On Fenchel's Problem for the Projective Plane

A. Muhammed Uludağ

Abstract. We consider the problem of existence of finite branched Galois coverings of $\mathbb{P}^2$. Let $p, q > 1$ be two integers and $C \subset \mathbb{P}^2$ be an irreducible curve. We prove that if there is a surjection $\pi_1(\mathbb{P}^2 \backslash C) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, then there is a finite Galois covering of $\mathbb{P}^2$ branched along C with any given branching index.

Introduction. Let M be a complex manifold, and $C_1, C_2, \cdots C_n \subset M$ be irreducible hypersurfaces. A morphism $X \to M$ is said to be a *Galois covering of M branched at the divisor* $D := r_1 C_1 + r_2 C_2 + \cdots + r_k C_k$ if it is a Galois covering of $M \backslash (C_1 \cup C_2 \cup \cdots \cup C_n)$ in the usual sense, and is branched along C_i with branching index $r_i \geq 2$ for $0 \leq i \leq n$.

Given a divisor D on M, is there a finite Galois covering $X \to M$ branched at D? This problem was proposed by Fenchel in the case where M is a Riemann surface and is completely solved in this form: With two exceptions (I) $M = \mathbb{P}^1$, $D = rp$ and (II) $M = \mathbb{P}^1$, $D = rp + sq$, $r \neq s$, there always exists such a covering, see [2], [4]. Here, we discuss the case $M = \mathbb{P}^2$.

By the Grauert-Remmert theorem [5], any unbranched finite covering $X' \to \mathbb{P}^2 \backslash C$ extends to a finite covering $X \to \mathbb{P}^2$ branched along C, which is unique up to isomorphism. Hence, there is a one-to-one correspondance between the normal subgroups of finite index in $\pi_1(\mathbb{P}^2 \backslash C)$ and the Galois coverings $X \to \mathbb{P}^2$ branched along C. The covering space X is a possibly singular algebraic surface.

The map $X \to \mathbb{P}^2$ being branched at D leads one to study the quotient $Gr(D)$ of the fundamental group $\pi_1(\mathbb{P}^2 \backslash C)$, defined as follows: First take a small analytic disc Δ intersecting C_i transversally at a smooth point of C, and define a *meridian* of C_i to be the homotopy class in $\pi_1(\mathbb{P}^2 \backslash C, *)$ of a loop obtained by joining $*$ to a point in $\partial\Delta$ along a path ω, turning once around $\partial\Delta$ in the positive sense, and going back to $*$ along ω. It is well known that the group $\pi_1(\mathbb{P}^2 \backslash C)$ is generated by the meridians of C (see e.g. [14]). Define *the group of* D to be the group

$$Gr(D) := \pi_1(\mathbb{P}^2 \backslash C) / \ll \mu_1^{r_1}, \mu_2^{r_2} \dots, \mu_k^{r_k} \gg .$$

Since any two meridians of an irreducible component of C are conjugate elements in $\pi_1(\mathbb{P}^2 \backslash C)$, the group $Gr(D)$ do not depend on the particular choice of the meridians μ_i, so $Gr(D)$ is a projective invariant of the curve C. Moreover, Fenchel's problem has a simple formulation in terms of this invariant: Is there a surjection $\phi : Gr(D) \twoheadrightarrow K$ onto a finite group K such that $|\phi(\mu_i)| = r_i$? In what follows, such a surjection will be called a *good image* of $Gr(D)$.

There is not much hope for a complete solution of the problem for a general complex manifold M, due to two main difficulties, first being topological, the other group theoretical: Firstly, it is not easy to determine the group $\pi_1(M \backslash C)$. Even when $M = \mathbb{P}^2$, there is no effective algorithm to compute this group. Secondly, if M is a Riemann surface, then $\pi_1(M \backslash C)$ is a free group unless $C = \emptyset$, whereas even for $M = \mathbb{P}^2$, this group can be very complicated. The group $Gr(D)$ may even be trivial, consider for example $D = 2L_1 + 3L_2 + 5L_3$, where $L_i \subset \mathbb{P}^2$ intersect generically. [1] For arbitrary M, the group $\pi_1(M \backslash C)$ can also be trivial, e.g. take M to be a simply connected surface and C to be a contractible curve. This is why we shall consider Fenchel's problem in the surface $M = \mathbb{P}^2$ only. Note that, in the algebraic case, the problem in dimension ≥ 3 can be reduced to the problem in dimension 2 by Zariski's hyperplane section theorem.

Many solutions to Fenchel's problem can be obtained by considering abelian coverings. For example, if C is a smooth curve of degree d and $D = nC$, then $\pi_1(\mathbb{P}^2 \backslash C) \simeq \mathbb{Z}/d\mathbb{Z}$ and $Gr(D) \simeq \mathbb{Z}/(d,n)\mathbb{Z}$, so that one can say that our problem is solved for smooth curves. At the other extreme, one can consider C to be an arrangement of d lines, one has then $H_1(\mathbb{P}^2 \backslash C) \simeq \mathbb{Z}^{d-1}$. However, if one takes $D = 2L_1 + 3L_2 + 5L_3$, where this time both three of the lines L_i pass through a common point, then as above there is no abelian solution, whereas it is readily seen that

$$Gr(D) \simeq < \mu_1, \mu_2, \mu_3 \,|\, \mu_1^2 = \mu_2^3 = \mu_3^5 = \mu_1\mu_2\mu_3 = 1 > \simeq T_{2,3,5},$$

the latter group being the triangle group, which is finite of order 60. This suggests to look for the non-abelian solutions to the problem. Note however

[1] The group $\pi_1(\mathbb{P}^2 \backslash C)$ is then the abelian group generated by μ_1 and μ_2, with $\mu_3 = \mu_1\mu_2$, the elements μ_i being meridians of L_i.

that the corresponding affine problem in $\mathbb{C}^2$ has always a positive solution, given by an abelian covering.

Non-abelian solutions to Fenchel's problem have been studied mainly by Kato [6] and Namba [10]. The following result of Kato on line arrangements is well known:

Theorem (Kato [6]) *Let $\mathcal{A} = L_1 \cup L_2 \cup \cdots \cup L_n$ be a line arrangement. If on each L_i lies at least one triple or higher point of $\mathcal{A}$, then there is a finite Galois covering of $\mathbb{P}^2$ branched at $D = r_1L_1 + r_2L_2 + \cdots + r_nL_n$ for any $r_i \geq 2$, $1 \leq i \leq n$.*

For a version of this theorem with arrangements of conics, see [10]. To the author's knowledge, the rest of the literature available on Fenchel's problem are [7], [8], [9].

Results. Perhaps the first fact to notice for Fenchel's problem is the following trivial proposition.

Proposition 1 *Let D_1, D_2 be two divisors in $\mathbb{P}^2$ without any common component. If there are finite Galois coverings $X_i \to \mathbb{P}^2$ branched at D_i for $i = 1, 2$, then there is a finite Galois covering $X \to \mathbb{P}^2$ branched at $D_1 + D_2$.*

The covering $X \to \mathbb{P}^2$ can be constructed as the fibered product $X_1 \times_{\mathbb{P}^2} X_2$. Observe that Kato's theorem can not be derived from Theorem 1, since there are no coverings of $\mathbb{P}^2$ branched along a unique line L; obviously, $\mathbb{P}^2 \backslash L$ is simply connected.

In view of Proposition 1, it is natural to take a closer look at Fenchel's problem for divisors $D = rC$ with C being irreducible. Unfortunately, for such divisors we are still at the point where Zariski gave the complete solution for the three-cuspidal quartic curve [14]. The group $\pi_1(\mathbb{P}^2\backslash C)$ for this curve is a non-abelian group of order 12, so that all the Galois coverings branched along it can be characterized. For curves with an infinite non-abelian group, we have the result below.

Theorem 1 *Let $C \subset \mathbb{P}^2$ be an irreducible curve. If there is a surjection $\pi_1(\mathbb{P}^2\backslash C) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ for some $p \geq 2$, $q \geq 2$, then there is a finite Galois covering of $\mathbb{P}^2$ branched at rC for any $r \in \mathbb{N}$.*

Observe that there are irreducible curves C with $\pi_1(\mathbb{P}^2\backslash C) \simeq \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ by a result of Oka [11]. Examples of curves with non-trivial surjections as in the hypothesis of the theorem are given in [13].

Proof of Theorem 2 makes use of the following result of Namba. Let $D = r_1C_1 + r_2C_2 \cdots + r_kC_k$ be a divisor, with meridians μ_i of C_i, and let $\rho : Gr(D) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ be a representation of $Gr(D)$. We say that ρ is *essential* if $|\rho(\mu_i)| = r_i$.

Lemma (Namba [7]) *If $Gr(D)$ has an essential representation $\rho : Gr(D) \hookrightarrow GL_n(\mathbb{C})$, then $Gr(D)$ has a good image $Gr(D) \twoheadrightarrow K$. In other words, there is a finite Galois covering of $\mathbb{P}^2$ branched at D.*

This lemma is a direct consequence of the following result:

Theorem (Selberg [12]) *Let R be a non-trivial, finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then there exists a torsion-free normal subgroup N of R of finite index.*

Indeed, putting $R := \rho(Gr(D))$ and $K := R/N$ yields Namba's lemma.

Proof of Theorem 1. Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d, with a surjection $\phi : \pi_1(\mathbb{P}^2 \backslash C) \to \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, with $p, q \geq 2$. Observe that ϕ induce a surjection of the abelianized groups

$$\mathbb{Z}/d\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}.$$

As the latter group should be cyclic, one has $(p, q) = 1$. Now let μ be a meridian of C, and put $w := \phi(\mu)$. Then there is a surjection

$$Gr(rC) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}/ \ll w \gg \simeq \langle a, b |\, a^p = b^q = w^r = 1\rangle.$$

In the letter presentation, w cannot be conjugate to a nor to b. Indeed, if w were conjugate to, say, a, then setting $a = 1$ we would obtain a surjection

$$\pi_1(\mathbb{P}^2 \backslash C)/ \ll \mu \gg \twoheadrightarrow \mathbb{Z}/q\mathbb{Z},$$

which contradict the fact that the conjugacy class of the meridian μ generate $\pi_1(\mathbb{P}^2 \backslash C)$. This implies that the word w, which can be assumed to be cyclically reduced, involves both of the letters a and b. This matches with the following definition.

Definition. A *generalized triangle group* is a group given by the presentation

$$G := \langle a, b |\, a^p = b^q = w^r = 1\rangle,$$

where $2 \leq p, q, r \leq \infty$ and w is a cyclically reduced word involving both of a, b.

Theorem 2 follows then by an application of the following theorem to the generalized triangle group G, and by Namba's lemma. $\square$

Theorem (Baumslag, Morgan, Shalen [1]) *The generalized triangle group G has a representation $\rho : G \to PSL(2, \mathbb{C})$, such that the orders of $\rho(a)$, $\rho(b)$ and $\rho(w)$ are p, q, and r respectively. Moreover, G has a non-abelian free subgroup if*

$$\kappa := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

and is infinite if $\kappa = 1$.

Remark 1. The following direct consequence of the Baumslag, Morgan, Shalen theorem is noteworthy. If $C_{p,q}$ is an Oka curve (see [11]), with $\pi_1(\mathbb{P}^2 \backslash C_{p,q}) \simeq \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, then the group $Gr(rC_{p,q})$ contains a non-abelian free subgroup for any $r \geq 2$ provided that $p, q \geq 5$. On the other hand, for an irreducible curve C, the group $Gr(rC)$ may be trivial for infinitely many $r \in \mathbb{N}$, even if the group $\pi_1(\mathbb{P}^2 \backslash C)$ contains a non-abelian free subgroup. Such examples are discussed in [13], where the following question is raised:
Question. Let $C \subset \mathbb{P}^2$ be an irreducible curve, such that the group $\pi_1(\mathbb{P}^2 \backslash C)$ is infinite. Is it true that there are are infinitely many $r \in \mathbb{N}$ such that there exists a finite Galois covering of $\mathbb{P}^2$ branched at rC?

In contrast with the above remark, it can be proved that the group $Gr(2C)$ is finite under some rather restrictive hypothesis:

Proposition 2 *If C is an irreducible curve such that the group $\pi_1(\mathbb{P}^2 \backslash C)$ is generated by only two meridians of C, then $Gr(2C)$ is a finite group (it can be trivial).*

Proof. Suppose that the meridians μ and ν generate $\pi_1(\mathbb{P}^2 \backslash C)$. Then, since any two meridians are conjugate elements of $\pi_1(\mathbb{P}^2 \backslash C)$, one has $\mu = x\nu x^{-1}$, where x is a word in μ and ν. This implies that $Gr(2C)$ is a quotient of the group

$$K := \langle \mu, \nu \mid \mu^2 = \nu^2 = 1, \quad \mu = x\nu x^{-1} \rangle.$$

Since $\mu^2 = \nu^2 = 1$ in this latter group, the relation $\mu = x\nu x^{-1}$ can be written in the form $(\mu\nu)^n \mu (\mu\nu)^{-n} = \nu$ for some n. Hence,

$$K = \langle \mu, \nu \mid (\mu\nu)^{2n+1} = \mu^2 = \nu^2 = 1 \rangle,$$

that is, K is the dihedral group of order $4n+2$. $\square$

A direct application of the Zariski-Van Kampen theorem [14] shows that if an irreducible curve C of degree d has a flex F or a singular point p of order $(d-2)$, then the group $G=\pi_1(\mathbb{P}^2\backslash C)$ is generated by two meridians. Indeed, in the former case, considering projection with center $O\in F\backslash C$, one sees that $d-2$ of the generators of $\pi_1(\mathbb{P}^2\backslash C)$ are equal, so that there remains 3 generators. One of these generators can be eliminated by the projective relation. In the latter case, putting the center of projection at the singular point p yields the result.

Fenchel's problem under equisingular deformations. Another basic fact concerning Fenchel's problem will be obtained as a corollary to the following theorem.

Theorem (Zariski [14]) *If the family of curves $\{C_t\}_{0<|t|\leq 1}$ is equisingular, and the limit curve C_0 is reduced, then there is a surjection*

$$\phi:\pi_1(\mathbb{P}^2\backslash C_0)\twoheadrightarrow\pi_1(\mathbb{P}^2\backslash C_1).$$

The surjection ϕ is "natural" in the sense that ϕ sends meridians to meridians. Hence, under the hypothesis of Zariski's theorem, one has the induced surjections

$$Gr(rC_0)\twoheadrightarrow Gr(rC_1)$$

for any $r\in\mathbb{N}$. Assume that C_0, C_1 are irreducible. If we suppose that $Gr(rC_1)$ has a good image $Gr(rC_1)\twoheadrightarrow K$, we obtain the following corollary:

Corollary 1 *Suppose that C_0 is an irreducible curve. Under the hypothesis of Zariski's theorem, if there is a finite Galois covering of $\mathbb{P}^2$ branched at rC_1, then there is a finite Galois covering of $\mathbb{P}^2$ branched at rC_0.*

Remark 2. To conclude, let us give an example illustrating the utility of the group $Gr(D)$ as a projective invariant. In [3], Dimca gives an equisingular deformation of the Oka curve $C_{2,3}$ of degree $d=6$ to a sextic with a unique singular point of multiplicity $d-2=4$. Let $p,q\in\mathbb{N}$ be two coprime numbers with $1/p+1/q+1/2\leq 1$. *Then the Oka curve $C_{p,q}$ (of degree $d=pq$) cannot be equisingularly deformed to a reduced irreducible curve C' with a singular point of multiplicity $d-2$.* Indeed, by Corollary 1, such a deformation would induce a surjection $Gr(2C') \twoheadrightarrow Gr(2C_{p,q})$. By Proposition 2, $Gr(2C')$ is finite, whereas by the Baumslag-Morgan-Shalen theorem, $Gr(2C_{p,q})$ is infinite, contradiction.

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