ON FINITE BRANCHED UNIFORMIZATIONS OF THE PROJECTIVE PLANE

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We give a brief survey of the so-called Fenchel’s problem for the projective plane, that is the problem of existence of finite Galois coverings of the complex projective plane branched along a given divisor and prove the following result: Let \( p, q \) be two integers greater than 1 and \( C \) be an irreducible plane curve. If there is a surjection of the fundamental group of the complement of \( C \) into a free product of cyclic groups of orders \( p \) and \( q \), then there is a finite Galois covering of the projective plane branched along \( C \) with any given branching index.

Keywords: Fenchel problem; Uniformization; Branched covering; Generalized triangle group

Mathematics Subject Classification 2010: 32Q30, 57M12

1. Introduction

Let \( M \) be a complex manifold, \( C_1, C_2, \cdots, C_k \subset M \) be irreducible hypersurfaces, and \( C := \bigcup_{i=1}^{k} C_i \). A morphism \( X \to M \) is said to be a Galois covering of \( M \) branched at the divisor \( D := \sum_{i=1}^{k} r_i C_i \) if it is a Galois covering of \( M \setminus C \) in the usual sense, and is branched along \( C_i \) with branching index \( r_i \geq 2 \) for \( 1 \leq i \leq k \).

Given a divisor \( D \) on \( M \), is there a finite Galois covering \( X \to M \) branched at \( D \)? This problem was proposed by Fenchel in the case where \( M \) is a Riemann surface and is completely solved in this form: With two exceptions ("bad orbifolds" of Thurston) (I) \( M = \mathbb{P}^1, D = rp \) and (II) \( M = \mathbb{P}^1, D = r_1p_1 + r_2p_2, r_1 \neq r_2 \), there always exists such a covering, see [2] and [5]. Here, we discuss the case \( M = \mathbb{P}^2 \).

Note that we are not concerned with the smoothness of the covering space \( X \). Almost all pairs \((\mathbb{P}^2, D)\) that we consider in this paper does not admit finite smooth uniformizations; this is why we avoid the orbifold terminology.

By the Grauert-Remmert theorem [7], any unbranched finite covering \( X' \to \mathbb{P}^2 \setminus C \) extends to a finite covering \( X \to \mathbb{P}^2 \) branched along \( C \), which is unique up
to isomorphism. Hence, there is a one-to-one correspondence between the normal subgroups of finite index in $\pi_1(\mathbb{P}^2 \setminus C)$ and the Galois coverings $X \to \mathbb{P}^2$ branched along $C$. The covering space $X$ is a possibly singular algebraic surface.

Note that any finite branched cover is dominated by a finite branched Galois cover, but this point of view seems not to be very helpful in this problem, where we want to have a control on the branch curve and the ramification indices.

The map $X \to \mathbb{P}^2$ being branched at $D$ leads one to study the orbifold fundamental group $\pi_1^{orb}(\mathbb{P}^2, D)$ defined as follows: First take a small analytic disc $\Delta$ intersecting $C_i$ transversally at a smooth point of $C$, and define a meridian of $C_i$ to be the homotopy class in $\pi_1(\mathbb{P}^2 \setminus C, \ast)$ of a loop obtained by joining $\ast$ to a point in $\partial \Delta$ along a path $\omega$, turning once around $\partial \Delta$ in the positive sense, and going back to $\ast$ along $\omega$ (See Fig. 1). It is well known that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is generated by the meridians of $C_i$ (see e.g. [22]). Define orbifold fundamental group of $(\mathbb{P}^2, D)$ as

$$\pi_1^{orb}(\mathbb{P}^2, D) := \pi_1(\mathbb{P}^2 \setminus C)/\langle \langle \mu_1^{r_1}, \mu_2^{r_2}, \ldots, \mu_k^{r_k} \rangle \rangle.$$

Since any two meridians of an irreducible component of $C$ are conjugate elements in $\mathbb{P}^2 \setminus C$, the group $\pi_1^{orb}(\mathbb{P}^2, D)$ does not depend on the particular choice of the meridians $\mu_i$, so $\pi_1^{orb}(\mathbb{P}^2, D)$ is a projective invariant of the curve $C$. Moreover, Fenchel’s problem has a simple formulation in terms of this invariant: Is there a surjection $\phi : \pi_1^{orb}(\mathbb{P}^2, D) \to K$ onto a finite group $K$ such that $|\phi(\mu_i)| = r_i$? In what follows, such a surjection will be called a good image of $\pi_1^{orb}(\mathbb{P}^2, D)$.

There is not much hope for a complete solution of the problem for a general complex manifold $M$, due to two main difficulties, first being topological, the other group theoretical: Firstly, it is not easy to determine the group $\pi_1(M \setminus C)$. The Zariski-Van Kampen method provides an algorithm to compute this group for $M = \mathbb{P}^2$, but does not give any further information on the group. Secondly, if $M$ is a Riemann surface, then $\pi_1(M \setminus C)$ is a free group unless $C = \emptyset$, whereas even for $M = \mathbb{P}^2$, this group can be very complicated. The group $\pi_1^{orb}(\mathbb{P}^2, D)$ may even be trivial, consider for example $D = 2L_1 + 3L_2 + 5L_3$, where $L_i \subset \mathbb{P}^2$ intersect generically. The group $\pi_1(\mathbb{P}^2 \setminus C)$ in this case is the abelian group generated by $\mu_1$ and $\mu_2$, with $\mu_3 = \mu_1 \mu_2$, the elements $\mu_i$ being meridians of $L_i$. Hence, the group

![Fig. 1: Meridians of $C_i$](image-url)
\(\pi_1^{orb}(\mathbb{P}^2, D)\) is the trivial group with the presentation

\[\langle \mu_1, \mu_2, \mu_3 | \mu_1^2 = \mu_2^3 = \mu_3^2 = \mu_1 \mu_2 \mu_3 = [\mu_1, \mu_2] = [\mu_2, \mu_3] = [\mu_3, \mu_1] = 1 \rangle\]

For arbitrary \(M\), the group \(\pi_1(M \setminus C)\) can also be trivial, e.g. take \(M\) to be a simply connected surface and \(C\) to be a contractible curve. This is why we shall consider Fenchel’s problem for the surface \(M = \mathbb{P}^2\) only. Note that, in the algebraic case, the problem in dimension \(\geq 3\) can be reduced to the problem in dimension 2 by Zariski’s hyperplane section theorem.

Many solutions to Fenchel’s problem can be obtained by considering abelian coverings. For example, if \(C\) is a smooth curve of degree \(d\) and \(D = nC\), then \(\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}/d\mathbb{Z}\) and \(\pi_1^{orb}(\mathbb{P}^2, D) \simeq \mathbb{Z}/(d, n)\mathbb{Z}\), so that one can say that our problem is solved for smooth curves. All abelian finite smooth uniformizations of projective spaces of arbitrary dimension have been effectively classified in [20]. At the other extreme, one can consider \(C\) to be an arrangement of \(d\) lines, one has then \(H_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}^{d-1}\). However, if one takes \(D = 2L_1 + 3L_2 + 5L_3\), where this time both three of the lines \(L_i\) pass through a common point, then as above there is no abelian solution, whereas it is readily seen that

\[\pi_1^{orb}(\mathbb{P}^2, D) \simeq \langle \mu_1, \mu_2, \mu_3 | \mu_1^2 = \mu_2^3 = \mu_3^5 = \mu_1 \mu_2 \mu_3 = 1 \rangle \simeq T_{2,3,5},\]

the latter group being the triangle group, which is finite of order 60. This suggests to look for the non-abelian solutions to the problem. Note however that the corresponding affine problem in \(\mathbb{C}^2\) has always a positive solution, given by an abelian covering.

Non-abelian solutions to Fenchel’s problem have been studied mainly by Kato [9] and Namba [14]. The following result of Kato on line arrangements is well known:

**Theorem 1.1 (Kato [9]).** Let \(\mathcal{A} = \{L_1, L_2, \ldots, L_k\}\) be a line arrangement. If on each \(L_i\) lies at least one triple or higher point of \(\cup_{i=1}^k L_i\), then there is a finite Galois covering of \(\mathbb{P}^2\) branched at \(D = \sum_{i=1}^k r_i L_i\) for any \(r_i \geq 2, 1 \leq i \leq k\).

Note that applying some birational transformations to \(\mathcal{A}\) we get a divisor \(D\) whose support consists of curves of higher degree and Fenchel’s problem is solvable under the same conditions. There is a version of Kato’s theorem for conics, proved by Namba.

**Theorem 1.2 (Namba [14]).** Let \(C_1, C_2, \ldots, C_k\) be distinct irreducible conics in \(\mathbb{P}^2\). Suppose that for each \(C_i\) there is another \(C_j\) such that they are tangent at two distinct points (See Fig. 2). Then, for any integers \(r_i \geq 2, 1 \leq i \leq k\), there is a finite Galois covering \(X \to \mathbb{P}^2\) branching at \(D = \sum_{i=1}^k r_i C_i\).

There is very special conic-line arrangement which has been studied in depth, namely the Apollonius configuration, which is an arrangement of a smooth conic \(Q\) together with its \(k\)-distinct tangent lines \(T_i\). This configuration is special since the fundamental group of this space is the second braid group of a \(k\)-times punctured
sphere. First result of uniformization problem branching along Apollonius configuration, stated in Theorem 1.3, is obtained by Namba [12]. Later, it has been studied in detail by Ueno [17] and Uludag [19], and more general result, stated in Theorem 1.4, was obtained.

Theorem 1.3 (Namba [12]). Let $T_1$, $T_2$ and $T_3$ be 3 distinct lines on $\mathbb{P}^2$ circumscribing an irreducible conic $Q$. Then for any integers $r, s \geq 2$, there is a finite Galois covering $X \to \mathbb{P}^2$ branching at $D = r(\sum_{i=1}^{3} T_i) + sQ$

Theorem 1.4 (Ueno [17], Uludag [19]). Let $T_1, T_2, \ldots, T_k$ be $k$-distinct tangent lines of a smooth conic $Q$ in $\mathbb{P}^2$, and $r_i, s \geq 2$ be integers ($1 \leq i \leq k$). Then, there is a finite Galois covering $X \to \mathbb{P}^2$ branching at $D = \sum_{i=1}^{k} r_i T_i + 2sQ$ if $(r_1, \ldots, r_k, s)$ is one of the followings:

(i) $k = 2$ and $r_1 = r_2 \leq \infty$
(ii) $k = 3$ and $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \leq 1$
(iii) $k = 3$, $s = 1$ and $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} > 1$
(iv) $k \geq 4$

To the authors’ knowledge, the rest of the literature available on Fenchel’s problem are [11], [12] and [13].

Finally, note that the real version of this problem is also interesting and well-studied, i.e. in the theory of knots.
2. A result on coverings branched along an irreducible curve.

The first fact to notice about Fenchel’s problem is the following trivial proposition.

**Proposition 2.1.** Let $D_1$, $D_2$ be two divisors in $\mathbb{P}^2$ without any common component. If there are finite Galois coverings $X_i \rightarrow \mathbb{P}^2$ branched at $D_i$ for $i = 1, 2$, then there is a finite Galois covering $X \rightarrow \mathbb{P}^2$ branched at $D_1 + D_2$.

The covering $X \rightarrow \mathbb{P}^2$ can be constructed as the fibered product $X_1 \times_{\mathbb{P}^2} X_2$. Observe that Kato’s theorem can not be derived from Theorem 2.1, since there are no coverings of $\mathbb{P}^2$ branched along a unique line $L$; obviously, $\mathbb{P}^2 \setminus L$ is simply connected.

In view of Proposition 2.1, it is natural to study Fenchel’s problem for divisors $D = rC$ with $C$ being irreducible. Unfortunately, for such divisors we are still at the point where Zariski gave the complete solution for the three-cuspidal quartic curve [22]. The group $\pi_1(\mathbb{P}^2 \setminus C)$ for this curve is a non-abelian group of order 12, so that all the Galois coverings branched along it can be characterized. For curves with an infinite non-abelian group, we have the result below.

**Theorem 2.1.** Let $C \subset \mathbb{P}^2$ be an irreducible curve. If there is a surjection $\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \ast \mathbb{Z}/q\mathbb{Z}$ for some $p \geq 2$, $q \geq 2$, then there is a finite Galois covering of $\mathbb{P}^2$ branched at $rC$ for any $r \in \mathbb{N}$.

Observe that there are irreducible curves $C$ with $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}/p\mathbb{Z} \ast \mathbb{Z}/q\mathbb{Z}$ by a result of Oka [15]. Examples of curves with non-trivial surjections as in the hypothesis of the theorem are given in [18].

Proof of Theorem 2.1 makes use of the following result of Namba. Let $D = \sum_{i=1}^{k} r_i C_i$ be a divisor, with meridians $\mu_i$ of $C_i$, and let $\rho : \pi_1^{\text{orb}}(\mathbb{P}^2, D) \hookrightarrow \text{GL}_n(\mathbb{C})$ be a representation of $\pi_1^{\text{orb}}(\mathbb{P}^2, D)$. We say that $\rho$ is *essential* if $|\rho(\mu_i)| = r_i$.

**Lemma 2.1 (Namba [11]).** If $\pi_1^{\text{orb}}(\mathbb{P}^2, D)$ has an essential representation $\rho : \pi_1^{\text{orb}}(\mathbb{P}^2, D) \hookrightarrow \text{GL}_n(\mathbb{C})$, then $\pi_1^{\text{orb}}(\mathbb{P}^2, D)$ has a good image $\pi_1^{\text{orb}}(\mathbb{P}^2, D) \twoheadrightarrow K$. In other words, there is a finite Galois covering of $\mathbb{P}^2$ branched at $D$.

This lemma is a direct consequence of the following result:

**Theorem 2.2 (Selberg [16]).** Let $R$ be a non-trivial, finitely generated subgroup of $\text{GL}_n(\mathbb{C})$. Then there exists a torsion-free normal subgroup $N$ of $R$ of finite index.

Indeed, putting $R := \rho(\pi_1^{\text{orb}}(\mathbb{P}^2, D))$ and $K := R/N$ yields Namba’s lemma.

**Definition 2.1.** A *generalized triangle group* is a group given by the presentation

$$G_{p,q,r} := \langle a, b \mid a^p = b^q = w^r = 1 \rangle,$$

where $2 \leq p, q, r \leq \infty$ and $w$ is a cyclically reduced word involving both of $a, b$.

**Remark 2.1.** In the definition of generalized triangle group, we have omitted the case $r = 1$. But, the group $G_{p,q,1}$ is still non-trivial and interesting. One has $G_{2,2,1} \simeq \cdots$
The following direct consequence of the Theorem 2.3 is noteworthy.

\textbf{Remark 2.2.} The following direct consequence of the Theorem 2.3 is noteworthy. If \( C_{p,q} \) is an Oka curve (see [15]), with \( \pi_1(\mathbb{P}^2 \setminus C_{p,q}) \simeq \mathbb{Z}/p\mathbb{Z} \ast \mathbb{Z}/q\mathbb{Z} \), then the group \( \pi_1^{\text{orb}}(\mathbb{P}^2, rC_{p,q}) \) contains a non-abelian free subgroup for any \( r \geq 2 \) provided that \( p, q \geq 5 \). On the other hand, for an irreducible curve \( C \), the group \( \pi_1^{\text{orb}}(\mathbb{P}^2, rC) \) may be trivial for infinitely many \( r \in \mathbb{N} \), even if the group \( \pi_1(\mathbb{P}^2 \setminus C) \) contains a non-abelian free subgroup. Such examples are discussed in [18], where the following question is raised:

\textit{How do we determine the existence of a non-abelian free subgroup for a given curve \( C \) in \( \mathbb{P}^2 \)?}

\textbf{Theorem 2.3 (Baumslag, Morgan, Shalen [1]).} The generalized triangle group \( \Gamma = \langle a, b \mid a^p = b^q = (ab)^r = 1 \rangle \) has a representation \( \rho : \Gamma \to \text{PSL}(2, \mathbb{C}) \), such that the orders of \( \rho(a) \), \( \rho(b) \) and \( \rho(w) \) are \( p, q \), and \( r \), respectively. Moreover, the group \( \Gamma \)

(i) has a non-abelian free subgroup if \( \kappa := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \).

(ii) is infinite if \( \kappa = 1 \).

(iii) is infinite if \( \kappa = 1 \).

The following direct consequence of the Theorem 2.3 is noteworthy. If \( C_{p,q} \) is an irreducible curve in \( \mathbb{P}^2 \), with \( \pi_1(\mathbb{P}^2 \setminus C_{p,q}) \simeq \mathbb{Z}/p\mathbb{Z} \ast \mathbb{Z}/q\mathbb{Z} \), then the group \( \pi_1^{\text{orb}}(\mathbb{P}^2, rC_{p,q}) \) contains a non-abelian free subgroup for any \( r \geq 2 \) provided that \( p, q \geq 5 \). On the other hand, for an irreducible curve \( C \), the group \( \pi_1^{\text{orb}}(\mathbb{P}^2, rC) \) may be trivial for infinitely many \( r \in \mathbb{N} \), even if the group \( \pi_1(\mathbb{P}^2 \setminus C) \) contains a non-abelian free subgroup. Such examples are discussed in [18], where the following question is raised:
Question 2.1. Let $C \subset \mathbb{P}^2$ be an irreducible curve, such that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is infinite. Is it true that there are infinitely many $r \in \mathbb{N}$ such that there exists a finite Galois covering of $\mathbb{P}$ branched at $rC$?

In contrast with the Remark 2.2, it can be proved that the group $\pi_{orb}^1(\mathbb{P}^2, 2C)$ is finite under some rather restrictive hypothesis:

**Proposition 2.2.** If $C$ is an irreducible curve such that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is generated by only two meridians of $C$, then $\pi_{orb}^1(\mathbb{P}^2, 2C)$ is a finite group (it can be trivial).

**Proof.** Suppose that the meridians $\mu$ and $\nu$ generate $\pi_1(\mathbb{P}^2 \setminus C)$. Then, since any two meridians are conjugate elements of $\pi_1(\mathbb{P}^2 \setminus C)$, one has $\mu = x\nu x^{-1}$, where $x$ is a word in $\mu$ and $\nu$. This implies that $\pi_{orb}^1(\mathbb{P}^2, 2C)$ is a quotient of the group $K := \langle \mu, \nu | \mu^2 = \nu^2 = 1, \mu = x\nu x^{-1} \rangle$.

Since $\mu^2 = \nu^2 = 1$ in this latter group, the relation $\mu = x\nu x^{-1}$ can be written in the form $(\mu\nu)^{2n+1} = \mu^2 = \nu^2 = 1$, that is, $K$ is the dihedral group of order $4n + 2$.

A direct application of the Zariski-Van Kampen theorem [22] shows that if an irreducible curve $C$ of degree $d$ has a flex $F$ or a singular point $p$ of order $(d - 2)$, then the group $G = \pi_1(\mathbb{P}^2 \setminus C)$ is generated by two meridians. Indeed, in the former case, considering projection with center $O \in F \setminus C$, one sees that $d - 2$ of the generators of $\pi_1(\mathbb{P}^2 \setminus C)$ are equal, so that there remains 3 generators. One of these generators can be eliminated by the projective relation. In the latter case, putting the center of projection at the singular point $p$ yields the result.

3. Fenchel’s problem under equisingular deformations

Another basic fact concerning Fenchel’s problem will be obtained as a corollary to the following theorem.

**Theorem 3.1 (Zariski [22]).** If the family of curves $\{C_t\}_{0 < |t| \leq 1}$ is equisingular, and the limit curve $C_0$ is reduced, then there is a surjection

$$\phi : \pi_1(\mathbb{P}^2 \setminus C_0) \rightarrow \pi_1(\mathbb{P}^2 \setminus C_1).$$

The surjection $\phi$ is **natural** in the sense that $\phi$ sends meridians to meridians. Hence, under the hypothesis of Zariski’s theorem, one has the induced surjections

$$\pi_{orb}^1(\mathbb{P}^2, rC_0) \rightarrow \pi_{orb}^1(\mathbb{P}^2, rC_1)$$

for any $r \in \mathbb{N}$. Assume that $C_0$, $C_1$ are irreducible. If we suppose that $\pi_{orb}^1(\mathbb{P}^2, rC_0)$ has a good image $\pi_{orb}^1(\mathbb{P}^2, rC_1) \rightarrow K$, we obtain the following corollary:
Corollary 3.1. Suppose that $C_0$ is an irreducible curve. Under the hypothesis of Zariski’s theorem, if there is a finite Galois covering of $\mathbb{P}^2$ branched at $rC_1$, then there is a finite Galois covering of $\mathbb{P}^2$ branched at $rC_0$.

Remark 3.1. To conclude, let us give an example illustrating the utility of the group $\pi_1^{orb}(\mathbb{P}^2, D)$ as a projective invariant. In [4], Dimca gives an equisingular deformation of the Oka curve $C_{2,3}$ of degree $d = 6$ to a sextic with a unique singular point of multiplicity $d - 2 = 4$. Let $p, q \in \mathbb{N}$ be two coprime numbers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{2} \leq 1$. Then the Oka curve $C_{p,q}$ (of degree $d = pq$) cannot be equisingularly deformed to a reduced irreducible curve $C'$ with a singular point of multiplicity $d - 2$. Indeed, by Corollary 3.1, such a deformation would induce a surjection $\pi_1^{orb}(\mathbb{P}^2, 2C') \twoheadrightarrow \pi_1^{orb}(\mathbb{P}^2, 2C_{p,q})$. By Proposition 2.2, $\pi_1^{orb}(\mathbb{P}^2, 2C')$ is finite, whereas by the Theorem 2.3, $\pi_1^{orb}(\mathbb{P}^2, 2C_{p,q})$ is infinite, contradiction.

Acknowledgments

The first named author was supported by TÜBİTAK grant 110T690 and the Galatasaray University Research Fund project 09.504.001.

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