative aspects, Contemp. Math. 276 (2001) 43–79.
[16] A.M. Uludağ, On finite smooth uniformizations of the plane along line arrangements, to be published.
[17] A.M. Uludağ, Covering relations between ball quotient orbifolds, to be published.
[18] E.R. van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math. 55 (1933) 255–260.
[19] O. Zariski, On the Poincare group of rational plane curves, Amer. J. Math. 58 (1936) 607–619.